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Research Article

Adjoining a Constant Function to n-Dimensional Chebyshev Space

Mansour Alyazidi-Asiry

Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Mansour Alyazidi-Asiry; yazidi@ksu.edu.sa

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This paper is concerned with extending a Chebyshev system of n continuous nonconstant functions into a set of $n+1$ functions including a constant function. Necessary and sufficient conditions for the new set to be a Chebyshev system are discussed and some results are obtained.

1. Introduction

Most of the material in this section and Section 2 can be found in any standard book in approximation theory and related topics; see, for example, [1–5]. The finite set of functions $\{g_1,\ldots,g_n\}\subset \mathrm{C}[a,b]$ is called a Chebyshev system on $[a,b]$ if it is linearly independent and $\mathrm{D}\binom{g_1,\ldots,g_n}{x_1,\ldots,x_n}=\mathrm{Det}[g_i(x_j)]\neq 0$, $i,j=1,\ldots,n$, for all $\{x_j\}_{j=1}^{n}$ such that $a\le x_1<x_2<\cdots<x_n\le b$, and the n-dimensional subspace $G=\langle g_1,\ldots,g_n\rangle$ of $\mathrm{C}[a,b]$ will be called a Chebyshev subspace or Haar subspace. Using the continuity of the determinant, it can be shown that the sign of the determinant is constant (see [6]), so we will assume that the determinant is always positive throughout this paper (replace g_1 by $-g_1$ if necessary). If each g_i is continuously differentiable function on $[a,b]$, $i=1,\ldots,n$ and $a\le x_1\le x_2\le\cdots\le x_n\le b$, then the determinant $\mathrm{D}^*\binom{g_1,\ldots,g_n}{x_1,\ldots,x_n}$ is defined as follows:

$$\mathrm{D}^*\begin{pmatrix} g_1,\ldots,g_n\\ x_1,\ldots,x_n\end{pmatrix}=\begin{bmatrix} g_1(t_1) & \cdots & g_n(t_1)\\ g_1'(t_1) & \cdots & g_n'(t_1)\\ \vdots & & \vdots\\ g_1^{(r_1-1)}(t_1) & \cdots & g_n^{(r_1-1)}(t_1)\\ \vdots & & \vdots\\ g_1(t_p) & \cdots & g_n(t_p)\\ \vdots & & \vdots\\ g_1^{(r_p-1)}(t_p) & \cdots & g_n^{(r_p-1)}(t_p)\end{bmatrix}, \quad (1)$$

where x_i is repeated r_i times, $i=1,\ldots,p$, $a\le t_1<t_2<\cdots<t_p\le b$, and $\{t_1,t_2,\ldots,t_p\}=\{x_1,x_2,\ldots,x_n\}$. The set of functions $\{g_1,\ldots,g_n\}$ is called an extended Chebyshev system on $[a,b]$ if $\mathrm{D}^*\binom{g_1,\ldots,g_n}{x_1,\ldots,x_n}>0$, and the n-dimensional subspace $G=\langle g_1,\ldots,g_n\rangle$ of $\mathrm{C}[a,b]$ will be called an extended Chebyshev subspace.

In this paper we will consider the following problem.

If $G=\langle g_1,\ldots,g_n\rangle$ is a Chebyshev subspace of $\mathrm{C}[a,b]$ such that $1\notin G$ then what property must G have so that the subspace $U=\langle u_0,u_1,\ldots,u_n\rangle$ is $(n+1)$-dimensional Chebyshev subspace of $\mathrm{C}[a,b]$, where $u_0=1$, $u_i=g_i$, $i=1,\ldots,n$? We will present some results in Section 3 which give a partial answer to this question.

2. Preliminary

Let $G=\langle g_1,\ldots,g_n\rangle$ be a Chebyshev subspace of $\mathrm{C}[a,b]$ and let $\{x_j\}_{j=1}^{n+1}$ be a set of points such that $a\le x_1<x_2<\cdots<x_{n+1}\le b$, and then for any $g\in G$ we have

$$0=\mathrm{D}\begin{pmatrix} g,g_1,\ldots,g_n\\ x_1,\ldots,x_{n+1}\end{pmatrix}=\sum_{i=1}^{n+1}(-1)^{i+1}\Delta_i g(x_i), \quad (2)$$

where

$$\Delta_1=\mathrm{D}\begin{pmatrix} g_1,\ldots,g_n\\ x_2,\ldots,x_{n+1}\end{pmatrix},$$

$$\Delta_{n+1} = \mathrm{D}\begin{pmatrix} g_1, \dots, g_n \\ x_1, \dots, x_n \end{pmatrix},$$
$$\Delta_i = \mathrm{D}\begin{pmatrix} g_1, \dots, g_n \\ x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1} \end{pmatrix}, \quad i = 2, \dots, n. \tag{3}$$

Taking $\theta_i = \Delta_i / \sum_{j=1}^{n+1} \Delta_j$ then $\theta_i > 0$ for all $i = 1, \dots, n+1$ with $\sum_{i=1}^{n+1} \theta_i = 1$ and $\sum_{i=1}^{n+1} (-1)^i \theta_i g(x_i) = 0$ for every $g \in G$.

This discussion proves the existence part of the following lemma.

Lemma 1. *Let $G = \langle g_1, \dots, g_n \rangle$ be a Chebyshev subspace of $\mathrm{C}[a,b]$ and let $\{x_j\}_{j=1}^{n+1}$ be a set of points such that $a \le x_1 < x_2 < \cdots < x_{n+1} \le b$, and then there exists a unique set of positive numbers $\{\theta_j\}_{j=1}^{n+1}$ with $\sum_{i=1}^{n+1} \theta_i = 1$ such that $\sum_{i=1}^{n+1} (-1)^i \theta_i g(x_i) = 0$ for every $g \in G$.*

Proof. We only need to prove the uniqueness part of this lemma. Suppose that there are two sets of positive real numbers $\{\theta_j\}_{j=1}^{n+1}$ and $\{\lambda_j\}_{j=1}^{n+1}$ with $\sum_{i=1}^{n+1} \theta_i = \sum_{i=1}^{n+1} \lambda_i = 1$ such that

$$\sum_{i=1}^{n+1} (-1)^i \theta_i g(x_i) = \sum_{i=1}^{n+1} (-1)^i \lambda_i g(x_i) = 0 \quad \text{for every } g \in G. \tag{4}$$

Since G is a Chebyshev subspace, then for each $k \in \{2, \dots, n+1\}$ there exists a unique function $h^{(k)} \in G$ such that $h^{(k)}(x_1) = 1$ and $h^{(k)}(x_l) = 0$, $l \in \{2, \dots, n+1\} \setminus \{k\}$ (see [6]), and from (4) we have

$$\begin{aligned} -\theta_1 + (-1)^k \theta_2 h^{(k)}(x_k) &= 0, \\ -\lambda_1 + (-1)^k \lambda_2 h^{(k)}(x_k) &= 0, \end{aligned} \tag{5}$$

where $k = 2, \dots, n+1$.

Clearly $h^{(k)}(x_k) \neq 0$, and therefore (5) yield

$$\frac{\theta_k}{\theta_1} = \frac{\lambda_k}{\lambda_1}, \quad k = 2, \dots, n+1. \tag{6}$$

Hence $(1/\theta_1)\sum_{k=2}^{n+1} \theta_k = (1/\lambda_1)\sum_{k=2}^{n+1} \lambda_k \Rightarrow (1-\theta_1)/\theta_1 = (1-\lambda_1)/\lambda_1 \Rightarrow \theta_1 = \lambda_1$, and, by (6), $\theta_i = \lambda_i$, $i = 2, \dots, n+1$, and the proof is complete. □

3. The Main Result

We start this section by the following theorem.

Theorem 2. *Let $G = \langle g_1, \dots, g_n \rangle$, where $\{g_i\}_{i=1}^n \subset \mathrm{C}[a,b]$ is a Chebyshev system on $[a,b]$. Then $\{g_0 = 1, g_1, \dots, g_n\}$ is a Chebyshev system on $[a,b]$ if and only if, for each set of points $\{t_i\}_{i=0}^n$ such that $a \le t_0 < t_1 < \cdots < t_n \le b$ and the corresponding set of positive real numbers $\{\theta_j\}_{j=0}^n$ with $\sum_{i=0}^n \theta_i = 1$ satisfying $\sum_{i=0}^n (-1)^i \theta_i g(t_i) = 0$ for every $g \in G$, we have*

$$\sum_{i \in I} \theta_i \neq \sum_{j \in J} \theta_j, \tag{7}$$

where $I = \{i \in A : i \text{ is odd}\}$, $A = \{0, 1, \dots, n\}$, and $J = A \setminus I$.

Proof. Let $\{t_i\}_{i=0}^n$ be a set of points such that $a \le t_0 < t_1 < \cdots < t_n \le b$ and the corresponding set of positive real numbers $\{\theta_j\}_{j=0}^n$ with $\sum_{i=0}^n \theta_i = 1$ satisfying $\sum_{i=0}^n (-1)^i \theta_i g(t_i) = 0$ for every $g \in G$, where $G = \langle g_1, \dots, g_n \rangle$ is a Chebyshev subspace of $\mathrm{C}[a,b]$. From Lemma 1 $\theta_i = \Delta_i / d$, where $d = \sum_{j=0}^n \Delta_j$ and $\Delta_i = \mathrm{D}\begin{pmatrix} g_1, \dots, g_n \\ t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_n \end{pmatrix}$, $i = 0, \dots, n$. Hence

$$\begin{aligned} \mathrm{D}\begin{pmatrix} g_0, g_1, \dots, g_n \\ t_0, \dots, t_n \end{pmatrix} &= \mathrm{Det}\begin{bmatrix} 1 & g_1(t_0) & \cdots & g_n(t_0) \\ 1 & g_1(t_1) & \cdots & g_n(t_1) \\ \vdots & \vdots & \cdots & \vdots \\ 1 & g_1(t_n) & \cdots & g_n(t_n) \end{bmatrix}, \\ &= \sum_{i=0}^n (-1)^i \Delta_i = \sum_{i=0}^n (-1)^i d\theta_i \\ &= d\left[\sum_{i \in I} \theta_i - \sum_{j \in J} \theta_j\right] \neq 0 \end{aligned} \tag{8}$$

if and only if $\sum_{i \in I} \theta_i \neq \sum_{j \in J} \theta$, where I and J are as defined above, and the theorem is proved. □

Assumption 3. Let $g_i \in \mathrm{C}^1[a,b]$ for all $i = 1, \dots, n$ and let $G = \langle g_1, \dots, g_n \rangle$ be a Chebyshev subspace of $\mathrm{C}[a,b]$. We say that G satisfies Assumption 3 if, for each nontrivial element g of G, g' can have at most $n-1$ distinct zeros on (a,b). That is, if $g'(x_i) = 0$, $a < x_1 < x_2 < \cdots < x_k < b$, and $k \ge n$, then g is identically zero.

Theorem 4. *Suppose $g_i \in \mathrm{C}^1[a,b]$ for all $i = 1, \dots, n$ and $G = \langle g_1, \dots, g_n \rangle$ is a Chebyshev subspace of $\mathrm{C}[a,b]$. If G satisfies Assumption 3, then $\{g_0 = 1, g_1, \dots, g_n\}$ is a Chebyshev system on $[a,b]$.*

Proof. If there is a function $\overline{h} = \overline{\alpha}_0 + \overline{\alpha}_1 g_1 + \cdots + \overline{\alpha}_n g_n$ such that $\overline{h}(t_i) = 0$ at some set of points $a \le t_0 < t_1 < \cdots < t_k \le b$; then by Rolle's theorem there exits a set of points $\{x_j\}_{j=1}^k$, $x_j \in (t_{j-1}, t_j)$, such that $(\overline{h})'(x_j) = 0 = (\overline{g})'(x_j)$, $j = 1, \dots, k$, where $\overline{g} = \overline{\alpha}_1 g_1 + \cdots + \overline{\alpha}_n g_n \in G$. So if $k \ge n$ then $\overline{g} = 0$, and hence $\overline{h} = \overline{\alpha}_0 \Rightarrow \overline{h} = \overline{\alpha}_0 = 0$ and this shows that $\{g_0 = 1, g_1, \dots, g_n\}$ is a Chebyshev system on $[a,b]$. □

When $n = 1$, $u \in \mathrm{C}[a,b]$, and $G = \langle u \rangle$ is a Chebyshev subspace of $\mathrm{C}[a,b]$ if $u(x) \neq 0$ for all $x \in [a,b]$. For this special case we have the following results which can be found in [3].

Proposition 5. *Let $u \in \mathrm{C}[a,b]$, and then $H = \langle 1, u \rangle$ is a Chebyshev subspace of $\mathrm{C}[a,b]$ if and only if u is strictly monotonic function on $[a,b]$.*

Remark 6. If u is an even function on $[-a, a]$, $a > 0$, then $\{1, u\}$ is not a Chebyshev system on $[-a, a]$, that is, since $\mathrm{D}\begin{pmatrix} 1,u \\ -t,t \end{pmatrix} = 0$, $\forall t \in (0, a]$.

Remark 7. Assumption 3 is not a necessary condition for Theorem 4 as the following example illustrates.

Example 8. Take $u = e^{x^3}$ and then $\{1, u\}$ is a Chebyshev system on $[-a, a]$, $a > 0$ although $u'(0) = 0$.

Finally, we will give an example of a set of continuously differentiable functions $\{g_i\}_{i=1}^{n}$ which is an extended Chebyshev system on $[a, b]$ and $\{g_0 = 1, g_1, \ldots, g_n\}$ is a Chebyshev system on $[a, b]$ but not an extended Chebyshev system.

Example 9. Let $g_1 = x$ and $g_2 = -\cos x$, then $\mathrm{D}^*\begin{pmatrix} g_1,g_2 \\ t,t \end{pmatrix} = t\sin t + \cos t > 0$, $\forall t \in [0, \pi/2]$, and hence $G = \langle g_1, g_2 \rangle$ is an extended Chebyshev system on $[0, \pi/2]$. And if $g \in G$ then g' can have at most one zero on $[0, \pi/2]$; this means that G satisfies Assumption 3 and by Theorem 4 $\{1, g_1, g_2\}$ is a Chebyshev system on $[0, \pi/2]$. Taking $v(x) = -\pi/2 + x + \cos x$, then $v(\pi/2) = v'(\pi/2) = v''(\pi/2) = 0$; that is, $\{1, g_1, g_2\}$ is not an extended Chebyshev system on $[0, \pi/2]$.

Competing Interests

The author declares no competing interests.

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