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Research Article

The Zeros of the Bergman Kernel for Some Reinhardt Domains

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We consider the Reinhardt domain $D_n = \{(\zeta, z) \in \mathbb{C} \times \mathbb{C}^n : |\zeta|^2 < (1 - |z_1|^2) \cdots (1 - |z_n|^2)\}$. We express the explicit closed form of the Bergman kernel for D_n using the exponential generating function for the Stirling number of the second kind. As an application, we show that the Bergman kernel K_n for D_n has zeros if and only if $n \geq 3$. The study of the zeros of K_n is reduced to some real polynomial with coefficients which are related to Bernoulli numbers. This result is a complete characterization of the existence of zeros of the Bergman kernel for D_n for all positive integers n.

1. Introduction

Let D be a bounded domain in $\mathbb{C}^n$. The space $L_a^2(D)$ is denoted by the set of all holomorphic functions f on D satisfying

$$\int_D |f(z)|^2 \, dV(z) < \infty, \tag{1}$$

where dV is the volume measure on D. For any $z \in D$, $\Phi_z : L_a^2(D) \to \mathbb{C}$ defined by $\Phi_z(f) = f(z)$ is a bounded linear functional on $L_a^2(D)$. By Riesz representation theorem, there exists the unique element $K_z(\cdot) \in L_a^2(D)$ such that $\Phi_z(f) = \langle f(\cdot), K_z(\cdot)\rangle$; namely,

$$f(z) = \int_D f(w) \overline{K_z(w)} dV(w), \tag{2}$$

for all $f \in L_a^2(D)$. Define the Bergman kernel function $K_D(z, w) := \overline{K_z(w)}$ for D. It is defined for arbitrary bounded domains, but it is difficult to obtain the explicit form of the Bergman kernel for general bounded domains. Recently, the Bergman kernels for various domains have been computed explicitly in [1–6].

Over the last decade the Hartogs domain

$$\widehat{\Omega}_m := \left\{(\zeta, z) \in \mathbb{C}^m \times \Omega : \|\zeta\|^2 < p(z)\right\} \tag{3}$$

was investigated, where p is a suitably chosen continuous function on a bounded domain Ω. The Bergman kernel for $\widehat{\Omega}_m$ was obtained explicitly in [7] when Ω is an irreducible bounded symmetric domain. This result was generalized to the cases when Ω is the product of bounded symmetric domains in [8] and when Ω is a bounded homogeneous domain in [9].

Also the problem of determining whether the Bergman kernels are zero-free has been a well-known open problem in several complex variables ever since Lu Qi-Keng raised the question related to the existence of Bergman representative coordinates. If the Bergman kernel $K_D(z, w)$ for a bounded domain D is *zero-free* for all $(z, w) \in D \times D$, then D is called the *Lu Qi-Keng domain*. One can see many examples of Lu Qi-Keng domains and non-Lu Qi-Keng domains in [8–13].

If Ω is symmetric [8] or homogeneous [9], then the main part of the Bergman kernel for $\widehat{\Omega}_m$ is the *polynomial* whose coefficients are written as the forms containing the Stirling number of the second kind. The Routh-Hurwitz theorem (see Lemma 11) gives the condition that a real polynomial has no zeros in the closed right half-plane, and using this criterion we have the algorithmic method of determining whether the Hartogs domain $\widehat{\Omega}_m$ is a Lu Qi-Keng domain or not. The existence of zeros of all Hartogs domains $\widehat{\Omega}_m$ is classified in [8, 9] only when the dimension of the base domain Ω is low (less than 4). However it looks hard to study Lu Qi-Keng problem for *all* dimensions, since Routh-Hurwitz theorem involves too many terms when the order of the polynomial is *large*.

In this paper we consider the Reinhardt domain $D_n \subset \mathbb{C}^{n+1}$ defined by

$$D_n := \left\{ (\zeta, z) = (\zeta, z_1, z_2, \ldots, z_n) \in \mathbb{C} \times \mathbb{C}^n : |\zeta|^2 < \prod_{j=1}^{n} \left(1 - |z_j|^2\right) \right\}. \tag{4}$$

From Theorem 2.5 in [8], the Bergman kernel for D_n can be obtained explicitly as the following.

Theorem 1. *The Bergman kernel K_n for D_n is written as*

$$K_n((\zeta, z), (\eta, w)) = \frac{1}{\pi^n} \prod_{l=1}^{n} \frac{1}{(1 - z_l \overline{w}_l)^3} F_n\left(\frac{\langle \zeta, \eta \rangle}{\prod_{l=1}^{n} (1 - z_l \overline{w}_l)}\right), \tag{5}$$

where

$$F_n(t) = \sum_{j=1}^{n} (-1)^{n-j} S(n, j) \frac{(j+1)!}{(1-t)^{j+2}}, \tag{6}$$

where $S(n, j)$ is the Stirling number of the second kind.

In Section 2, we prove Theorem 1 using the result in [8] and express $a(n, \ell)$ in terms of the coefficients of a certain generating function (see Theorem 8). We use the well-known formal series

$$\frac{(e^t - 1)^j}{j!} = \sum_{n=0}^{\infty} S(n, j) \frac{t^n}{n!}, \tag{7}$$

for exponential generating function, where $S(n, j)$ is the Stirling number of the second kind.

For the study of the existence of zeros of $K_n((\zeta, z), (\eta, w))$, we need to define $\widetilde{F}_n(t)$ and $G_n(\tau)$ by

$$\begin{aligned} \widetilde{F}_n(t) &= (1-t)^2 F_n(t), \\ G_n(\tau) &= \widetilde{F}_n\left(1 - \frac{1}{\tau + 1/2}\right) \\ &= \sum_{j=1}^{n} (-1)^{n-j} S(n, j) (j+1)! \left(\tau + \frac{1}{2}\right)^j. \end{aligned} \tag{8}$$

The zero set of the Bergman kernel $K_n((\zeta, z), (\eta, w))$ with $(\zeta, z), (\eta, w) \in D_n$ reduces to the zero set of the polynomial $G_n(\tau)$ with $\operatorname{Re} \tau < 0$. Now we write

$$\begin{aligned} G_n(\tau) &= \sum_{k=0}^{n} a(n, n-k) \tau^k \\ &= a(n, n) + a(n, n-1) \tau + \cdots + a(n, 1) \tau^{n-1} \\ &\quad + a(n, 0) \tau^n. \end{aligned} \tag{9}$$

In Section 3, we introduce the Routh-Hurwitz theorem that is efficient on checking whether the real polynomial $G_n(\tau)$ has zeros in the right half plane. Using the generating form of coefficients of $G_n(\tau)$ (see Proposition 12), we will show that if $n \geq 3$, then $G_n(\tau)$ does not satisfy Routh-Hurwitz conditions, so we obtain the following main result of this paper.

Theorem 2. *The Bergman kernel for D_n is zero-free if and only if $n \leq 2$.*

For the proof of Theorem 2, we will show that if $n \geq 3$, then at least one of $a(n, n)$, $a(n, n-1)$, or $a(n, n-2)$ is negative (see Theorem 13). In Section 4, we discuss the properties of $a(n, \ell)$ and prove Theorem 13 using properties of the Bernoulli numbers and Genocchi numbers.

Remark 3. In Theorem 5.2(ii) and Theorem 5.3 of [8], one can see that K_2 has no zeros and K_3 has zeros. The main contribution, Theorem 2 in this paper, is the complete classification of the answer to the Lu Qi-Keng problem for D_n for all dimensions n.

2. Explicit Form of the Bergman Kernel for D_n

In [8], we know the explicit form of the Bergman kernel for Cartan-Hartogs domain $\widehat{\Omega}_m$ in the case when

$$\begin{gathered} \Omega = \Omega_1 \times \cdots \times \Omega_n, \\ p(z_1, \ldots, z_n) = \prod_{l=1}^{n} N_{\Omega_l}(z_l, z_l)^{\mu_l}, \quad \mu_l > 0, \end{gathered} \tag{10}$$

where N_{Ω_l} is the generic norm with respect to the bounded symmetric domain Ω_l. Using the numerical invariants a, b, r with respect to the bounded symmetric domain, we define the Hua polynomial

$$\chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}, \tag{11}$$

where $(s)_k = s(s+1)(s+2)\cdots(s+k-1)$ for $k \geq 1$ and $(s)_0 = 1$. Then for any $\mu > 0$, we define $c(\mu, j)$ by

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c(\mu, j)(k+1)_j, \tag{12}$$

where $d := r + rb + r(r-1)a/2$. We also define $d_j^{j_1, \ldots, j_n}$ satisfying

$$\prod_{l=1}^{n} (k+1)_{j_l} = \sum_{j=\max(j_1, \ldots, j_n)}^{j_1 + \cdots + j_n} d_j^{j_1, \ldots, j_n} (k+1)_j. \tag{13}$$

Theorem 4 (see [8]). *Let $\widehat{\Omega}_m := \{(\zeta, z) \in \mathbb{C}^m \times \Omega : \|\zeta\|^2 < p(z)\}$, where Ω and p are defined as in (10). Then the Bergman kernel for $\widehat{\Omega}_m$ is given by*

$$\widehat{K}_m((\zeta, z), (\eta, w)) = \frac{1}{m!} \prod_{l=1}^{n} \frac{K_{\Omega_l}(z_l, w_l)}{N_{\Omega_l}(z_l, w_l)^{m\mu_l}} F_n\left(\frac{\langle \zeta, z \rangle}{\prod_{l=1}^{n} N_{\Omega_l}(z_l, w_l)^{\mu_l}}\right), \tag{14}$$

where

$$F_n(t) = \sum_{j_1=0}^{\dim\Omega_1} \cdots \sum_{j_n=0}^{\dim\Omega_n} \prod_{l=1}^{n} c(\mu_l, j_l) \cdot \sum_{j=\max(j_1,\ldots,j_n)}^{j_1+\cdots+j_n} d_j^{j_1,\ldots,j_n} \frac{(j+m)!}{(1-t)^{j+m+1}}. \quad (15)$$

For any $n \geq k$, the number of the partitions of $\{1, 2, \ldots, n\}$ into k blocks is denoted by $S(n,k)$ and called the *Stirling number of the second kind.*

Lemma 5 (see [14]). *For any positive integer p, it holds that*

$$x^p = \sum_{j=1}^{p} S(p,j)(x-j+1)_j. \quad (16)$$

Proof of Theorem 1. For any $1 \leq \ell \leq n$, let $\Omega_l = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $m = \mu_1 = \cdots = \mu_n = 1$. Then $\widehat{\Omega}_m = D_n$ and

$$\begin{aligned} r &= 1, \\ a &= 2, \\ b &= 0, \\ g &= 2, \\ N_{\Omega_l}(z,w) &= 1 - z\overline{w}, \\ K_{\Omega_l}(z,w) &= (1 - z\overline{w})^{-2}. \end{aligned} \quad (17)$$

Since the Hua polynomial is $\chi(s) = s + 1$, by (12) we have

$$k + 1 = c(1,0) + c(1,1)(k+1), \quad (18)$$

so that $c(\mu, 0) = 0$ and $c(\mu, 1) = 1$.

Thus the function $F_n(t)$ in Theorem 4 is reduced to

$$F_n(t) = \sum_{j=1}^{n} d_j^{1,\ldots,1} \frac{(j+1)!}{(1-t)^{j+2}}. \quad (19)$$

Now we claim that $d_j^{1,\ldots,1}$ is equal to $(-1)^{n-j}S(n,j)$. If $j_1 = \cdots = j_n = 1$, then, by (13), we have

$$(k+1)^n = \sum_{j=1}^{n} d_j^{1,\ldots,1}(k+1)_j. \quad (20)$$

By Lemma 5, we have

$$\begin{aligned} (-x)^n &= \sum_{j=1}^{n} S(n,j)(-x-j+1)_j \\ &= \sum_{j=1}^{n} S(n,j)(-1)^j (x_j), \end{aligned} \quad (21)$$

since $(-x-j+1)_j = (-x-j+1)(-x-j+2)\cdots(-x) = (-1)^j (x)_j$. Thus we obtain

$$x^n = \sum_{j=1}^{n} (-1)^{n-j} S(n,j)(x)_j. \quad (22)$$

If we compare (20) and (22), then $d_j^{1,\ldots,1} = (-1)^{n-j}S(n,j)$ for all $1 \leq j \leq n$. Then from (19), it completes the proof of Theorem 1. □

Then the polynomial $G_n(\tau)$ can be written as

$$\begin{aligned} G_n(\tau) &= \sum_{j=1}^{n} (-1)^{n-j} S(n,j)(j+1)! \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)^{j-k} \tau^k \\ &= \sum_{k=0}^{n} \left\{ \sum_{j=k}^{n} (-1)^{n-j} S(n,j)(j+1)! \binom{j}{k} \left(\frac{1}{2}\right)^{j-k} \right\} \tau^k \\ &= \sum_{k=0}^{n} a(n, n-k) \tau^k, \end{aligned} \quad (23)$$

where

$$a(n,\ell) := \sum_{j=n-\ell}^{n} (-1)^{n-j} S(n,j)(j+1)! \binom{j}{n-\ell} \left(\frac{1}{2}\right)^{j-n+\ell}. \quad (24)$$

Note that

$$\begin{aligned} \binom{j}{n-\ell} &= \frac{j(j-1)\cdots(j-n+\ell+1)}{(n-\ell)!} \\ &= \frac{j!}{(n-\ell)!(j-n+\ell)!} \\ &= \frac{(n-\ell+1)(n-\ell+2)\cdots j}{(j-n+\ell)!} \\ &= (-1)^{j-n+\ell} \frac{(\ell-n-1)(\ell-n-2)\cdots(-j)}{(j-n+\ell)!} \\ &= (-1)^{j-n+\ell} \binom{\ell-n-1}{\ell-n+j}. \end{aligned} \quad (25)$$

Thus we have

$$a(n,\ell) = (-1)^{\ell} \sum_{j=n-\ell}^{n} S(n,j)(j+1)! \binom{\ell-n-1}{\ell-n+j} \left(\frac{1}{2}\right)^{j-n+\ell}. \quad (26)$$

Note that $S(n,j) = 0$ for $j > n$ and $\binom{\ell-n-1}{\ell-n+j} = 0$ for $j < n - \ell$; we have

$$a(n,\ell) = (-1)^{\ell} \sum_{j=0}^{\infty} S(n,j)\, j!\, (j+1) \binom{\ell-n-1}{\ell-n+j} \left(\frac{1}{2}\right)^{j-n+\ell}. \quad (27)$$

Lemma 6. *For any nonnegative integer t, one has*

$$\sum_{j=0}^{\infty}(j+1)\binom{-t-1}{-t+j}x^{j}=\frac{d}{dx}\left(\frac{x^{t+1}}{(1+x)^{t+1}}\right). \tag{28}$$

Proof. Note that for any nonnegative integer t,

$$\begin{aligned}\sum_{j=0}^{\infty}(j+1)\binom{-t-1}{-t+j}x^{-t+j}\\ =\sum_{j=0}^{\infty}(j+t+1)\binom{-t-1}{j}x^{j}.\end{aligned} \tag{29}$$

Note that

$$\begin{aligned}\frac{d}{dx}\left(\sum_{j=0}^{\infty}\binom{-t-1}{j}x^{j+t+1}\right)\\ =\sum_{j=0}^{\infty}\binom{-t-1}{j}(j+t+1)x^{j+t}\\ =\sum_{j=0}^{\infty}(j+t+1)\binom{-t-1}{j}x^{j+t}.\end{aligned} \tag{30}$$

Thus

$$\sum_{j=0}^{\infty}(j+t+1)\binom{-t-1}{j}x^{j}=\frac{1}{x^{t}}\frac{d}{dx}\left(\frac{x^{t+1}}{(1+x)^{t+1}}\right), \tag{31}$$

which completes the proof. □

By Lemma 6, we have

$$\begin{aligned}\sum_{j=0}^{\infty}(j+1)\binom{\ell-n-1}{\ell-n+j}\left(\frac{1}{2}\right)^{\ell-n+j}u^{j}\\ =\left(\frac{1}{2}\right)^{\ell-n}\sum_{j=0}^{\infty}(j+1)\binom{\ell-n-1}{\ell-n+j}\left(\frac{u}{2}\right)^{j}\\ =\left(\frac{1}{2}\right)^{\ell-n}\frac{d}{dx}\left(\frac{x^{n-\ell+1}}{(1+x)^{n-\ell+1}}\right)\Bigg|_{x=u/2}.\end{aligned} \tag{32}$$

Definition 7. Define

$$H_{n,\ell}(u):=\left(\frac{1}{2}\right)^{\ell-n}\frac{d}{dx}\left(\frac{x^{n-\ell+1}}{(1+x)^{n-\ell+1}}\right)\Bigg|_{x=u/2}. \tag{33}$$

Let $[t^{n}]$ be the operator which gives the nth coefficient in the series expansion of a generating function. It is well-known that the exponential generating function of $S(n,j)$ is the formal power series

$$\frac{(e^{t}-1)^{j}}{j!}=\sum_{n=0}^{\infty}S(n,j)\frac{t^{n}}{n!}. \tag{34}$$

Using the above generating function, we prove the following.

Theorem 8. *Let $G_n(\tau)$ be the polynomial defined as in (8) with*

$$G_{n}(\tau)=\sum_{k=0}^{n}a_{n}(n-k)\tau^{k}. \tag{35}$$

Then the coefficient $a(n,\ell)$ is written as

$$a(n,\ell)=(-1)^{\ell}\left[\frac{t^{n}}{n!}\right]H_{n,\ell}\left(e^{t}-1\right), \tag{36}$$

where $0\le\ell\le n$.

Proof. By (27) and (32), the coefficient $a(n,\ell)$ of $G_n(\tau)$ can be expressed as

$$a(n,\ell)=(-1)^{\ell}\sum_{j=0}^{\infty}S(n,j)\,j!\left[u^{j}\right]H_{n,\ell}(u). \tag{37}$$

Note that, by (34), we have

$$\begin{aligned}(-1)^{\ell}\sum_{j=0}^{\infty}\left(e^{t}-1\right)^{j}\left[u^{j}\right]H_{n,\ell}(u)\\ =(-1)^{\ell}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}S(n,j)\,j!\left[u^{j}\right]H_{n,\ell}(u)\frac{t^{n}}{n!}\\ =\sum_{n=0}^{\infty}\left((-1)^{\ell}\sum_{j=0}^{\infty}S(n,j)\,j!\left[u^{j}\right]H_{n,\ell}(u)\right)\frac{t^{n}}{n!}\\ =\sum_{n=0}^{\infty}a(n,\ell)\frac{t^{n}}{n!},\end{aligned} \tag{38}$$

which follows that

$$\begin{aligned}a(n,\ell)=(-1)^{\ell}\left[\frac{t^{n}}{n!}\right]\sum_{j=0}^{\infty}\left(e^{t}-1\right)^{j}\left[u^{j}\right]H_{n,\ell}(u)\\ =(-1)^{\ell}\left[\frac{t^{n}}{n!}\right]H_{n,\ell}\left(e^{t}-1\right).\end{aligned} \tag{39}$$

It completes the proof. □

3. Lu Qi-Keng Domains

In this section we investigate the explicit form of $a(n,\ell)$ and prove that the Bergman kernel for D_n has zeros for any positive integer n.

Note that if $(\zeta,z),(\eta,w)\in D_n$, then

$$\begin{aligned}\left|\frac{\langle\zeta,\eta\rangle}{\prod_{l=1}^{n}(1-z_{l}\overline{w}_{l})}\right|\le\frac{|\zeta|\,|\eta|}{\prod_{l=1}^{n}\sqrt{\left(1-|z_{l}|^{2}\right)\left(1-|w_{l}|^{2}\right)}}\\ =\sqrt{\frac{|\zeta|^{2}}{\prod_{l=1}^{n}\left(1-|z_{l}|^{2}\right)}\frac{|\eta|^{2}}{\prod_{l=1}^{n}\left(1-|w_{l}|^{2}\right)}}<1.\end{aligned} \tag{40}$$

Thus by Theorem 1 and (8), we obtain the following.

Lemma 9. *The zero set*

$$\{((\zeta,z),(\eta,w)) \in D_n \times D_n : K_n((\zeta,z),(\eta,w)) = 0\} \quad (41)$$

is equal to the zero set

$$\left\{((\zeta,z),(\eta,w)) \in D_n \times D_n : \widetilde{F}_n(t) = 0,\ |t| < 1\right\}, \quad (42)$$

where $t := \langle\zeta,\eta\rangle/\prod_{l=1}^{n}(1 - z_l\overline{w}_l)$.

Since the holomorphic function $t \mapsto 1/(1-t) - 1/2$ maps the unit disk onto the right half plane, we obtain the following consequence of Theorem 1 and Lemma 9.

Lemma 10. *For any positive integers n, the domain D_n is a Lu Qi-Keng domain if and only if all zeros of the polynomial $G_n(\tau)$ lie in the closed left half plane $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$.*

The Routh-Hurwitz criterion is the most efficient method for determining whether the polynomial $G_n(\tau)$ has zeros in the open left half plane. Let

$$f(\tau) = a_0\tau^n + a_1\tau^{-1} + \cdots + a_{n-1}\tau + a_n, \quad (43)$$

with real coefficients and $a_0 > 0$, and define Δ_j^n for $j = 1, \ldots, n$ by

$$\Delta_j^n := \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2j-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2j-2} \\ 0 & a_1 & a_3 & \cdots & a_{2j-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_j \end{vmatrix}, \quad (44)$$

where $a_j = 0$ if $j < 0$ or $j > n$.

Lemma 11 (Routh-Hurwitz/Liénard-Chipart [15]). *All zeros of given polynomial $f(\tau)$ lie in the open left half plane $\{\tau \in \mathbb{C} : \operatorname{Re}\tau < 0\}$ if and only if*

$$\Delta_1^n > 0, \ldots, \Delta_n^n > 0. \quad (45)$$

This condition is also equivalent to any one of the following four forms:

(i) $a_n > 0,\ a_{n-2} > 0,\ a_{n-4} > 0, \ldots;\ \Delta_1^n > 0,\ \Delta_3^n > 0, \ldots$,

(ii) $a_n > 0,\ a_{n-2} > 0,\ a_{n-4} > 0, \ldots;\ \Delta_2^n > 0,\ \Delta_4^n > 0, \ldots$,

(iii) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0, \ldots;\ \Delta_1^n > 0,\ \Delta_3^n > 0, \ldots$,

(iv) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0, \ldots;\ \Delta_2^n > 0,\ \Delta_4^n > 0, \ldots$.

Proposition 12. *Let $[t^n]$ be the operator which gives the nth coefficient in the series expansion of a generating function*

$$a(n,\ell) = (-1)^\ell (n-\ell+1)\, 2^{n-\ell+2} \left[\frac{t^n}{n!}\right] \frac{(e^t-1)^{n-\ell}}{(e^t+1)^{n-\ell+2}}. \quad (46)$$

Proof. Note that

$$\begin{aligned} H_{n,\ell}(u) &= \left(\frac{1}{2}\right)^{\ell-n} \frac{d}{dx}\left(\frac{x^{n-\ell+1}}{(1+x)^{n-\ell+1}}\right)\Bigg|_{x=u/2} \\ &= \left(\frac{1}{2}\right)^{\ell-n} (n-\ell+1) \frac{x^{n-\ell}}{(1+x)^{n-\ell+2}}\Bigg|_{x=u/2} \\ &= (n-\ell+1)\, 2^{n-\ell+2} \frac{u^{n-\ell}}{(u+2)^{n-\ell+2}}. \end{aligned} \quad (47)$$

By Theorem 8,

$$a(n,\ell) = (-1)^\ell (n-\ell+1)\, 2^{n-\ell+2} \left[\frac{t^n}{n!}\right] \frac{(e^t-1)^{n-\ell}}{(e^t+1)^{n-\ell+2}}. \quad (48)$$

□

By Proposition 12, we have

$$\begin{aligned} a(n,n) &= 4(-1)^n \left[\frac{t^n}{n!}\right] \frac{1}{(e^t+1)^2}, \\ a(n,n-1) &= 16(-1)^{n-1} \left[\frac{t^n}{n!}\right] \frac{e^t-1}{(e^t+1)^3}, \\ a(n,n-2) &= 48(-1)^{n-2} \left[\frac{t^n}{n!}\right] \frac{(e^t-1)^2}{(e^t+1)^4}. \end{aligned} \quad (49)$$

Theorem 13. *For each $n \geq 3$, at least one of $a(n,n)$, $a(n,n-1)$, or $a(n,n-2)$ is negative. More precisely, one has*

(i) $a(n,n) < 0$ *if* $n = 3, 4, 7, 8, 11, 12, 15, 16, \ldots$,

(ii) $a(n,n-1) < 0$ *if* $n = 5, 9, 13, \ldots$,

(iii) $a(n,n-2) < 0$ *if* $n = 6, 10, 14, \ldots$.

Proof. We will prove it in Section 4. □

Remark 14. (i) Note that

$$G_2(\tau) = \frac{1}{2} + 6\tau + 4\tau^2 \quad (50)$$

has two negative real zeros. Thus, D_2 is a Lu Qi-Keng domain.

(ii) In fact, we see that

$$G_3(\tau) = -\frac{1}{2} + 2\tau + 18\tau^2 + 24\tau^3 \quad (51)$$

has one positive real zero $-1/8 + \sqrt{33}/24 > 0$. Thus, we conclude that the Bergman kernel for D_3 has zeros, so D_3 is not a Lu Qi-Keng domain.

(iii) By using a computer program (Maple or Mathematica), we computed the explicit values of $a(n,n)$, $a(n,n-1)$, and $a(n,n-2)$ for $2 \leq n \leq 14$. One can check Theorem 13 holds for $3 \leq n \leq 14$ in Tables 1 and 2.

Now we can prove the main theorem of this paper using Theorem 13.

Table 1: $2 \le n \le 8$.

	G_2	G_3	G_4	G_5	G_6	G_7	G_8
$a(n,n)$	1/2	**−1/2**	**−1**	1	17/4	**−17/4**	−31
$a(n,n-1)$	6	2	−8	**−13**	34	107	**−248**
$a(n,n-2)$	4	18	6	−90	**−129**	693	1896

Table 2: $9 \le n \le 14$.

G_9	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}
131	691/2	**−691/2**	**−5461**	5461	929569/8
−1258	2764	20462	−43688	**−885881/2**	929569
−7920	**−34014**	126918	776661	−2723175	**−44729673/2**

Theorem 2 (again). *The Bergman kernel for*

$$D_n = \left\{ (\zeta, z_1, z_2, \ldots, z_n) \in \mathbb{C}^{n+1} : \|\zeta\|^2 < \prod_{j=1}^{n} \left(1 - |z_j|^2\right) \right\} \quad (52)$$

is zero-free if and only if $n \le 2$.

Proof. (i) If $n = 4k - 1$ or $n = 4k$ for $k \in \mathbb{N}$, then $a(n,n) < 0$ by Theorem 13. So, the polynomial $G_n(\tau)$ does not satisfy any condition in Lemma 11. It follows that D_n is not a Lu Qi-Keng domain.

(ii) If $n = 4k + 1$ for $k \in \mathbb{N}$, then $a(n, n-1) < 0$ by Theorem 13. So, the polynomial $G_n(\tau)$ does not satisfy conditions (iii) and (iv) in Lemma 11. It follows that D_n is not a Lu Qi-Keng domain.

(iii) If $n = 4k + 1$ for $k \in \mathbb{N}$, then $a(n, n-2) < 0$ by Theorem 13. So, the polynomial $G_n(\tau)$ does not satisfy conditions (i) and (ii) in Lemma 11. It follows that D_n is not a Lu Qi-Keng domain.

By (i), (ii), and (iii), $G_n(\tau)$ does not satisfy any condition of Routh-Hurwitz theorem, so the Bergman kernel for D_n has zeros for all $n \ge 3$. □

4. Proof of Theorem 13

In this section, we investigate the properties of $a(n,n)$, $a(n,n-1)$, and $a(n,n-2)$ and prove Theorem 13. For convenience, we denote the functions A, B, C, D by

$$\begin{aligned} A(t) &:= \frac{1}{e^t + 1} = \sum_{k=0}^{\infty} a_k t^k, \\ B(t) &:= \frac{1}{(e^t + 1)^2} = \sum_{k=0}^{\infty} b_k t^k, \\ C(t) &:= \frac{1}{(e^t + 1)^3} = \sum_{k=0}^{\infty} c_k t^k, \\ D(t) &:= \frac{1}{(e^t + 1)^4} = \sum_{k=0}^{\infty} d_k t^k. \end{aligned} \quad (53)$$

It is interesting that the numbers a_k's are related to the following Bernoulli numbers. Bernoulli [16] introduced *Bernoulli numbers* B_{2n} for $n \ge 1$ satisfying the identity

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} B_{2n} \frac{t^{2n}}{(2n)!}, \quad (54)$$

where $B_2 = 1/6$, $B_4 = 1/30$, and $B_6 = 1/42, \ldots$. The *Genocchi numbers* G_{2n} for $n \ge 1$ are defined [17] by

$$G_{2n} := 2\left(2^{2n} - 1\right) B_{2n}. \quad (55)$$

Lemma 15 (see [18]). *Let n be any positive integer. Then*

(i) *B_{2n}'s are positive for all positive integers n;*

(ii) $2t/(e^t + 1) = t + \sum_{n=1}^{\infty} (-1)^n G_{2n} (t^{2n}/(2n)!)$.

Lemma 16. *For any $k \in \mathbb{N}$, one has*

$$\begin{aligned} a_{2k} &= 0, \\ a_{4k-1} &> 0, \\ a_{4k-3} &< 0. \end{aligned} \quad (56)$$

Moreover $a_0 = 1/2$.

Proof. By Lemma 15(ii), we have

$$\frac{1}{e^t + 1} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n G_{2n}}{2 \cdot (2n)!} t^{2n-1}. \quad (57)$$

It follows that

$$\begin{aligned} a_0 &= \frac{1}{2}, \\ a_{2k} &= 0 \quad (k \ge 1), \\ a_{2n-1} &= (-1)^n \frac{G_{2n}}{2 \cdot (2n)!}. \end{aligned} \quad (58)$$

Note that $G_{2k} > 0$ for all $k \ge 1$ by Lemma 15(i). Thus $a_{4k-1} > 0$ and $a_{4k-3} < 0$ for all $k \ge 1$. □

Lemma 17. *For any $k \in \mathbb{N}$, one has*

(i) $b_{2k+1} = a_{2k+1}$,

(ii) $b_k = (k+1) a_{k+1} + a_k$,

(iii) $2c_{2k} = 3b_{2k}$,

(iv) $2c_k = (k+1) b_{k+1} + 2b_k$,

(v) $2d_{2k+1} = 4a_{2k+1} - 6b_{2k+1} + 4c_{2k+1}$,

(vi) $3d_k = (k+1) c_{k+1} + 3c_k$.

Proof. From the identity

$$B(-t) = \frac{e^{2t}}{(e^t + 1)^2} = \frac{(e^t + 1)^2 - 2(e^t + 1) + 1}{(e^t + 1)^2}, \quad (59)$$

we obtain

$$B(t) - B(-t) = 2A(t) - 1. \tag{60}$$

It follows that

$$2\sum_{k=0}^{\infty} b_{2k+1} t^{2k+1} = 2\sum_{k=0}^{\infty} a_k t^k + 1. \tag{61}$$

Thus we obtain (i). From the identity

$$A'(t) = \frac{-e^t}{(e^t+1)^2} = \frac{-(e^t+1)+1}{(e^t+1)^2}, \tag{62}$$

we obtain

$$A(t) + A'(t) = B(t). \tag{63}$$

It follows that

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} b_k t^k. \tag{64}$$

Thus we obtain (ii). From the identity

$$C(t) + C(-t) = 1 - 3A(t) + 3B(t), \tag{65}$$

we have $2c_{2k} = -3a_{2k} + 3b_{2k}$. Since $a_{2k} = 0$ for $k \geq 1$ in Lemma 16, we obtain (iii). From the identity

$$B'(t) + 2B(t) = 2C(t), \tag{66}$$

we obtain (iv). Similarly as the previous proofs, we can easily see that (v) comes from the identity

$$D(t) - D(-t) = -1 + 4A(t) - 6B(t) + 4C(t), \tag{67}$$

and (vi) comes from the identity

$$C'(t) + 3C(t) = 3D(t). \tag{68}$$

□

Proposition 18. *For any $k \in \mathbb{N}$, it holds that*

(i) $a_{2k} = 0,\ a_{4k-1} > 0,\ a_{4k+1} < 0,$

(ii) $b_{4k} < 0,\ b_{4k+1} < 0,\ b_{4k+2} > 0,\ b_{4k+3} > 0,$

(iii) $c_{4k+1} > 0,\ c_{4k+2} > 0,$

(iv) $d_{4k+2} < 0$ *for* $k \geq 2$.

Proof. (ii) Note that

$$\begin{aligned} b_{4k} &= (4k+1)\, a_{4k+1} < 0, \\ b_{4k+1} &= a_{4k+1} < 0, \\ b_{4k+2} &= (4k+3)\, a_{4k+3} > 0, \\ b_{4k+3} &= a_{4k+3} > 0. \end{aligned} \tag{69}$$

(iii) Note that $c_{4k+2} = (3/2) b_{4k+2} > 0$. Since $C(t) = A(t) \cdot B(t)$, we have

$$\begin{aligned} c_{4k+1} &= \sum_{l=0}^{4k+1} a_l b_{4k+1-l} \\ &= a_0 b_{4k+1} + a_1 b_{4k} + a_{4k+1} b_0 + \sum_{l=1}^{2k-1} a_{2l+1} b_{4k-2l} \\ &> a_0 b_{4k+1} + a_1 b_{4k} + a_{4k+1} b_0 \\ &= \frac{1}{2} b_{4k+1} - \frac{1}{4} b_{4k} + \frac{1}{4} a_{4k+1} \\ &= \frac{1}{2} a_{4k+1} - \frac{1}{4}(4k+1)\, a_{4k+1} + \frac{1}{4} a_{4k+1} \\ &= \frac{1}{4}(2-4k)\, a_{4k+1} > 0. \end{aligned} \tag{70}$$

Here by Proposition 18(i) and (ii),

$$\sum_{l=1}^{2k-1} a_{2l+1} b_{4k-2l} = \sum_{l=1}^{k-1} a_{4l+1} b_{4k-4l} + \sum_{l=1}^{k} a_{2l-1} b_{4k-4l+2} > 0. \tag{71}$$

(iv) Similarly as the proof of (iii), we obtain $d_{4k+2} < 0$ for $k \geq 2$. □

Now we prove Theorem 13 using the above proposition.

Theorem 13 (again). *For any positive integers k, one has*

(i) $a(n,n) < 0$ *if* $n = 4k-1$ *or* $n = 4k$,

(ii) $a(n,n-1) < 0$ *if* $n = 4k+1$,

(iii) $a(n,n-2) < 0$ *if* $n = 4k+2$.

Proof. (i) Note that

$$a(n,n) = 4(-1)^n \left[\frac{t^n}{n!}\right] \frac{1}{(e^t+1)^2} = 4(-1)^n b_n. \tag{72}$$

By Proposition 18(ii), it follows that

$$\begin{gathered} a(4k-1, 4k-1) = -4b_{4k-1} < 0, \\ a(4k, 4k) = 4b_{4k} < 0. \end{gathered} \tag{73}$$

(ii) Note that

$$\begin{aligned} a(n, n-1) &= 16(-1)^{n-1} \left[\frac{t^n}{n!}\right] \frac{e^t - 1}{(e^t+1)^3} \\ &= 16(-1)^{n-1} (b_n - 2c_n), \end{aligned} \tag{74}$$

since

$$\frac{e^t-1}{(e^t+1)^3} = \frac{(e^t+1)-2}{(e^t+1)^3} = \frac{1}{(e^t+1)^2} - \frac{2}{(e^t+1)^3}. \tag{75}$$

By Proposition 18(ii) and (iii), it follows that

$$a(4k+1, 4k) = 16(b_{4k+1} - 2c_{4k+1}) < 0. \tag{76}$$

(iii) Note that

$$a(n,n-2) = 48(-1)^{n-2}\left[\frac{t^n}{n!}\right]\frac{(e^t-1)^2}{(e^t+1)^4} = 48(-1)^{n-2}(b_n - 4c_n + 4d_n), \tag{77}$$

since

$$\frac{(e^t-1)^2}{(e^t+1)^4} = \frac{(e^t+1-2)^2}{(e^t+1)^4} = \frac{1}{(e^t+1)^2} - \frac{4}{(e^t+1)^3} + \frac{4}{(e^t+1)^4}. \tag{78}$$

By Lemma 17(iii), we have

$$a(4k+2,4k) = 48(b_{4k+2} - 4c_{4k+2} + 4d_{4k+2}) = 48(-5b_{4k+2} + 4d_{4k+2}) < 0 \quad \text{for } k \geq 2. \tag{79}$$

Moreover $a(6,4) = 48(-5b_6 + 4d_6) = 48\cdot(-43/11520) < 0$. □

Appendix

We add the explicit forms of Taylor expansions for some functions which have been discussed in Section 4.

$$\begin{aligned}
\frac{1}{e^x+1} &= \frac{1}{2} - \frac{x}{4} + \frac{x^3}{48} - \frac{x^5}{480} + \frac{17x^7}{80640} - \frac{31x^9}{1451520} + \frac{691x^{11}}{319334400} - \cdots, \\
\frac{1}{(e^x+1)^2} &= \frac{1}{4} - \frac{x}{4} + \frac{x^2}{16} + \frac{x^3}{48} - \frac{x^4}{96} - \frac{x^5}{480} + \frac{17x^6}{11520} + \frac{17x^7}{80640} - \frac{31x^8}{161280} - \frac{31x^9}{1451520} + \frac{691x^{10}}{29030400} + \frac{691x^{11}}{319334400} - \frac{5461x^{12}}{1916006400} - \cdots, \\
\frac{1}{(e^x+1)^3} &= \frac{1}{8} - \frac{3x}{16} + \frac{3x^2}{32} - \frac{x^4}{64} + \frac{3x^5}{1280} + \frac{17x^6}{7680} - \frac{x^7}{1792} - \frac{31x^8}{107520} + \frac{x^9}{10240} + \frac{691x^{10}}{19353600} - \frac{53x^{11}}{3548160} - \frac{5461x^{12}}{1277337600} + \frac{8507x^{13}}{4025548800} + \frac{929569x^{14}}{1859803545600} - \cdots, \\
\frac{1}{(e^x+1)^4} &= \frac{1}{16} - \frac{x}{8} + \frac{3x^2}{32} - \frac{x^3}{48} - \frac{3x^4}{256} + \frac{13x^5}{1920} + \frac{7x^6}{7680} - \frac{107x^7}{80640} + \frac{x^8}{215040} + \frac{629x^9}{2903040} - \frac{41x^{10}}{2150400} - \frac{10231x^{11}}{319334400} + \frac{7127x^{12}}{1459814400} + \frac{885881x^{13}}{199264665600} - \frac{1710341x^{14}}{1859803545600} - \frac{24688759x^{15}}{41845579776000} + \cdots, \\
\frac{e^x-1}{(e^x+1)^3} &= \frac{x}{8} - \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{48} - \frac{13x^5}{1920} - \frac{17x^6}{5760} + \frac{107x^7}{80640} + \frac{31x^8}{80640} - \frac{629x^9}{2903040} - \frac{691x^{10}}{14515200} + \frac{10231x^{11}}{319334400} + \frac{5461x^{12}}{958003200} - \frac{885881x^{13}}{199264665600} - \cdots, \\
\frac{(e^x-1)^2}{(e^x+1)^4} &= \frac{x^2}{16} - \frac{x^3}{16} + \frac{x^4}{192} + \frac{x^5}{64} - \frac{43x^6}{11520} - \frac{11x^7}{3840} + \frac{79x^8}{80640} + \frac{11x^9}{24192} - \frac{5669x^{10}}{29030400} - \frac{641x^{11}}{9676800} + \frac{258887x^{12}}{7664025600} + \frac{133x^{13}}{14598144} - \frac{14909891x^{14}}{2789705318400} - \cdots.
\end{aligned} \tag{A.1}$$

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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