Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 2734947, 5 pages
http://dx.doi.org/10.1155/2016/2734947

Research Article

A Remark on the Stability of Approximative Compactness

Zhenghua Luo,[1] Longfa Sun,[2] and Wen Zhang[2]

[1]*School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China*
[2]*School of Mathematical Sciences, Xiamen University, Xiamen 361005, China*

Correspondence should be addressed to Wen Zhang; wenzhang@xmu.edu.cn

Received 26 November 2015; Accepted 12 January 2016

Academic Editor: Giuseppe Marino

We study the stability of approximative τ-compactness, where τ is the norm or the weak topology. Let Λ be an index set and for every $\lambda \in \Lambda$, let Y_λ be a subspace of a Banach space X_λ. For $1 \le p < \infty$, let $X = \oplus_{l_p} X_\lambda$ and $Y = \oplus_{l_p} Y_\lambda$. We prove that Y (resp., B_Y) is approximatively τ-compact in X if and only if, for every $\lambda \in \Lambda$, Y_λ (resp., B_{Y_λ}) is approximatively τ-compact in X_λ.

1. Introduction

Let X be a real Banach space and let K be a subset of X. We denote by τ either the norm or the weak topology on X. The metric projection of X onto K is the set valued map defined by $P_K(x) = \{y \in K : \|x-y\| = d(x,K)\}$ for $x \in X$, where $d(x,K)$ denotes the distance from x to K. If, for every $x \in X$, $P_K(x) \neq \emptyset$, we say that K is a proximinal subset of X. A sequence $\{y_n\} \subset K$ is called *minimizing* for $x \in X$, if $\|x - y_n\| \to d(x,K)$.

The notion of approximative compactness was introduced by Efimov and Stechkin [1] in connection with the study of Chebyshev sets in Banach spaces and plays an important role in approximation theory (see, e.g., [2, 3]). Deutsch [4] extended this notion to define approximative τ-compactness.

Definition 1. Let K be a τ-closed subset of X and $x_0 \in X$. We say that K is approximatively τ-compact for x_0 if every minimizing sequence $\{y_n\} \subset K$ for x_0 has a τ-convergent subsequence. If K is approximatively τ-compact for every $x \in X$, we say that K is approximatively τ-compact in X.

It is easy to verify that approximative τ-compactness implies proximinality. Clearly, compact sets or finite-dimensional subspaces of a Banach space are approximatively compact; weakly compact sets or reflexive subspaces of Banach spaces are approximatively weakly compact. Approximative τ-compactness has been studied in detail in [1, 3–7].

When it comes to the stability of approximative τ-compactness, we suppose that Λ is an index set and for every $\lambda \in \Lambda$, Y_λ is a subspace of a Banach space X_λ. And let $X = \oplus_{l_p} X_\lambda$, $Y = \oplus_{l_p} Y_\lambda$, where $1 \le p < \infty$. Bandyopadhyay et al. [5] proved that if Y is approximatively τ-compact in X, then Y_λ is approximatively τ-compact in X_λ for every $\lambda \in \Lambda$. In this paper, we prove that the converse is also true. On the other hand, the proximinality of the unit ball of subspaces has been the subject in many recent papers (see, e.g., [8–11]). In this paper, under the above assumption, we also prove that the unit ball of Y is approximatively τ-compact in X if and only if, for every $\lambda \in \Lambda$, the unit ball of Y_λ is approximatively τ-compact in X_λ.

For a real Banach space X, we denote by B_X the unit ball of X and denote by X^* the dual space of X. Before we prove the main conclusions we first show a simple property on approximative τ-compactness of the unit ball of subspaces.

Proposition 2. *Let Y be a subspace of a Banach space X. If B_Y is approximatively τ-compact in X, then so is Y. But the converse is not true.*

Proof. Suppose that $x \in X$ and $\{y_n\} \subset Y$ is a minimizing sequence of x in Y; that is, $\|x - y_n\| \to d(x,Y)$. Then $\{y_n\} \subset \lambda B_Y$ for sufficiently large $\lambda > 0$. This means that $d(x,Y) = d(x,\lambda B_Y)$ and $\{y_n\}$ is also a minimizing sequence of x in λB_Y. By approximative τ-compactness of λB_Y (which is equivalent to the one of B_Y), $\{y_n\}$ has a τ-convergent subsequence.

To illustrate that the converse is not true, first, we show that B_{c_0} is not approximatively weakly compact in c_0. Take $x = (2, 0, 0, \ldots)$ and for every $n \in \mathbb{N}$, $y_n = (1, 1, 1, \ldots, 1, 0, 0, \ldots)$, where 1 appears n times. Then $d(x, B_{c_0}) = 1$ and $\{y_n\}$ is a minimizing sequence of x in B_{c_0}. But $\{y_n\}$ has no weakly convergent subsequence. Hence B_{c_0} is not approximatively weakly compact in c_0.

Next, let $X = c_0 \oplus_{l_1} \mathbb{R}$ and $Y = c_0 \oplus_{l_1} \{0\}$. For any $\alpha = (z, r) \in X$, it is easy to see that $d(\alpha, Y) = |r|$ and $P_Y(\alpha) = \{(z, 0)\}$. Now, suppose $\{\beta_n\} = \{(z_n, 0)\} \subset Y$ is a minimizing sequence of α in Y; that is,

$$\|\alpha - \beta_n\| = \|z - z_n\| + |r| \longrightarrow d(\alpha, Y) = |r|. \tag{1}$$

This implies that $z_n \to z$. Hence $\beta_n = (z_n, 0) \to (z, 0)$. Therefore Y is approximatively compact in X. But, by the above discussion, B_Y is not approximatively weakly compact in Y, and not in X either. □

In order to prove our conclusions, we need the following lemmas.

Lemma 3. *Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of Banach spaces and let Y_i be a subspace of X_i, respectively, for $i \in \mathbb{N}$. Consider $X = \oplus_{l_p} X_i$ and $Y = \oplus_{l_p} Y_i$, where $1 \le p < \infty$. Let $x = (x_i) \in X$ and $\{y_n = (y_{n,i})\} \subset B_Y$ be a minimizing sequence of x in B_Y. Then, for every $\varepsilon > 0$, there exists some $j \in \mathbb{N}$ such that, for all n, $\sum_{i>j} \|y_{n,i}\|^p < \varepsilon^p$.*

Proof. If the conclusion does not hold, then, for every j, there exists infinitely many n such that $\sum_{i>j} \|y_{n,i}\|^p \ge \varepsilon^p$. We can choose some j_0 such that $\sum_{i>j_0} \|x_i\|^p < (\varepsilon/3)^p$ and infinite subset $\{n_k\}$ of $\mathbb{N}$ such that $\sum_{i>j_0} \|y_{n_k,i}\|^p \ge \varepsilon^p$ for every k. Therefore for every k,

$$\begin{aligned}
\|x - y_{n_k}\|^p &= \sum_{i \le j_0} \|x_i - y_{n_k,i}\|^p + \sum_{i>j_0} \|x_i - y_{n_k,i}\|^p \\
&= \left(\sum_{i \le j_0} \|x_i - y_{n_k,i}\|^p + \sum_{i>j_0} \|x_i\|^p \right) \\
&\quad + \sum_{i>j_0} \|x_i - y_{n_k,i}\|^p - \sum_{i>j_0} \|x_i\|^p \\
&\ge (d(x, B_Y))^p \\
&\quad + \left[\left(\sum_{i>j_0} \|y_{n_k,i}\|^p \right)^{1/p} - \left(\sum_{i>j_0} \|x_i\|^p \right)^{1/p} \right]^p \\
&\quad - \sum_{i>j_0} \|x_i\|^p \ge (d(x, B_Y))^p + \left(\frac{2\varepsilon}{3}\right)^p - \left(\frac{\varepsilon}{3}\right)^p \\
&\ge (d(x, B_Y))^p + \left(\frac{\varepsilon}{3}\right)^p.
\end{aligned} \tag{2}$$

But $\|x - y_n\| \to d(x, B_Y) (n \to \infty)$; then $\|x - y_n\|^p < (d(x, B_Y))^p + (\varepsilon/3)^p$ for sufficiently large n. This is a contradiction. □

Remark 4. In Lemma 3, if we replace B_Y by Y, that is, $\{y_n = (y_{n,i})\} \subset Y$ is a minimizing sequence of x in Y, then the conclusion still holds.

Lemma 5. *Under the assumption in Lemma 3, if, moreover, $\lim_{n\to\infty} \|y_n\| = r$ and for every $i \in \mathbb{N}$, $\lim_{n\to\infty} \|y_{n,i}\| = r_i$, then*

(1) $r = (\sum_{i\in\mathbb{N}} r_i^p)^{1/p}$;

(2) $d(x, B_Y) = d(x, \oplus_{l_p} r_i B_{Y_i}) = (\sum_{i\in\mathbb{N}} [d(x_i, r_i B_{Y_i})]^p)^{1/p}$.

Proof. (1) For every $\varepsilon > 0$, by Lemma 3, there exists $j \in \mathbb{N}$ such that, for all n, $\sum_{i>j} \|y_{n,i}\|^p < \varepsilon^p$. For every fixed $j' > j$, we can choose n' such that $\sum_{i\le j'} |\|y_{n',i}\|^p - r_i^p| < \varepsilon^p$ and $|\|y_{n'}\|^p - r^p| < \varepsilon^p$. Then

$$\begin{aligned}
\left| \sum_{i\le j'} r_i^p - r^p \right| &\le \left| \sum_{i\le j'} r_i^p - \|y_{n'}\|^p \right| + \left| \|y_{n'}\|^p - r^p \right| \\
&= \left| \sum_{i\le j'} r_i^p - \sum_{i\in\mathbb{N}} \|y_{n',i}\|^p \right| + \left| \|y_{n'}\|^p - r^p \right| \\
&\le \sum_{i\le j'} \left| r_i^p - \|y_{n',i}\|^p \right| + \sum_{i>j'} \|y_{n',i}\|^p \\
&\quad + \left| \|y_{n'}\|^p - r^p \right| < 3\varepsilon^p.
\end{aligned} \tag{3}$$

By the arbitrariness of ε, we have $r = (\sum_{i\in N} r_i{}^p)^{1/p}$.

(2) Note that $r \le 1$; hence $\oplus_{l_p} r_i B_{Y_i} \subset B_Y$. This implies that $d(x, B_Y) \le d(x, \oplus_{l_p} r_i B_{Y_i})$. To prove that $d(x, B_Y) \ge d(x, \oplus_{l_p} r_i B_{Y_i})$, for every n, we define $z_n = (z_{n,i})$, where $z_{n,i} = y_{n,i}$ for $\|y_{n,i}\| \le r_i$, and $z_{n,i} = (r_i/\|y_{n,i}\|) y_{n,i}$ for $\|y_{n,i}\| > r_i$. Then $\{z_n\} \subset \oplus_{l_p} r_i B_{Y_i}$.

For arbitrary $\varepsilon > 0$, by Lemma 3, there exists j such that $\sum_{i>j} r_i^p < \varepsilon^p$, and for all n, $\sum_{i>j} \|y_{n,i}\|^p < \varepsilon^p$. Further, we can choose some n_0 such that, for all $n > n_0$, $\sum_{i\le j} |\|y_{n,i}\| - r_i|^p < \varepsilon^p$. Then for all $n > n_0$, we have

$$\begin{aligned}
\|y_n - z_n\| &= \left(\sum_{i\in\mathbb{N}} \|y_{n,i} - z_{n,i}\|^p \right)^{1/p} \\
&\le \left(\sum_{i\in\mathbb{N}} |\|y_{n,i}\| - r_i|^p \right)^{1/p} \\
&\le \left(\sum_{i\le j} |\|y_{n,i}\| - r_i|^p \right)^{1/p} \\
&\quad + \left(\sum_{i>j} |\|y_{n,i}\| - r_i|^p \right)^{1/p} \\
&\le \left(\sum_{i\le j} |\|y_{n,i}\| - r_i|^p \right)^{1/p} + \left(\sum_{i>j} \|y_{n,i}\|^p \right)^{1/p} \\
&\quad + \left(\sum_{i>j} r_i^p \right)^{1/p} < 3\varepsilon.
\end{aligned} \tag{4}$$

By the arbitrariness of ε, we have $\|y_n - z_n\| \to 0$. This implies that

$$\lim_{n\to\infty}\|x - z_n\| = \lim_{n\to\infty}\|x - y_n\| = d(x, B_Y). \tag{5}$$

Therefore $d(x, B_Y) \geq d(x, \oplus_{l_p} r_i B_{Y_i})$. So we have $d(x, B_Y) = d(x, \oplus_{l_p} r_i B_{Y_i})$.

For the second equality, first, it is obvious that

$$d\left(x, \oplus_{l_p} r_i B_{Y_i}\right) \geq \left(\sum_{i\in\mathbb{N}}\left[d\left(x_i, r_i B_{Y_i}\right)\right]^p\right)^{1/p}. \tag{6}$$

On the other hand, let $\varepsilon > 0$ be given. For every i, we can choose $z_i \in r_i B_{Y_i}$ such that $\|x_i - z_i\|^p < [d(x_i, r_i B_{Y_i})]^p + \varepsilon/2^i$. Let $z = (z_i) \in \oplus_{l_p} r_i B_{Y_i}$; then

$$\begin{aligned}\|x - z\| &= \left(\sum_{i\in\mathbb{N}}\|x_i - z_i\|^p\right)^{1/p}\\ &< \left(\sum_{i\in\mathbb{N}}\left[d\left(x_i, r_i B_{Y_i}\right)\right]^p + \varepsilon\right)^{1/p}.\end{aligned} \tag{7}$$

By the arbitrariness of ε, we have $d(x, \oplus_{l_p} r_i B_{Y_i}) \leq (\sum_{i\in\mathbb{N}}[d(x_i, r_i B_{Y_i})]^p)^{1/p}$. Therefore the second equality holds. □

The following is our main result.

Theorem 6. *Let Λ be an index set. For every $\lambda \in \Lambda$, let Y_λ be a subspace of a Banach space X_λ. For $1 \leq p < \infty$, let $X = \oplus_{l_p} X_\lambda$ and $Y = \oplus_{l_p} Y_\lambda$. Then*

(1) *Y is approximatively τ-compact in X if and only if, for every $\lambda \in \Lambda$, Y_λ is approximatively τ-compact in X_λ;*

(2) *B_Y is approximatively τ-compact in X if and only if, for every $\lambda \in \Lambda$, B_{Y_λ} is approximatively τ-compact in X_λ.*

Proof. (1) Necessity has been proven in [5].

Sufficiency: let $x \in X$ and $\{y_n\} \subset Y$ be a minimizing sequence for x. We will show that $\{y_n\}$ has a τ-convergent subsequence. Without loss of generality, we can assume $\Lambda = \mathbb{N}$ and $x = (x_i)$, $y_n = (y_{n,i})$.

First, notice that if $z_i \in P_{Y_i}(x_i)$ for every i, then $z = (z_i) \in P_Y(x)$. Hence $(d(x, Y))^p = \sum_{j\in\mathbb{N}}(d(x_j, Y_j))^p$. And for every i,

$$\begin{aligned}\|x_i - y_{n,i}\|^p + \sum_{j\neq i}\left(d\left(x_j, Y_j\right)\right)^p &\leq \|x - y_n\|^p\\ &= (d(x, Y))^p + \left[\|x - y_n\|^p - (d(x, Y))^p\right]\\ &= \sum_{j\in\mathbb{N}}\left(d\left(x_j, Y_j\right)\right)^p + \left[\|x - y_n\|^p - (d(x, Y))^p\right].\end{aligned} \tag{8}$$

So

$$\begin{aligned}\|x_i - y_{n,i}\|^p &\leq (d(x_i, Y_i))^p + \left[\|x - y_n\|^p - (d(x, Y))^p\right]\\ &\longrightarrow (d(x_i, Y_i))^p \ (n \longrightarrow \infty).\end{aligned} \tag{9}$$

This implies that, for every i, $\{y_{n,i}\}$ is a minimizing sequence for x_i in Y_i. Then $\{y_{n,i}\}$ has a τ-convergent subsequence by the approximative τ-compactness of Y_i. By employing the diagonal process, we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that, for every i, $\{y_{n_k,i}\}$ has a τ-convergent to some $y_i \in Y_i$. Obviously, $y_i \in P_{Y_i}(x_i)$, and $y = (y_i) \in Y$, $\|x - y\| = d(x, Y)$.

We still denote the subsequence $\{y_{n_k}\}$ as $\{y_n\}$. Next, to complete the proof, we will prove that $\{y_n\}$ has a τ-convergent to y.

Case 1. τ is the norm topology. For every $\varepsilon > 0$, by Remark 4, there exists some j such that $\sum_{i>j}\|y_i\|^p < \varepsilon^p$ and for all n, $\sum_{i>j}\|y_{n,i}\|^p < \varepsilon^p$. Then we can choose n_0 such that, for $n > n_0$, $\sum_{i\leq j}\|y_{n,i} - y_i\|^p < \varepsilon^p$. Hence for all $n > n_0$,

$$\begin{aligned}\|y_n - y\| &= \left(\sum_{i\leq j}\|y_{n,i} - y_i\|^p + \sum_{i>j}\|y_{n,i} - y_i\|^p\right)^{1/p}\\ &\leq \left(\sum_{i\leq j}\|y_{n,i} - y_i\|^p\right)^{1/p} + \left(\sum_{i>j}\|y_{n,i}\|^p\right)^{1/p}\\ &\quad + \left(\sum_{i>j}\|y_i\|^p\right)^{1/p} < 3\varepsilon.\end{aligned} \tag{10}$$

Therefore, by the arbitrariness of ε, we have that $\{y_n\}$ converges to y.

Case 2. τ is the weak topology. Suppose $f = (f_i) \in \oplus_{l_q} X_i^* = X^*$ with $\|f\| = 1$, where $1/p + 1/q = 1$ when $p > 1$ and $q = \infty$ when $p = 1$. For every $\varepsilon > 0$, again by Remark 4, we can choose some j such that $\sum_{i>j}\|y_i\|^p < \varepsilon^p$, and for all n, $\sum_{i>j}\|y_{n,i}\|^p < \varepsilon^p$. Note that, for every $1 \leq i \leq j$, $\{y_{n,i}\}$ weakly converges to y_i; hence there exists n_0 such that, for $n > n_0$, $|\sum_{i\leq j} f_i(y_{n,i} - y_i)| < \varepsilon$. Then for all $n > n_0$,

$$\begin{aligned}|f(y_n - y)| &= \left|\sum_{i\in\mathbb{N}} f_i(y_{n,i} - y_i)\right|\\ &\leq \left|\sum_{i\leq j} f_i(y_{n,i} - y_i)\right| + \left|\sum_{i>j} f_i(y_{n,i} - y_i)\right|\\ &\leq \varepsilon + \|f\|\left(\sum_{i>j}\|y_{n,i} - y_i\|^p\right)^{1/p}\\ &\leq \varepsilon + \left(\sum_{i>j}\|y_{n,i}\|^p\right)^{1/p} + \left(\sum_{i>j}\|y_i\|^p\right)^{1/p}\\ &< 3\varepsilon.\end{aligned} \tag{11}$$

Again by the arbitrariness of ε, we have that $\{y_n\}$ weakly converges to y.

(2) Necessity: fix $\lambda_0 \in \Lambda$. Suppose that $x_{\lambda_0} \in X_{\lambda_0}$, and $\{y_{n,\lambda_0}\} \subset B_{Y_{\lambda_0}}$ is a minimizing sequence of x_{λ_0} in $B_{Y_{\lambda_0}}$. Let $x = (x_\lambda)$, $y_n = (y_{n,\lambda})$, where $x_\lambda = 0$ and $y_{n,\lambda} = 0$ for $\lambda \neq \lambda_0$. Then $x \in X$, $\{y_n\} \subset B_Y$, and

$$\|x - y_n\| = \|x_{\lambda_0} - y_{n,\lambda_0}\| \longrightarrow d\left(x_{\lambda_0}, B_{Y_{\lambda_0}}\right). \tag{12}$$

Notice that $d(x, B_Y) \geq d(x_{\lambda_0}, B_{Y_{\lambda_0}})$. Hence

$$\|x - y_n\| \longrightarrow d\left(x_{\lambda_0}, B_{Y_{\lambda_0}}\right) = d\left(x, B_Y\right), \tag{13}$$

which implies that $\{y_n\}$ is a minimizing sequence of x in B_Y. By approximative τ-compactness of B_Y in X, $\{y_n\}$ has a τ-convergent subsequence $\{y_{n_k}\}$. Therefore $\{y_{n_k,\lambda_0}\}$ is τ-convergent.

Sufficiency: suppose that $x \in X \setminus B_Y$ and $\{y_n\} \subset B_Y$ is a minimizing sequence of x in B_Y. Like the proof in (1), we will prove that $\{y_n\}$ has a τ-convergent subsequence and we can assume $\Lambda = \mathbb{N}$, $x = (x_i)$, and $y_n = (y_{n,i})$. By employing the diagonal process, we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ (we still denote the subsequence as $\{y_n\}$) such that $\lim_{n\to\infty}\|y_n\| = r$, and for every $i \in \mathbb{N}$, $\lim_{n\to\infty}\|y_{n,i}\| = r_i$. Then by Lemma 5, we have $r = (\sum_{i\in\mathbb{N}} r_i^p)^{1/p}$, and

$$\begin{aligned} d\left(x, B_Y\right) &= d\left(x, \oplus_{l_p} r_i B_{Y_i}\right) \\ &= \left(\sum_{i\in\mathbb{N}} \left[d\left(x_i, r_i B_{Y_i}\right)\right]^p\right)^{1/p}. \end{aligned} \tag{14}$$

Next, for every $i \in \mathbb{N}$, we will show that $\{y_{n,i}\}$ has a τ-convergent subsequence. We can assume that, for all n and i, $\|y_{n,i}\| \leq r_i$. Otherwise, we can replace $\{y_n\}$ with $\{z_n\}$ which we define in the proof of Lemma 5(2).

Note that, for every $i \in \mathbb{N}$,

$$\begin{aligned} &\|x_i - y_{n,i}\|^p + \sum_{j\neq i}\left[d\left(x_j, r_j B_{Y_j}\right)\right]^p \leq \|x - y_n\|^p \\ &\quad = \left[d\left(x, B_Y\right)\right]^p + \|x - y_n\|^p - \left[d\left(x, B_Y\right)\right]^p \\ &\quad = \sum_{j\in N}\left[d\left(x_j, r_j B_{Y_j}\right)\right]^p + \|x - y_n\|^p - \left[d\left(x, B_Y\right)\right]^p. \end{aligned} \tag{15}$$

Then

$$\begin{aligned} \left[d\left(x_i, r_i B_{Y_i}\right)\right]^p &\leq \|x_i - y_{n,i}\|^p \\ &\leq \left[d\left(x_i, r_i B_{Y_i}\right)\right]^p + \|x - y_n\|^p \\ &\quad - \left[d\left(x, B_Y\right)\right]^p. \end{aligned} \tag{16}$$

Hence, when $n \to \infty$,

$$\|x_i - y_{n,i}\| \longrightarrow d\left(x_i, r_i B_{Y_i}\right). \tag{17}$$

This implies that $\{y_{n,i}\}$ is a minimizing sequence of x_i in $r_i B_{Y_i}$. By approximative τ-compactness of $r_i B_{Y_i}$ in X_i, $\{y_{n,i}\}$ has a τ-convergent subsequence.

Employing the diagonal process again, we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that, for evrey i, $\{y_{n_k,i}\}$ has a τ-convergents to some $y_i \in r_i B_{Y_i}$. Let $y = (y_i)$; then $y \in \oplus_{l_p} r_i B_{Y_i}$. We still denote $\{y_{n_k}\}$ as $\{y_n\}$. Finally, just like the proof in (1), we can prove that $\{y_n\}$ has a τ-convergent to y. □

Remark 7. The above theorem does not hold for $p = \infty$. Indeed, suppose that Z is an infinite-dimensional proper closed subspace of l_2. By Theorem 1.4 in [12], B_Z is approximatively compact in l_2. Next, we show that $Z\oplus_{l_\infty} Z$ is not approximatively compact in $l_2\oplus_{l_\infty} l_2$. Choose $x_0 \in l_2 \setminus Z$ and $y_0 \in Z$ such that $d(x_0, Z) = 1 = \|x_0 - y_0\|$. Furthermore, we take a sequence $\{z_n\} \subset Z$ with $\|z_n\| = 1$ satisfying that $\{z_n\}$ has no convergent subsequence. Note that

$$\|(0, x_0) - (z_n, y_0)\|_\infty = \max\{\|z_n\|, \|x_0 - y_0\|\} = 1; \tag{18}$$

and for any $(z, y) \in Z\oplus_{l_\infty} Z$,

$$\begin{aligned} \|(0, x_0) - (z, y)\|_\infty &= \max\{\|z\|, \|x_0 - y\|\} \geq \|x_0 - y\| \\ &\geq 1. \end{aligned} \tag{19}$$

This means that $d((0, x_0), Z\oplus_{l_\infty} Z) = 1$ and $\{(z_n, y_0)\}$ is a minimizing sequence of $(0, x_0)$ in $Z\oplus_{l_\infty} Z$. But $\{(z_n, y_0)\}$ has no convergent subsequence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank colleagues and graduate students in the Functional Analysis group of Xiamen University for their very helpful conversations and suggestions. Zhenghua Luo was supported partially by the Natural Science Foundation of China, Grant no. 11201160, and the Natural Science Foundation of Fujian Province, Grant no. 2012J05006. Wen Zhang was supported in part by the Natural Science Foundation of China, Grant no. 11471270, and by the Natural Science Foundation of Fujian Province, Grant no. 2015J01022.

References

[1] N. V. Efimov and S. B. Stechkin, "Approximative compactness and Chebyshev sets," *Soviet Mathematics—Doklady*, vol. 2, pp. 1226–1228, 1961.

[2] V. S. Balaganskii and L. P. Vlasov, "The problem of the convexity of Chebyshev sets," *Uspekhi Matematicheskikh Nauk*, vol. 51, pp. 125–188, 1996.

[3] I. Singer, "Some remarks on approximative compactness," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 9, pp. 167–177, 1964.

[4] F. Deutsch, "Existence of best approximations," *Journal of Approximation Theory*, vol. 28, no. 2, pp. 132–154, 1980.

[5] P. Bandyopadhyay, Y. Li, B.-L. Lin, and D. Narayana, "Proximinality in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 309–317, 2008.

[6] P. A. Borodin and I. A. Pyatyshev, "An example of non-approximatively-compact existence set with finite-valued metric projection," *Mathematical Notes*, vol. 86, no. 2, pp. 159–162, 2009.

[7] I. A. Pyatyshev, "Operation on approximatively compact sets," *Mathematical Notes*, vol. 82, no. 5, pp. 653–659, 2007.

[8] P. Bandyopadhyay, B.-L. Lin, and T. S. S. R. K. Rao, "Ball proximinality in Banach spaces," in *Banach Spaces and Their Applications in Analysis. Conference in Honor of Nigel Kalton's 60th Birthday*, pp. 251–264, De Gruyter, Berlin, Germany, 2007.

[9] V. Indumathi, S. Lalithambigai, and B.-L. Lin, "Ball proximinality of closed $*$ subalgebras in C(Q)," *Extracta Mathematicae*, vol. 22, pp. 1–17, 2007.

[10] S. Lalithambigai, "Ball proximinality of equable spaces," *Collectanea Mathematica*, vol. 60, no. 1, pp. 79–88, 2009.

[11] F. B. Saidi, "On the proximinality of the unit ball of proximinal subspaces in Banach spaces: a counterexample," *Proceedings of the American Mathematical Society*, vol. 133, no. 9, pp. 2697–2703, 2005.

[12] S. Dutta and P. Shunmugaraj, "Strong proximinality of closed convex sets," *Journal of Approximation Theory*, vol. 163, no. 4, pp. 547–553, 2011.