Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 3805804, 12 pages
http://dx.doi.org/10.1155/2016/3805804

Research Article

Multiple Positive Solutions of Third-Order BVP with Advanced Arguments and Stieltjes Integral Conditions

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Received 24 March 2016; Revised 26 June 2016; Accepted 5 July 2016

Academic Editor: Gennaro Infante

We consider the following third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions: $u'''(t)+f(t,u(\alpha(t)))=0,\ t\in(0,1),\ u(0)=\gamma u(\eta_1)+\lambda_1[u]$ and $u''(0)=0,\ u(1)=\beta u(\eta_2)+\lambda_2[u]$, where $0<\eta_1<\eta_2<1$, $0\le\gamma,\beta\le 1$, $\alpha:[0,1]\to[0,1]$ is continuous, $\alpha(t)\ge t$ for $t\in[0,1]$, and $\alpha(t)\le\eta_2$ for $t\in[\eta_1,\eta_2]$. Under some suitable conditions, by applying a fixed point theorem due to Avery and Peterson, we obtain the existence of multiple positive solutions to the above problem. An example is also included to illustrate the main results obtained.

1. Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity driven flows, and so on [1].

Recently, third-order boundary value problems (BVPs for short) have received much attention from many authors; see [2–20] and the references therein. However, it is necessary to point out that all the unknown functions in the above-mentioned papers do not depend on advanced arguments.

In 2012, Jankowski [21] studied the existence of multiple positive solutions to the BVP

$$\begin{gathered} u'''(t)+h(t)f(t,u(\alpha(t)))=0,\quad t\in(0,1),\\ u(0)=u''(0)=0,\\ u(1)=\beta u(\eta)+\lambda[u], \end{gathered}\tag{1}$$

where the unknown function u depended on an advanced argument α satisfying the following condition:

(A_0) $\alpha:[0,1]\to[0,1]$ was continuous and $\alpha(t)\ge t$ for $t\in[0,1]$.

λ denoted a linear functional on $C[0,1]$ given by

$$\lambda[u]=\int_0^1 u(t)\,d\Lambda(t)\tag{2}$$

involving a Stieltjes integral with a suitable function Λ of bounded variation. The measure $d\Lambda$ could be a signed one. The situation with a signed measure $d\Lambda$ was first discussed in [22, 23] for second-order differential equations; it was also discussed in [24, 25] for second-order impulsive differential equations.

Among the boundary conditions in (1), only $u(1)$ was related to $u(\eta)$ and a Stieltjes integral. When $u(0)$ was also related to $u(\eta)$ and the Stieltjes integral, the authors in [26, 27] obtained the existence and multiplicity of positive solutions to the BVP

$$\begin{gathered} u'''(t)+f(t,u(\alpha(t)))=0,\quad t\in(0,1),\\ u(0)=\gamma u(\eta)+\lambda[u],\\ u''(0)=0,\\ u(1)=\beta u(\eta)+\lambda[u], \end{gathered}\tag{3}$$

where $\lambda[u]$ was defined as in (2) and the condition (A_0) was imposed on the advanced argument α. The main tools used were the Guo-Krasnoselskii fixed point theorem [28, 29] and a fixed point theorem due to Avery and Peterson [30].

In this paper, we are concerned with the following third-order BVP with advanced arguments and Stieltjes integral boundary conditions:

$$\begin{aligned} u'''(t) + f(t, u(\alpha(t))) &= 0, \quad t \in (0,1), \\ u(0) &= \gamma u(\eta_1) + \lambda_1[u], \\ u''(0) &= 0, \\ u(1) &= \beta u(\eta_2) + \lambda_2[u], \end{aligned} \tag{4}$$

where

$$\lambda_i[u] = \int_0^1 u(t)\, d\Lambda_i(t), \quad i = 1, 2, \tag{5}$$

here Λ_1 and Λ_2 are suitable functions of bounded variation. It is important to indicate that it is not assumed that $\lambda_i[u]$ $(i = 1, 2)$ is positive to all positive u. Throughout this paper, we always assume that $0 < \eta_1 < \eta_2 < 1$, $0 \le \gamma, \beta \le 1$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, and the advanced argument α satisfies the following condition:

(A_0') $\alpha : [0,1] \to [0,1]$ is continuous, $\alpha(t) \ge t$ for $t \in [0,1]$ and $\alpha(t) \le \eta_2$ for $t \in [\eta_1, \eta_2]$.

In order to obtain our main results, we need the following concepts and fixed point theorem [30].

Let E be a real Banach space and let K be a cone in E.

A map Θ is said to be a nonnegative continuous convex functional on K if $\Theta : K \to [0, \infty)$ is continuous and $\Theta(tu + (1-t)v) \le t\Theta(u) + (1-t)\Theta(v)$ for all $u, v \in K$ and $t \in [0,1]$.

Similarly, a map Φ is said to be a nonnegative continuous concave functional on K if $\Phi : K \to [0, \infty)$ is continuous and $\Phi(tu + (1-t)v) \ge t\Phi(u) + (1-t)\Phi(v)$ for all $u, v \in K$ and $t \in [0,1]$.

Let φ and Θ be nonnegative continuous convex functionals on K, let Φ be a nonnegative continuous concave functional on K, and let Ψ be a nonnegative continuous functional on K. For positive numbers a, b, c, d, we define the following sets:

$$\begin{aligned} K(\varphi, d) &= \{u \in K : \varphi(u) < d\}, \\ K(\varphi, \Phi, b, d) &= \{u \in K : b \le \Phi(u),\ \varphi(u) \le d\}, \\ K(\varphi, \Theta, \Phi, b, c, d) & \\ &= \{u \in K : b \le \Phi(u),\ \Theta(u) \le c,\ \varphi(u) \le d\}, \\ R(\varphi, \Psi, a, d) &= \{u \in K : a \le \Psi(u),\ \varphi(u) \le d\}. \end{aligned} \tag{6}$$

Theorem 1 (Avery and Peterson fixed point theorem). *Let E be a real Banach space and let K be a cone in E. Let φ and Θ be nonnegative continuous convex functionals on K, let Φ be a nonnegative continuous concave functional on K, and let Ψ be a nonnegative continuous functional on K satisfying $\Psi(ku) \le k\Psi(u)$ for $0 \le k \le 1$, such that, for some positive numbers M and d,*

$$\begin{aligned} \Phi(u) &\le \Psi(u), \\ \|u\| &\le M\varphi(u) \end{aligned} \tag{7}$$

for all $u \in \overline{K(\varphi, d)}$. Suppose $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

(C1) $\{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\} \ne \phi$ and $\Phi(Su) > b$ for $u \in K(\varphi, \Theta, \Phi, b, c, d)$;

(C2) $\Phi(Su) > b$ for $u \in K(\varphi, \Phi, b, d)$ with $\Theta(Su) > c$;

(C3) $\theta \notin R(\varphi, \Psi, a, d)$ and $\Psi(Su) < a$ for $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = a$.

Then S has at least three fixed points $u_1, u_2, u_3 \in \overline{K(\varphi, d)}$, such that

$$\begin{aligned} b &< \Phi(u_1), \\ a &< \Psi(u_2) \quad \text{with } \Phi(u_2) < b, \\ \Psi(u_3) &< a. \end{aligned} \tag{8}$$

2. Main Results

For convenience, we denote

$$\begin{aligned} \Delta &= (1-\gamma)(1-\eta_2\beta) + (1-\beta)\eta_1\gamma, \\ k(t,s) &= \frac{1}{2}\begin{cases} (1-t)(t-s^2), & 0 \le s \le t \le 1, \\ t(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \\ \rho_i &= (1-\eta_2\beta)\int_0^1 d\Lambda_i(t) - (1-\beta)\int_0^1 t\, d\Lambda_i(t), \\ & \qquad i = 1, 2, \\ \tau_i &= \eta_1\gamma\int_0^1 d\Lambda_i(t) + (1-\gamma)\int_0^1 t\, d\Lambda_i(t), \\ & \qquad i = 1, 2. \end{aligned} \tag{9}$$

In the remainder of this paper, we always assume that $\Delta - \rho_1 > 0$, $\Delta - \tau_2 > 0$, and $(\Delta - \rho_1)(\Delta - \tau_2) > \rho_2\tau_1$, and for Λ_i $(i = 1, 2)$, the following conditions are fulfilled:

$$\begin{aligned} \int_0^1 d\Lambda_i(t) &\ge \int_0^1 t\, d\Lambda_i(t) \ge 0, \\ \kappa_i(s) &= \int_0^1 k(t,s)\, d\Lambda_i(t) \ge 0, \\ & \qquad s \in [0,1],\ i = 1, 2. \end{aligned} \tag{10}$$

Then, $\rho_i \ge 0$, $\tau_i \ge 0$ $(i = 1, 2)$ and $\Delta > 0$.

Lemma 2 (see [21]). *One has $0 \le k(t,s) \le (1/2)(1+s)(1-s)^2$, $(t,s) \in [0,1] \times [0,1]$.*

Lemma 3. *For any $y \in C[0,1]$, the BVP*

$$\begin{aligned} u'''(t) &= -y(t), \quad t \in (0,1), \\ u(0) &= \gamma u(\eta_1) + \lambda_1[u], \\ u''(0) &= 0, \\ u(1) &= \beta u(\eta_2) + \lambda_2[u] \end{aligned} \tag{11}$$

has the unique solution

$$\begin{aligned} u(t) = {} & \frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] \\ & + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta}\lambda_2[u] \\ & + \frac{(1-\eta_2\beta)\gamma - t\gamma(1-\beta)}{\Delta}\int_0^1 k(\eta_1,s)\,y(s)\,ds \\ & + \frac{\eta_1\beta\gamma + t\beta(1-\gamma)}{\Delta}\int_0^1 k(\eta_2,s)\,y(s)\,ds \\ & + \int_0^1 k(t,s)\,y(s)\,ds, \quad t\in[0,1]. \end{aligned} \tag{12}$$

Proof. By integrating the differential equation in (82) three times from 0 to t and using the boundary condition $u''(0) = 0$, we know that

$$u(t) = u(0) + u'(0)\,t - \frac{1}{2}\int_0^t (t-s)^2\,y(s)\,ds, \quad t\in[0,1]. \tag{13}$$

And so,

$$u'(0) = u(1) - u(0) + \frac{1}{2}\int_0^1 (1-s)^2\,y(s)\,ds. \tag{14}$$

In view of (13), (14), and the boundary conditions $u(0) = \gamma u(\eta_1) + \lambda_1[u]$ and $u(1) = \beta u(\eta_2) + \lambda_2[u]$, we have

$$\begin{aligned} u(t) = {} & (1-t)\,\gamma u(\eta_1) + t\beta u(\eta_2) + (1-t)\,\lambda_1[u] \\ & + t\lambda_2[u] + \int_0^1 k(t,s)\,y(s)\,ds, \quad t\in[0,1]. \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned} u(\eta_1) = {} & \frac{1-\eta_1+\eta_1\beta-\eta_2\beta}{\Delta}\lambda_1[u] + \frac{\eta_1}{\Delta}\lambda_2[u] \\ & + \frac{1-\eta_2\beta}{\Delta}\int_0^1 k(\eta_1,s)\,y(s)\,ds \\ & + \frac{\eta_1\beta}{\Delta}\int_0^1 k(\eta_2,s)\,y(s)\,ds, \\ u(\eta_2) = {} & \frac{1-\eta_2}{\Delta}\lambda_1[u] + \frac{\eta_2+\eta_1\gamma-\eta_2\gamma}{\Delta}\lambda_2[u] \\ & + \frac{\gamma-\eta_2\gamma}{\Delta}\int_0^1 k(\eta_1,s)\,y(s)\,ds \\ & + \frac{1-\gamma+\eta_1\gamma}{\Delta}\int_0^1 k(\eta_2,s)\,y(s)\,ds. \end{aligned} \tag{16}$$

Substituting (16) into (15), we get

$$\begin{aligned} u(t) = {} & \frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] \\ & + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta}\lambda_2[u] \\ & + \frac{(1-\eta_2\beta)\gamma - t\gamma(1-\beta)}{\Delta}\int_0^1 k(\eta_1,s)\,y(s)\,ds \\ & + \frac{\eta_1\beta\gamma + t\beta(1-\gamma)}{\Delta}\int_0^1 k(\eta_2,s)\,y(s)\,ds \\ & + \int_0^1 k(t,s)\,y(s)\,ds, \quad t\in[0,1]. \end{aligned} \tag{17}$$

□

Let $C[0,1]$ be equipped with the maximum norm. Then $C[0,1]$ is a Banach space. If we let

$$\begin{aligned} K = \Big\{ & u\in C[0,1] : u(t)\ge 0,\ t\in[0,1],\ \min_{t\in[\eta_1,\eta_2]} u(t) \\ & \ge \Gamma\|u\|,\ \lambda_i[u]\ge 0,\ i=1,2\Big\}, \end{aligned} \tag{18}$$

where

$$\Gamma = \min\left\{\frac{\eta_1}{1-\gamma+\eta_1\gamma}, \frac{1-\eta_2}{1-\eta_2\beta}\right\}, \tag{19}$$

then K is a cone in $C[0,1]$. Now, we define operators T and S on K by

$$\begin{aligned} (Tu)(t) = {} & \frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta} \\ & \cdot\lambda_2[u] + (Fu)(t), \quad t\in[0,1], \\ (Su)(t) & \\ = {} & \frac{(1-\eta_2\beta - t(1-\beta))(\Delta-\tau_2) + (\eta_1\gamma + t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1} \\ & \cdot\lambda_1[Fu] \\ & + \frac{(1-\eta_2\beta - t(1-\beta))\tau_1 + (\eta_1\gamma + t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1} \\ & \cdot\lambda_2[Fu] + (Fu)(t), \quad t\in[0,1], \end{aligned} \tag{20}$$

where

$$\begin{aligned}(Fu)(t) = {} & \frac{(1-\eta_2\beta)\gamma - t\gamma(1-\beta)}{\Delta} \\ & \cdot \int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds \\ & + \frac{\eta_1\beta\gamma + t\beta(1-\gamma)}{\Delta} \\ & \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\ & + \int_0^1 k(t, s) f(s, u(\alpha(s)))\, ds, \quad t \in [0,1]. \end{aligned} \tag{21}$$

Lemma 4. $T, S : K \to K$.

Proof. Let $u \in K$. Then it is easy to know that

$$(Tu)''(t) = -\int_0^1 f(s, u(\alpha(s)))\, ds \le 0, \quad t \in [0,1], \tag{22}$$

which shows that Tu is concave down on $[0,1]$. In view of

$$\begin{aligned}&(Fu)(0) \\ &\quad = \frac{(1-\eta_2\beta)\gamma}{\Delta} \int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds \\ &\qquad + \frac{\eta_1\beta\gamma}{\Delta} \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \ge 0, \\ &(Fu)(1) \\ &\quad = \frac{(1-\eta_2)\beta\gamma}{\Delta} \int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds \\ &\qquad + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta} \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\ &\quad \ge 0, \end{aligned} \tag{23}$$

we have

$$\begin{aligned}(Tu)(0) &= \frac{1-\eta_2\beta}{\Delta}\lambda_1[u] + \frac{\eta_1\gamma}{\Delta}\lambda_2[u] + (Fu)(0) \\ &\ge 0, \\ (Tu)(1) &= \frac{(1-\eta_2)\beta}{\Delta}\lambda_1[u] + \frac{1-\gamma+\eta_1\gamma}{\Delta}\lambda_2[u] \\ &\quad + (Fu)(1) \ge 0. \end{aligned} \tag{24}$$

So, $(Tu)(t) \ge 0$, $t \in [0,1]$.

Now, we prove that $\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \ge \Gamma\|Tu\|$. To do it we consider two cases.

Case 1. Let $(Tu)(\eta_1) \le (Tu)(\eta_2)$. Then $\min_{t\in[\eta_1,\eta_2]}(Tu)(t) = (Tu)(\eta_1)$ and there exists $\bar{t} \in [\eta_1, 1]$ such that $\|Tu\| = (Tu)(\bar{t})$.

If $\bar{t} \in [\eta_1, \eta_2]$, then

$$\frac{(Tu)(\bar{t}) - (Tu)(0)}{\bar{t}} \le \frac{(Tu)(\eta_1) - (Tu)(0)}{\eta_1}. \tag{25}$$

So,

$$\|Tu\| \le \frac{\eta_2}{\eta_1}(Tu)(\eta_1) - \frac{\eta_2-\eta_1}{\eta_1}(Tu)(0), \tag{26}$$

which together with

$$(Tu)(0) = \gamma(Tu)(\eta_1) + \lambda_1[u] \tag{27}$$

implies that

$$\|Tu\| \le \frac{\eta_2 - (\eta_2-\eta_1)\gamma}{\eta_1}(Tu)(\eta_1); \tag{28}$$

that is,

$$\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \ge \frac{\eta_1}{\eta_2 - (\eta_2-\eta_1)\gamma}\|Tu\|. \tag{29}$$

If $\bar{t} \in (\eta_2, 1]$, then

$$\frac{(Tu)(\bar{t}) - (Tu)(\eta_1)}{\bar{t}-\eta_1} \le \frac{(Tu)(\eta_2) - (Tu)(\eta_1)}{\eta_2-\eta_1}. \tag{30}$$

So,

$$\|Tu\| \le \frac{1-\eta_1}{\eta_2-\eta_1}(Tu)(\eta_2) - \frac{1-\eta_2}{\eta_2-\eta_1}(Tu)(\eta_1). \tag{31}$$

On the other hand, it follows from

$$\frac{(Tu)(\eta_1) - (Tu)(0)}{\eta_1} \ge \frac{(Tu)(\eta_2) - (Tu)(0)}{\eta_2} \tag{32}$$

and (27) that

$$(Tu)(\eta_2) \le \frac{\eta_2 - (\eta_2-\eta_1)\gamma}{\eta_1}(Tu)(\eta_1), \tag{33}$$

which together with (31) implies that

$$\|Tu\| \le \frac{1-\gamma+\eta_1\gamma}{\eta_1}(Tu)(\eta_1); \tag{34}$$

that is,

$$\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \ge \frac{\eta_1}{1-\gamma+\eta_1\gamma}\|Tu\|. \tag{35}$$

Case 2. Let $(Tu)(\eta_1) > (Tu)(\eta_2)$. Then $\min_{t\in[\eta_1,\eta_2]}(Tu)(t) = (Tu)(\eta_2)$ and there exists $\bar{t} \in [0,\eta_2)$ such that $\|Tu\| = (Tu)(\bar{t})$.

If $\bar{t} \in [0,\eta_1]$, then

$$\frac{(Tu)(\eta_2) - (Tu)(\bar{t})}{\eta_2 - \bar{t}} \geq \frac{(Tu)(\eta_2) - (Tu)(\eta_1)}{\eta_2 - \eta_1}. \tag{36}$$

So,

$$\|Tu\| \leq \frac{\eta_2}{\eta_2 - \eta_1}(Tu)(\eta_1) - \frac{\eta_1}{\eta_2 - \eta_1}(Tu)(\eta_2). \tag{37}$$

At the same time, since

$$\frac{(Tu)(\eta_2) - (Tu)(\eta_1)}{\eta_2 - \eta_1} \geq \frac{(Tu)(1) - (Tu)(\eta_1)}{1 - \eta_1}, \tag{38}$$

we have

$$(Tu)(\eta_1) \leq \frac{1-\eta_1}{1-\eta_2}(Tu)(\eta_2) - \frac{\eta_2 - \eta_1}{1-\eta_2}(Tu)(1), \tag{39}$$

which together with

$$(Tu)(1) = \beta(Tu)(\eta_2) + \lambda_2[u] \tag{40}$$

implies that

$$(Tu)(\eta_1) \leq \frac{1-\eta_1-(\eta_2-\eta_1)\beta}{1-\eta_2}(Tu)(\eta_2). \tag{41}$$

In view of (37) and (41), we have

$$\|Tu\| \leq \frac{1-\eta_2\beta}{1-\eta_2}(Tu)(\eta_2); \tag{42}$$

that is,

$$\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \geq \frac{1-\eta_2}{1-\eta_2\beta}\|Tu\|. \tag{43}$$

If $\bar{t} \in (\eta_1,\eta_2)$, then

$$\frac{(Tu)(1) - (Tu)(\bar{t})}{1-\bar{t}} \geq \frac{(Tu)(1) - (Tu)(\eta_2)}{1-\eta_2}. \tag{44}$$

So,

$$\|Tu\| \leq \frac{1-\eta_1}{1-\eta_2}(Tu)(\eta_2) - \frac{\eta_2-\eta_1}{1-\eta_2}(Tu)(1), \tag{45}$$

which together with (40) implies that

$$\|Tu\| \leq \frac{1-\eta_1-(\eta_2-\eta_1)\beta}{1-\eta_2}(Tu)(\eta_2); \tag{46}$$

that is,

$$\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \geq \frac{1-\eta_2}{1-\eta_1-(\eta_2-\eta_1)\beta}\|Tu\|. \tag{47}$$

It follows from (29), (35), (43), and (47) that

$$\min_{t\in[\eta_1,\eta_2]}(Tu)(t) \geq \Gamma\|Tu\|. \tag{48}$$

Finally, we need to show that $\lambda_i[Tu] \geq 0$, $i = 1,2$. Since

$$\begin{aligned}
\lambda_i[Fu] = {} & \int_0^1 \frac{(1-\eta_2\beta)\gamma - t\gamma(1-\beta)}{\Delta} \\
& \cdot \int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds\,d\Lambda_i(t) \\
& + \int_0^1 \frac{\eta_1\beta\gamma + t\beta(1-\gamma)}{\Delta} \\
& \cdot \int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds\,d\Lambda_i(t) \\
& + \int_0^1\int_0^1 k(t,s) f(s,u(\alpha(s)))\,ds\,d\Lambda_i(t) = \frac{\gamma\rho_i}{\Delta} \\
& \cdot \int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds + \frac{\beta\tau_i}{\Delta} \\
& \cdot \int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds \\
& + \int_0^1 \kappa_i(s) f(s,u(\alpha(s)))\,ds \geq 0, \quad i = 1,2,
\end{aligned} \tag{49}$$

we have

$$\lambda_i[Tu] = \frac{\rho_i}{\Delta}\lambda_1[u] + \frac{\tau_i}{\Delta}\lambda_2[u] + \lambda_i[Fu] \geq 0, \quad i = 1,2. \tag{50}$$

Therefore, $T : K \to K$. Similarly, we may prove that $S : K \to K$. □

Lemma 5. *T and S have the same fixed points in K.*

Proof. On the one hand, if $u \in K$ is a fixed point of S, that is, $u = Su$, then

$$\begin{aligned}
\lambda_1[u] &= \lambda_1[Su] \\
&= \int_0^1 \left(\frac{(1-\eta_2\beta - t(1-\beta))(\Delta-\tau_2) + (\eta_1\gamma + t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \right. \\
&\quad \cdot \lambda_1[Fu] \\
&\quad + \frac{(1-\eta_2\beta - t(1-\beta))\tau_1 + (\eta_1\gamma + t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \left. \cdot \lambda_2[Fu] + (Fu)(t) \right) d\Lambda_1(t) \\
&= \frac{\Delta(\Delta-\tau_2)\lambda_1[Fu] + \Delta\tau_1\lambda_2[Fu]}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}, \\
\lambda_2[u] &= \lambda_2[Su] \\
&= \int_0^1 \left(\frac{(1-\eta_2\beta - t(1-\beta))(\Delta-\tau_2) + (\eta_1\gamma + t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \right. \\
&\quad \cdot \lambda_1[Fu] \\
&\quad + \frac{(1-\eta_2\beta - t(1-\beta))\tau_1 + (\eta_1\gamma + t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \left. \cdot \lambda_2[Fu] + (Fu)(t) \right) d\Lambda_2(t) \\
&= \frac{\Delta\rho_2\lambda_1[Fu] + \Delta(\Delta-\rho_1)\lambda_2[Fu]}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1},
\end{aligned} \tag{51}$$

which shows that

$$\begin{aligned}
\lambda_1[Fu] &= \frac{(\Delta-\rho_1)\lambda_1[u] - \tau_1\lambda_2[u]}{\Delta}, \\
\lambda_2[Fu] &= \frac{(\Delta-\tau_2)\lambda_2[u] - \rho_2\lambda_1[u]}{\Delta}.
\end{aligned} \tag{52}$$

So,

$$\begin{aligned}
u(t) &= (Su)(t) \\
&= \frac{(1-\eta_2\beta - t(1-\beta))(\Delta-\tau_2) + (\eta_1\gamma + t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \cdot \lambda_1[Fu] \\
&\quad + \frac{(1-\eta_2\beta - t(1-\beta))\tau_1 + (\eta_1\gamma + t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \cdot \lambda_2[Fu] + (Fu)(t) = \frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] \\
&\quad + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta}\lambda_2[u] + (Fu)(t) = (Tu)(t), \\
&\qquad t \in [0,1],
\end{aligned} \tag{53}$$

which indicates that u is a fixed point of T.

On the other hand, if $u \in K$ is a fixed point of T, that is, $u = Tu$, then

$$\begin{aligned}
\lambda_i[u] = \lambda_i[Tu] &= \int_0^1 \left(\frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] \right. \\
&\quad \left. + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta}\lambda_2[u] + (Fu)(t) \right) d\Lambda_i(t) \\
&= \frac{\rho_i}{\Delta}\lambda_1[u] + \frac{\tau_i}{\Delta}\lambda_2[u] + \lambda_i[Fu], \quad i = 1,2,
\end{aligned} \tag{54}$$

which shows that

$$\begin{aligned}
\lambda_1[u] &= \frac{\Delta(\Delta-\tau_2)\lambda_1[Fu] + \Delta\tau_1\lambda_2[Fu]}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}, \\
\lambda_2[u] &= \frac{\Delta\rho_2\lambda_1[Fu] + \Delta(\Delta-\rho_1)\lambda_2[Fu]}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}.
\end{aligned} \tag{55}$$

So,

$$\begin{aligned}
u(t) = (Tu)(t) &= \frac{1-\eta_2\beta - t(1-\beta)}{\Delta}\lambda_1[u] \\
&\quad + \frac{\eta_1\gamma + t(1-\gamma)}{\Delta}\lambda_2[u] + (Fu)(t) \\
&= \frac{(1-\eta_2\beta - t(1-\beta))(\Delta-\tau_2) + (\eta_1\gamma + t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \cdot \lambda_1[Fu] \\
&\quad + \frac{(1-\eta_2\beta - t(1-\beta))\tau_1 + (\eta_1\gamma + t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \\
&\quad \cdot \lambda_2[Fu] + (Fu)(t) = (Su)(t), \quad t \in [0,1],
\end{aligned} \tag{56}$$

which indicates that u is a fixed point of S. □

Lemma 6. *$T, S : K \to K$ is completely continuous.*

Proof. First, by Lemma 4, we know that $T(K) \subset K$.

Next, we show that T is compact.

Let $D \subset K$ be a bounded set. Then there exists $M_1 > 0$ such that $\|u\| \le M_1$ for any $u \in D$. Since Λ_1 and Λ_2 are functions of bounded variation, there exists $M_2 > 0$ such that

$$\sum_{j=1}^{n} \left| \Lambda_i(t_j) - \Lambda_i(t_{j-1}) \right| \le M_2, \quad i = 1,2 \tag{57}$$

for any partition $\Delta' : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Let

$$M_3 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,M_1]\}. \tag{58}$$

Then for any $u \in D$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\
&\le \frac{1-\eta_2\beta}{\Delta}\lambda_1[u] + \frac{1-\gamma+\eta_1\gamma}{\Delta}\lambda_2[u] \\
&\quad + \frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))\,ds \\
&\le \frac{1-\eta_2\beta}{\Delta}M_1M_2 + \frac{1-\gamma+\eta_1\gamma}{\Delta}M_1M_2 \\
&\quad + \frac{(1-\eta_2\beta)\gamma M_3}{\Delta}\int_0^1 k(\eta_1,s)\,ds \\
&\quad + \frac{(1-\gamma+\eta_1\gamma)\beta M_3}{\Delta}\int_0^1 k(\eta_2,s)\,ds + \frac{5}{24}M_3,
\end{aligned} \tag{59}$$

which shows that $T(D)$ is uniformly bounded.

On the other hand, for any $\varepsilon > 0$, since $k(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, there exists $\delta_1(\varepsilon) > 0$ such that, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1(\varepsilon)$,

$$|k(t_1,s) - k(t_2,s)| < \frac{\varepsilon}{5(M_3+1)}, \quad s \in [0,1]. \tag{60}$$

Let

$$\begin{aligned}
\delta = \min\Bigg\{ &\delta_1(\varepsilon), \frac{\varepsilon\Delta}{5((1-\beta)M_1M_2+1)}, \\
&\frac{\varepsilon\Delta}{5((1-\gamma)M_1M_2+1)}, \\
&\frac{\varepsilon\Delta}{5\left((1-\beta)\gamma M_3\int_0^1 k(\eta_1,s)\,ds+1\right)}, \\
&\frac{\varepsilon\Delta}{5\left((1-\gamma)\beta M_3\int_0^1 k(\eta_2,s)\,ds+1\right)}\Bigg\}.
\end{aligned} \tag{61}$$

Then for any $u \in D$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &= \Bigg|\frac{(t_1-t_2)(\beta-1)}{\Delta}\lambda_1[u] \\
&\quad + \frac{(t_1-t_2)(1-\gamma)}{\Delta}\lambda_2[u] + \frac{(t_1-t_2)(\beta-1)\gamma}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \frac{(t_1-t_2)(1-\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \int_0^1 (k(t_1,s) - k(t_2,s)) f(s,u(\alpha(s)))\,ds\Bigg| \\
&\le \frac{(1-\beta)|t_1-t_2|}{\Delta}\lambda_1[u] + \frac{(1-\gamma)|t_1-t_2|}{\Delta}\lambda_2[u] \\
&\quad + \frac{(1-\beta)\gamma|t_1-t_2|}{\Delta}\int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \frac{(1-\gamma)\beta|t_1-t_2|}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds \\
&\quad + \int_0^1 |k(t_1,s) - k(t_2,s)| f(s,u(\alpha(s)))\,ds \\
&\le \frac{(1-\beta)|t_1-t_2|M_1M_2}{\Delta} \\
&\quad + \frac{(1-\gamma)|t_1-t_2|M_1M_2}{\Delta} + \frac{(1-\beta)\gamma|t_1-t_2|M_3}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_1,s)\,ds + \frac{(1-\gamma)\beta|t_1-t_2|M_3}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_2,s)\,ds + M_3\int_0^1 |k(t_1,s) - k(t_2,s)|\,ds \\
&< \varepsilon,
\end{aligned} \tag{62}$$

which shows that $T(D)$ is equicontinuous. It follows from Arzela-Ascoli theorem that $T(D)$ is relatively compact. Thus, we have shown that T is a compact operator.

Finally, we prove that T is continuous.

Assume that u_n, $u \in K$ and $\lim_{n\to\infty} u_n = u$. Then there exists $M_4 > 0$ such that $\|u\| \le M_4$ and $\|u_n\| \le M_4$, $n = 1, 2, \ldots$. For any $\varepsilon > 0$, since $f(s,x)$ is uniformly continuous on $[0,1]\times[0,M_4]$, there exists $\delta > 0$ such that, for any $x_1, x_2 \in [0,M_4]$ with $|x_1 - x_2| < \delta$,

$$|f(s,x_1) - f(s,x_2)| < \frac{\varepsilon}{(3(2-\eta_2\beta-\beta)\gamma/\Delta)\int_0^1 k(\eta_1,s)\,ds + (3(1-\gamma+\eta_1\gamma)\beta/\Delta)\int_0^1 k(\eta_2,s)\,ds + 5/8}, \quad s \in [0,1]. \tag{63}$$

At the same time, since $\lim_{n\to\infty} u_n = u$, there exists positive integer N such that, for any $n > N$,

$$\|u_n - u\| < \min\left\{\delta, \frac{\varepsilon\Delta}{3(2-\eta_2\beta-\beta)M_2}, \frac{\varepsilon\Delta}{3(1-\gamma+\eta_1\gamma)M_2}\right\}. \tag{64}$$

It follows from (63) and (64) that, for any $n > N$,

$$\begin{aligned}
\|Tu_n - Tu\| &= \max_{t\in[0,1]} |(Tu_n)(t) - (Tu)(t)| \\
&\le \frac{2-\eta_2\beta-\beta}{\Delta}|\lambda_1[u_n] - \lambda_1[u]| \\
&+ \frac{1-\gamma+\eta_1\gamma}{\Delta}|\lambda_2[u_n] - \lambda_2[u]| + \frac{(2-\eta_2\beta-\beta)\gamma}{\Delta} \\
&\cdot \int_0^1 k(\eta_1, s)|f(s, u_n(\alpha(s))) - f(s, u(\alpha(s)))|\,ds \\
&+ \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2, s)|f(s, u_n(\alpha(s))) \\
&- f(s, u(\alpha(s)))|\,ds + \frac{1}{2}\int_0^1 (1+s)(1-s)^2 \\
&\cdot |f(s, u_n(\alpha(s))) - f(s, u(\alpha(s)))|\,ds \\
&\le \frac{2-\eta_2\beta-\beta}{\Delta}\|u_n - u\|M_2 + \frac{1-\gamma+\eta_1\gamma}{\Delta}\|u_n - u\| \\
&\cdot M_2 + \int_0^1 \left(\frac{(2-\eta_2\beta-\beta)\gamma}{\Delta}k(\eta_1, s)\right. \\
&\left. + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}k(\eta_2, s) + \frac{1}{2}(1+s)(1-s)^2\right) \\
&\cdot |f(s, u_n(\alpha(s))) - f(s, u(\alpha(s)))|\,ds < \varepsilon,
\end{aligned} \tag{65}$$

which indicates that T is continuous.

Therefore, $T : K \to K$ is completely continuous. Similarly, we can prove that $S : K \to K$ is also completely continuous. □

For convenience, we denote

$$\begin{aligned}
D_1 &= \frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1, s)\,ds + \frac{\beta\tau_1}{\Delta}\int_0^1 k(\eta_2, s)\,ds \\
&+ \int_0^1 \kappa_1(s)\,ds, \\
D_2 &= \frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1, s)\,ds + \frac{\beta\tau_2}{\Delta}\int_0^1 k(\eta_2, s)\,ds \\
&+ \int_0^1 \kappa_2(s)\,ds, \\
D_3 &= \frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1, s)\,ds \\
&+ \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2, s)\,ds + \frac{5}{24}, \\
D_4 &= \frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s)\,ds + \frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2, s)\,ds \\
&+ \int_{\eta_1}^{\eta_2} \kappa_1(s)\,ds, \\
D_5 &= \frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s)\,ds + \frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2, s)\,ds \\
&+ \int_{\eta_1}^{\eta_2} \kappa_2(s)\,ds, \\
D_6 &= \frac{\gamma(1-\eta_2)}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s)\,ds \\
&+ \frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2, s)\,ds.
\end{aligned} \tag{66}$$

Let

$$\begin{aligned}
\mu &> \frac{(1-\eta_2\beta)(\Delta-\tau_2) + (1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}D_1 \\
&+ \frac{(1-\eta_2\beta)\tau_1 + (1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}D_2 + D_3, \\
0 &< L \\
&< \frac{\eta_1}{\eta_2 - (\eta_2-\eta_1)\gamma}\left(\frac{(1-\eta_2)(\Delta-\tau_2) + (\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}\right. \\
&\left. \cdot D_4 + \frac{(1-\eta_2)\tau_1 + (\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}D_5 + D_6\right).
\end{aligned} \tag{67}$$

Theorem 7. *Suppose that there exist positive constants a, b, and d with $a < b < b/\Gamma \le d$ such that the following conditions are fulfilled:*

(A_1) $f(t,u) \le d/\mu$, $(t,u) \in [0,1]\times[0,d]$,

(A_2) $f(t,u) \ge b/L$, $(t,u) \in [\eta_1,\eta_2]\times[b,b/\Gamma]$,

(A_3) $f(t,u) \le a/\mu$, $(t,u) \in [0,1]\times[0,a]$.

Then the BVP (4) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_i\| \le d$ $(i = 1,2,3)$ and

$$\begin{aligned}
&\min_{t\in[\eta_1,\eta_2]} u_1(t) > b, \\
&\|u_2\| > a \quad \text{with} \min_{t\in[\eta_1,\eta_2]} u_2(t) < b, \\
&\|u_3\| < a.
\end{aligned} \tag{68}$$

Proof. For $u \in K$, we define

$$\begin{aligned}
\Phi(u) &= \min_{t\in[\eta_1,\eta_2]} u(t), \\
\varphi(u) &= \Theta(u) = \Psi(u) = \|u\|.
\end{aligned} \tag{69}$$

Then it is easy to know that Φ is a nonnegative continuous concave functional on K and φ, Θ and Ψ are nonnegative continuous convex functionals on K. In order to apply Theorem 1 to prove our main results, we use the operator S and take $c = b/\Gamma$.

First, we assert that $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$.

In fact, if $u \in \overline{K(\varphi, d)}$, then $0 \le u(t) \le d$, $t \in [0,1]$, which together with (A_1) implies that

$$\begin{aligned}
\lambda_i [Fu] &= \frac{\gamma\rho_i}{\Delta}\int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{\beta\tau_i}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \int_0^1 \kappa_i(s) f(s, u(\alpha(s)))\, ds \\
&\le \left(\frac{\gamma\rho_i}{\Delta}\int_0^1 k(\eta_1, s)\, ds + \frac{\beta\tau_i}{\Delta}\int_0^1 k(\eta_2, s)\, ds \right. \\
&\quad \left. + \int_0^1 \kappa_i(s)\, ds\right)\frac{d}{\mu} = \frac{D_i d}{\mu}, \quad i = 1, 2, \\
\|Fu\| &= \max_{t\in[0,1]} (Fu)(t) \le \frac{(1-\eta_2\beta)\gamma}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds + \frac{1}{2} \\
&\quad \cdot \int_0^1 (1+s)(1-s)^2 f(s, u(\alpha(s)))\, ds \\
&\le \left(\frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1, s)\, ds \right. \\
&\quad \left. + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2, s)\, ds + \frac{5}{24}\right)\frac{d}{\mu} \\
&= \frac{D_3 d}{\mu}.
\end{aligned} \tag{70}$$

It follows from (70) that

$$\begin{aligned}
\varphi(Su) &= \|Su\| \\
&\le \frac{(1-\eta_2\beta)(\Delta-\tau_2) + (1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}\lambda_1[Fu] \\
&\quad + \frac{(1-\eta_2\beta)\tau_1 + (1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1}\lambda_2[Fu] \\
&\quad + \|Fu\| \\
&\le \left(\frac{(1-\eta_2\beta)(\Delta-\tau_2) + (1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1} D_1 \right. \\
&\quad + \frac{(1-\eta_2\beta)\tau_1 + (1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2) - \rho_2\tau_1} D_2 \\
&\quad \left. + D_3\right)\frac{d}{\mu} \le d.
\end{aligned} \tag{71}$$

This shows that $S : \overline{K(\varphi, d)} \to \overline{K(\varphi, d)}$.

Next, we claim that $\{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\} \ne \phi$ and $\Phi(Su) > b$ for $u \in K(\varphi, \Theta, \Phi, b, c, d)$.

Indeed, the constant function $(b + c)/2 \in \{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\}$. Moreover, if $u \in K(\varphi, \Theta, \Phi, b, c, d)$, then $b \le u(t) \le c$, $t \in [\eta_1, \eta_2]$, which together with $\eta_1 \le t \le \alpha(t) \le \eta_2$ for $t \in [\eta_1, \eta_2]$ implies that $b \le u(\alpha(t)) \le c$, $t \in [\eta_1, \eta_2]$. In view of (A_2), we have

$$\begin{aligned}
\lambda_1[Fu] &= \frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{\beta\tau_1}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \int_0^1 \kappa_1(s) f(s, u(\alpha(s)))\, ds \ge \frac{\gamma\rho_1}{\Delta} \\
&\quad \cdot \int_{\eta_1}^{\eta_2} k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{\beta\tau_1}{\Delta} \\
&\quad \cdot \int_{\eta_1}^{\eta_2} k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \int_{\eta_1}^{\eta_2} \kappa_1(s) f(s, u(\alpha(s)))\, ds \\
&\ge \left(\frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s)\, ds + \frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2, s)\, ds \right. \\
&\quad \left. + \int_{\eta_1}^{\eta_2} \kappa_1(s)\, ds\right)\frac{b}{L} = \frac{D_4 b}{L},
\end{aligned}$$

$$\begin{aligned}
\lambda_2[Fu] &= \frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{\beta\tau_2}{\Delta} \\
&\quad \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \int_0^1 \kappa_2(s) f(s, u(\alpha(s)))\, ds \ge \frac{\gamma\rho_2}{\Delta} \\
&\quad \cdot \int_{\eta_1}^{\eta_2} k(\eta_1, s) f(s, u(\alpha(s)))\, ds + \frac{\beta\tau_2}{\Delta} \\
&\quad \cdot \int_{\eta_1}^{\eta_2} k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \int_{\eta_1}^{\eta_2} \kappa_2(s) f(s, u(\alpha(s)))\, ds \\
&\ge \left(\frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s)\, ds + \frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2, s)\, ds \right. \\
&\quad \left. + \int_{\eta_1}^{\eta_2} \kappa_2(s)\, ds\right)\frac{b}{L} = \frac{D_5 b}{L},
\end{aligned}$$

$$\begin{aligned}
(Fu)(\eta_2) &= \frac{\gamma(1-\eta_2)}{\Delta}\int_0^1 k(\eta_1, s) f(s, u(\alpha(s)))\, ds \\
&\quad + \frac{1-\gamma+\eta_1\gamma}{\Delta}\int_0^1 k(\eta_2, s) f(s, u(\alpha(s)))\, ds \\
&\ge \frac{\gamma(1-\eta_2)}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1, s) f(s, u(\alpha(s)))\, ds
\end{aligned}$$

$$+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2}k(\eta_2,s)f(s,u(\alpha(s)))\,ds$$
$$\geq\left(\frac{\gamma(1-\eta_2)}{\Delta}\int_{\eta_1}^{\eta_2}k(\eta_1,s)\,ds\right.$$
$$\left.+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2}k(\eta_2,s)\,ds\right)\frac{b}{L}=\frac{D_6b}{L}. \tag{72}$$

Since Su is concave down on $[0,1]$, we have

$$\frac{(Su)(\eta_2)-(Su)(0)}{\eta_2}\leq\frac{(Su)(\eta_1)-(Su)(0)}{\eta_1}. \tag{73}$$

So,

$$(Su)(\eta_2)\leq\frac{\eta_2}{\eta_1}(Su)(\eta_1)-\frac{\eta_2-\eta_1}{\eta_1}(Su)(0), \tag{74}$$

which together with

$$(Su)(0)=\gamma(Su)(\eta_1)+\frac{\Delta(\Delta-\tau_2)\lambda_1[Fu]+\Delta\tau_1\lambda_2[Fu]}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1} \tag{75}$$

implies that

$$(Su)(\eta_1)\geq\frac{\eta_1}{\eta_2-(\eta_2-\eta_1)\gamma}(Su)(\eta_2). \tag{76}$$

Therefore, it follows from (72) and (76) that

$$\begin{aligned}\Phi(Su)&=\min_{t\in[\eta_1,\eta_2]}(Su)(t)=\min\{(Su)(\eta_1),(Su)(\eta_2)\}\\&\geq\min\left\{\frac{\eta_1}{\eta_2-(\eta_2-\eta_1)\gamma}(Su)(\eta_2),(Su)(\eta_2)\right\}\\&=\frac{\eta_1}{\eta_2-(\eta_2-\eta_1)\gamma}(Su)(\eta_2)\\&=\frac{\eta_1}{\eta_2-(\eta_2-\eta_1)\gamma}\left(\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\right.\\&\quad\cdot\lambda_1[Fu]+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_2[Fu]\\&\quad\left.+(Fu)(\eta_2)\right)\\&\geq\frac{\eta_1}{\eta_2-(\eta_2-\eta_1)\gamma}\left(\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\right.\\&\quad\left.\cdot D_4+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_5+D_6\right)\frac{b}{L}\\&>b.\end{aligned} \tag{77}$$

Thirdly, we assert that $\Phi(Su)>b$ for $u\in K(\varphi,\Phi,b,d)$ with $\Theta(Su)>c$.

To see this, we suppose $u\in K(\varphi,\Phi,b,d)$ and $\Theta(Su)=\|Su\|>c$. Then

$$\Phi(Su)=\min_{t\in[\eta_1,\eta_2]}(Su)(t)\geq\Gamma\|Su\|>\Gamma c=b. \tag{78}$$

Finally, we prove that $\theta\notin R(\varphi,\Psi,a,d)$ and $\Psi(Su)<a$ for $u\in R(\varphi,\Psi,a,d)$ with $\Psi(u)=a$.

Indeed, it follows from $\Psi(\theta)=0<a$ that $\theta\notin R(\varphi,\Psi,a,d)$. Moreover, if $u\in R(\varphi,\Psi,a,d)$ and $\Psi(u)=a$, then $0\leq u(t)\leq a$, $t\in[0,1]$, which together with (A_3) implies that

$$\begin{aligned}\lambda_i[Fu]&=\frac{\gamma\rho_i}{\Delta}\int_0^1k(\eta_1,s)f(s,u(\alpha(s)))\,ds+\frac{\beta\tau_i}{\Delta}\\&\quad\cdot\int_0^1k(\eta_2,s)f(s,u(\alpha(s)))\,ds\\&\quad+\int_0^1\kappa_i(s)f(s,u(\alpha(s)))\,ds\\&\leq\left(\frac{\gamma\rho_i}{\Delta}\int_0^1k(\eta_1,s)\,ds+\frac{\beta\tau_i}{\Delta}\int_0^1k(\eta_2,s)\,ds\right.\\&\quad\left.+\int_0^1\kappa_i(s)\,ds\right)\frac{a}{\mu}=\frac{D_ia}{\mu},\quad i=1,2,\end{aligned} \tag{79}$$

$$\begin{aligned}\|Fu\|&=\max_{t\in[0,1]}(Fu)(t)\leq\frac{(1-\eta_2\beta)\gamma}{\Delta}\\&\quad\cdot\int_0^1k(\eta_1,s)f(s,u(\alpha(s)))\,ds+\frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\\&\quad\cdot\int_0^1k(\eta_2,s)f(s,u(\alpha(s)))\,ds+\frac{1}{2}\\&\quad\cdot\int_0^1(1+s)(1-s)^2f(s,u(\alpha(s)))\,ds\\&\leq\left(\frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1k(\eta_1,s)\,ds\right.\\&\quad\left.+\frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1k(\eta_2,s)\,ds+\frac{5}{24}\right)\frac{a}{\mu}=\frac{D_3a}{\mu}.\end{aligned} \tag{79}$$

In view of (79), we have

$$\begin{aligned}\Psi(Su)&=\|Su\|\\&\leq\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_1[Fu]\\&\quad+\frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_2[Fu]\\&\quad+\|Fu\|\\&\leq\left(\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_1\right.\\&\quad\left.+\frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_2+D_3\right)\\&\quad\cdot\frac{a}{\mu}<a,\end{aligned} \tag{80}$$

as required.

To sum up, all the hypotheses of Theorem 1 are satisfied. Hence, the BVP (4) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_i\| \le d$ $(i = 1, 2, 3)$ and

$$\begin{aligned} &\min_{t\in[\eta_1,\eta_2]} u_1(t) > b, \\ &\|u_2\| > a \quad \text{with} \min_{t\in[\eta_1,\eta_2]} u_2(t) < b, \\ &\|u_3\| < a. \end{aligned} \tag{81}$$

□

3. An Example

Example 1. Consider the following BVP:

$$\begin{aligned} &u'''(t) + f(t, u(\alpha(t))) = 0, \quad t \in (0,1), \\ &u(0) = \frac{1}{2}u\left(\frac{1}{4}\right) + \int_0^1 u(t)\, d\left(\frac{3}{2}t^2 - t\right), \\ &u''(0) = 0, \\ &u(1) = \frac{1}{4}u\left(\frac{1}{2}\right) + \int_0^1 u(t)\, d\left(\frac{1}{4}t^2\right), \end{aligned} \tag{82}$$

where

$$f(t,u) = \begin{cases} 20u^2 + \left(\frac{1}{20} - u\right)t(1-t), & (t,u) \in [0,1] \times \left[0, \frac{1}{20}\right], \\ \frac{1}{20} + 1980\left(u - \frac{1}{20}\right)^2 + \left(u - \frac{1}{20}\right)\left(\frac{1}{10} - u\right)t(1-t), & (t,u) \in [0,1] \times \left(\frac{1}{20}, \frac{1}{10}\right), \\ 5 + \frac{1}{10}\left(u - \frac{1}{10}\right)^2 t(1-t), & (t,u) \in [0,1] \times \left[\frac{1}{10}, +\infty\right), \end{cases}$$
$$\alpha(t) = \begin{cases} \frac{\sqrt{2t}}{2}, & t \in \left[0, \frac{1}{2}\right], \\ \frac{\sqrt{2t-1}+1}{2}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{83}$$

Since $\Lambda_1(t) = (3/2)t^2 - t$ and $\Lambda_2(t) = (1/4)t^2$, $t \in [0,1]$, a simple calculation shows that

$$\begin{aligned} &\int_0^1 d\Lambda_1(t) = \int_0^1 t\, d\Lambda_1(t) = \frac{1}{2}, \\ &\int_0^1 d\Lambda_2(t) = \frac{1}{4}, \\ &\int_0^1 t\, d\Lambda_2(t) = \frac{1}{6}, \\ &\kappa_1(s) = \frac{1}{8}s^4 - \frac{1}{6}s^3 + \frac{1}{24}, \\ &\kappa_2(s) = \frac{1}{48}\left(1 - s^2\right)^2, \\ &\qquad s \in [0,1]. \end{aligned} \tag{84}$$

At the same time, in view of $\eta_1 = \beta = 1/4$ and $\eta_2 = \gamma = 1/2$, we get

$$\begin{aligned} \Delta &= \frac{17}{32}, \\ \rho_1 &= \frac{1}{16}, \\ \tau_1 &= \frac{5}{16}, \\ \rho_2 &= \frac{3}{32}, \\ \tau_2 &= \frac{11}{96}, \\ \Gamma &= \frac{2}{5}, \\ D_1 &= \frac{397}{10880}, \\ D_2 &= \frac{3511}{195840}, \\ D_3 &= \frac{845}{3264}, \\ D_4 &= \frac{26561}{2088960}, \\ D_5 &= \frac{25487}{4177920}, \\ D_6 &= \frac{13}{408}. \end{aligned} \tag{85}$$

If we choose $\mu = 1$, $L = 1/50$, $a = 1/20, b = 1/10$, and $d = 6$, then all the conditions of Theorem 7 are fulfilled. Therefore, it follows from Theorem 7 that the BVP (82) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_i\| \le 6$ $(i = 1, 2, 3)$ and

$$\begin{aligned} &\min_{t\in[1/4,1/2]} u_1(t) > \frac{1}{10}, \\ &\|u_2\| > \frac{1}{20} \quad \text{with} \min_{t\in[1/4,1/2]} u_2(t) < \frac{1}{10}, \\ &\|u_3\| < \frac{1}{20}. \end{aligned} \tag{86}$$

Competing Interests

The authors declare that they have no competing interests.

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