Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 5161682, 18 pages
http://dx.doi.org/10.1155/2016/5161682

Research Article

Convergence Theorems for Bregman K-Mappings and Mixed Equilibrium Problems in Reflexive Banach Spaces

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Received 19 May 2016; Accepted 25 July 2016

Academic Editor: Adrian Petrusel

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We introduce a new mixed equilibrium problem with a relaxed monotone mapping in a reflexive Banach space and prove the existence of solution of the equilibrium problem. Using Bregman distance, we introduce the concept of Bregman K-mapping for a finite family of Bregman quasiasymptotically nonexpansive mappings and show the fixed point set of the Bregman K-mapping is the set of common fixed points of $\{T_i\}_{i=1}^N$. Using the Bregman K-mapping, we introduce an iterative sequence for finding a common point in the set of a common fixed points of the finite family of Bregman quasiasymptotically nonexpansive mappings and the set of solutions of some mixed equilibrium problems. Strong convergence of the iterative sequence is proved. Our results generalise and improve many recent results in the literature.

1. Introduction

Let E be a real Banach space and let E^* be the dual of E. Let C be a nonempty closed and convex subset of E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ $\forall x, y \in C$. A point $x \in C$ is a fixed point of T if $Tx = x$. We denote by $F(T)$ the fixed point set of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let $g : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem with respect to g and C in the sense of Blum and Oettli [1] is to find $z \in C$ such that

$$g(z, y) \ge 0 \quad \forall y \in C. \tag{1}$$

The set of solutions of equilibrium problem is denoted by $\mathrm{EP}(g)$; that is,

$$\mathrm{EP}(g) = \{z \in C : g(z, y) \ge 0 \ \forall y \in C\}. \tag{2}$$

In order to solve equilibrium problem (1), the bifunction g is usually assumed to satisfy the following conditions:

(C1) $g(x, x) = 0$ for all $x, y \in C$;

(C2) g is monotone; that is, $g(x, y) + g(y, x) \le 0$ for all $x, y \in C$;

(C3) for all $x, y, z \in C$, $\limsup_{t\to 0} g(tz + (1 - t)x, y) \le g(x, y)$;

(C4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

Fang and Huang [2] introduced the concept of relaxed $\eta - \alpha$ monotone mapping for solving mixed equilibrium problems.

A mapping $A : C \to E^*$ is said to be relaxed $\eta - \alpha$ monotone (see also [3]) if there exist a mapping $\eta : C \times C \to E$ and a function $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in E$, where $p > 1$ such that

$$\langle Ax - Ay, \eta(x, y)\rangle \ge \alpha(x - y) \quad \forall x, y \in C. \tag{3}$$

Particularly if $\eta(x, y) = x - y$ $\forall x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 1$ are two constants, then A is called p monotone; see [4, 5].

Fang and Huang [2] proved that under some suitable assumptions, the following variational inequality is solvable: find $z \in C$ such that

$$\langle Az, \eta(y, z)\rangle + \psi(y) - \psi(z) \ge 0 \quad \forall y \in C, \tag{4}$$

where ψ is a function from C to $\mathbb{R}\cup\{\infty\}$. They also proved that the following inequality is equivalent to variational inequality (4): find $z \in C$ such that

$$\langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) \geq \alpha(y-z) \quad \forall y \in C. \quad (5)$$

The mixed equilibrium problem (see [6, 7]) is to find $z \in C$ such that

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) \geq 0 \quad \forall y \in C. \quad (6)$$

We denote the set of solutions of mixed equilibrium problem (6) by $\mathrm{EP}(g, A)$. It is easily seen that if $g(z, y) = 0$ $\forall z, y \in C$, then mixed equilibrium problem (6) reduces to variational inequality (4). In the case of $A = 0$ and $\psi = 0$, then, $\mathrm{EP}(g, A)$ coincides with $\mathrm{EP}(g)$.

Equilibrium problems and mixed equilibrium problems have been used as tools for solving problems arising from linear and nonlinear programming, optimization problems, variational inequalities, fixed point problems, and also problems in physics, economics, engineering, and so forth (see, e.g., [1, 6, 8–15] and the references therein).

Let $J : E \to 2^{E^*}$ be the normalised duality mapping defined by

$$J(x) = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\right\}, \quad (7)$$

where $\langle\cdot,\cdot\rangle$ denotes the generalised duality pairing. It is well known that if E is smooth, strictly convex, and reflexive, then J is single-valued, one-to-one, and onto.

Let $f : E \to (-\infty, +\infty]$ be a convex function. We denote by $\operatorname{dom} f$ the domain of f; that is, $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$. The function f is said to be coercive if $\lim_{\|x\|\to\infty} f(x) = +\infty$. f is said to be strongly coercive if $\lim_{\|x\|\to\infty}(f(x)/\|x\|) = +\infty$. The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \quad (8)$$

The subdifferential of f is a mapping $\partial f : E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle x^*, y - x\rangle \ \ \forall y \in E\} \quad \forall x \in E. \quad (9)$$

It is well known that (see [16]) $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$ for all $x \in E$. It is also known that if $f : E \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, convex, and weak* lower semicontinuous function; see, for example, [17].

For any convex function $f : E \to (-\infty, +\infty]$, let $x \in \operatorname{int}\operatorname{dom} f$ and $y \in E$. The right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x,y) = \lim_{t\to 0}\frac{f(x+ty) - f(x)}{t}. \quad (10)$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0}((f(x+ty) - f(x))/t)$ exists for any $y \in E$. In this case $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \operatorname{int}\operatorname{dom} f$. If the limit in (10) is attained uniformly in $\|y\| = 1$, then f is said to be Fréchet differentiable at x. f is said to be uniformly Fréchet differentiable on a subset C of E if the limit in (10) is attained uniformly for every $x \in C$ and $\|y\| = 1$. We know that if f is uniformly Fréchet differentiable on bounded subset of E, then f is uniformly continuous on bounded set of E (see, e.g., [18]).

The function f is said to be essentially smooth if ∂f is both bounded and single-valued on its domain. It is called essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom}\partial f$. f is said to be a Legendre function if it is both essentially smooth and essentially strictly convex. When the subdifferential of f is single-valued, it coincides with the gradient; that is, $\partial f = \nabla f$; see, for example, [19].

For a Legendre function f, the following properties are well known:

(i) f is essentially smooth if and only if f^* is essentially strictly convex; see [16];

(ii) $(\partial f)^{-1} = \partial f^*$; see [20];

(iii) f is Legendre if and only if f^* is Legendre function; see [16];

(iv) if f is Legendre function, then ∇f is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$, and $\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$; see [16].

If E is smooth and strictly convex, the function $f(x) = (1/p)\|x\|^p$, $1 < p < \infty$, is Legendre function; see, for example, [21]. In this case $\nabla f = J_p$, $1 < p < \infty$. In particular if $E = H$ is a Hilbert space we have $\nabla f = I$, the identity mapping.

Let $f : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. The function $D_f : E \times E \to \mathbb{R}$ defined by

$$D_f(x,y) = f(x) - f(y) - \langle \nabla f(y), x - y\rangle \quad \forall x, y \in E \quad (11)$$

is called Bregman distance corresponding to f; see [22, 23]. It follows from the strict convexity of f that $D_f(x, y) \geq 0$ $\forall x, y \in E$ and $D_f(x, y) = 0$ if and only if $x = y$; see [24].

Bregman projection with respect to f of $x \in E$ onto the nonempty closed convex subset C of E is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\left(P_C^f(x), x\right) = \inf\left\{D_f(y,x) : y \in C\right\}. \quad (12)$$

Remark 1. If E is smooth and strictly convex Banach space and $f(x) = \|x\|^2$ $\forall x \in E$, then we have $\nabla f(x) = 2J(x)$ $\forall x \in E$ and hence $D_f(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 = \phi(x, y)$ $\forall x, y \in E$ which is the Lyapunov function introduced by Alber [25] and the Bregman projection $P_C^f(x)$ reduces to the generalised projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x), x) = \inf\{\phi(y,x) : y \in C\}. \quad (13)$$

If $E = H$, a Hilbert space, then the Bregman projection $P_C^f(x)$ reduces to the metric projection $P_C(x)$ of H onto C.

Observing (11), we have

$$D_f(z,x) - D_f(z,y) = D_f(y,x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle \quad \forall x, y, z \in E, \tag{14}$$

which is called the three-point identity.

Let $f : E \to (-\infty, +\infty]$ be a convex, Legendre, and Gâteaux differentiable function. Following [23, 25] we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*) \quad \forall x \in E,\ x^* \in E^*. \tag{15}$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ $\forall x \in E$ and $x^* \in E^*$. Also from definition (15), it is obvious that $D_f(x, y) = V_f(x, \nabla f(y))$ and V_f is convex in the second variable. Therefore for $t \in (0, 1)$ and $x, y \in E$, we have

$$D_f(z, \nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))) \le tD_f(z,x) + (1-t)D_f(z,y). \tag{16}$$

Moreover by subdifferential inequality [26], we have

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \le V_f(x, x^* + y^*) \quad \forall x \in E,\ x^* \in E^*. \tag{17}$$

Recall that a mapping $T : C \to C$ is said to be ϕ-quasi nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ $\forall x \in C$ and $p \in F(T)$. T is ϕ-quasiasymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{v_n\} \subset [0, \infty)$ such that $v_n \to 0$ as $n \to \infty$ and $\phi(p, T^n x) \le (1+v_n)\phi(p, x)$ $\forall x \in C$ and $p \in F(T)$. T is called Bregman quasi nonexpansive if $F(T) \neq \emptyset$ and $D_f(p, Tx) \le D_f(p, x)$ $\forall x \in C$ and $p \in F(T)$. T is Bregman quasiasymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{v_n\} \subset [0, \infty)$ such that $v_n \to 0$ as $n \to \infty$ and $D_f(p, T^n x) \le (1 + v_n)D_f(p, x)$ $\forall x \in C$ and $p \in F(T)$. T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $Tx = y$.

It is worth mentioning that several iterative schemes have been constructed and proposed for finding points which solve fixed point problems and mixed equilibrium problems with relaxed monotone mappings in various settings. In 2010 Wang et al. [7] introduced the following iterative scheme for finding a common element of the set of solutions of a mixed equilibrium problem with relaxed monotone mapping and the set of fixed points of nonexpansive mappings in Hilbert spaces:

$$\begin{aligned}
&x_1 \in C \quad \text{chosen arbitrarily},\\
&\Phi(u_n, y) + \langle Tu_n, \eta(y, u_n) \rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n \rangle\\
&\quad \ge 0 \quad \forall y \in C,\\
&y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n)u_n,\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\},\\
&D_n = \bigcap_{j=1}^{n} C_j,\\
&x_{n+1} = P_{D_n} x_1, \quad n \ge 1,
\end{aligned} \tag{18}$$

where T is a relaxed $\eta - \alpha$ monotone mapping and $S : C \to C$ is a nonexpansive mapping. Under some mild conditions on the three control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$, they obtained strong convergence of scheme (18) to common solution of mixed equilibrium problems and fixed point of nonexpansive mapping.

Recently, Chen et al. [27] introduced a new mixed equilibrium problem with the relaxed monotone mapping in uniformly convex and uniformly smooth Banach spaces and proved the existence of solutions of the mixed equilibrium problem. They also proposed the following iterative scheme to find the common element of the set of solutions of the mixed equilibrium problem and the set of fixed points of a quasi-ϕ-nonexpansive mapping:

$$\begin{aligned}
&x_1 = x \in C \quad \text{chosen arbitrarily},\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n),\\
&u_n \in C \quad \text{such that}\\
&\theta(u_n, y) + \langle Au_n, \eta(y, u_n) \rangle + f(y) - f(u_n)\\
&\quad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0 \quad \forall y \in C,\\
&C_n = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\},\\
&D_n = \bigcap_{j=1}^{n} C_j,\\
&x_{n+1} = \Pi_{D_n} x, \quad n \ge 1,
\end{aligned} \tag{19}$$

where S is a quasi-ϕ-nonexpansive mapping from C into itself and J is the normalised duality mapping. Under some assumptions on the parameter sequences $\{\alpha_n\}$ and $\{r_n\}$, they obtained strong convergence of scheme (19) to common solution of mixed equilibrium problems and fixed point of nonexpansive mapping.

Motivated and inspired by the above results, in this paper we introduce and prove the existence of solutions of the mixed equilibrium problem with relaxed monotone

mapping in reflexive Banach spaces. Using Bregman distance, we introduce the concept of Bregman K-mapping of a finite family of Bregman quasiasymptotically nonexpansive mappings and propose an iterative sequence for finding a common element of the set of fixed points of a finite family of Bregman quasiasymptotically nonexpansive mappings and the set of solutions of mixed equilibrium problem.

2. Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{int} \operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x,t) = \inf\left\{D_f(y,x) : y \in \operatorname{dom} f, \|y - x\| = t\right\}. \quad (20)$$

The function f is totally convex at x if $\nu_f(x,t) > 0$ for all $t > 0$ and f is totally convex if it is totally convex at each point $x \in \operatorname{dom} f$. Let B be a bounded subset of E. For $t > 0$, define a functional on B, $\nu_f : \operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B,t) = \inf\left\{\nu_f(x,t) : x \in B \cup \operatorname{dom} f\right\}. \quad (21)$$

f is totally convex on bounded set B if $\nu_f(B,t) > 0$ for any bounded subset D of E and $t > 0$, where $\nu_f(\cdot, t)$ is the total convexity of the function f on the set B.

Let $B_r = \{z \in E : \|z\| \le r\}$, for all $r > 0$ and $S_E = \{x \in E : \|x\| = 1\}$. The function f is bounded if $f(B_r)$ is bounded for all $r > 0$ and f is uniformly convex on bounded subsets of E [28] if the function $\rho_r : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)} \quad (22)$$

satisfies

$$\rho_r(t) > 0 \quad \forall r, t > 0, \quad (23)$$

where ρ_r is called the *gauge of uniform convexity of* f.

The *gauge of uniform smoothness of* f is the function $\sigma_r : [0, +\infty) \to [0, +\infty)$ defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha f(x + (1-\alpha) ty) + (1-\alpha) f(x - t\alpha y) - f(x)}{\alpha(1-\alpha)}. \quad (24)$$

The function f is said to be uniformly smooth on E if $\sigma_r(t) > 0$ $\forall r, t > 0$. We know that from [28, 29] f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.

Definition 2 (see [2]). Let E be a Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ and $\eta : C \times C \to E$ be two mappings. Then $A : C \to E^*$ is said to be *η-hemicontinuous* if for any fixed $x, y \in C$, the function $h : [0,1] \to \mathbb{R}$ defined by $h(t) = \langle A((1-t)x + ty), \eta(x,y)\rangle$ is continuous at 0^+.

Definition 3 (see [30]). Let E be a Banach space with the dual E^* and C be a nonempty subset of E. A mapping $H : C \to 2^E$ is called a KKM mapping if for any finite set $B \subset C$ one has $\operatorname{co} B \subset \bigcup_{x \in B} H(x)$.

Remark 4. Observe that from Definition 3, if H is a KKM mapping and $W : C \to 2^E$ such that $H(x) \subset W(x)$ for all $x \in C$, then W is a KKM mapping.

In the sequel we will need the following lemmas.

Lemma 5 (see [31]). *Let M be a nonempty subset of a Hausdorff topological vector space X and let $H : M \to 2^X$ be a KKM mapping. If $H(x)$ is closed in X for all $x \in M$ and compact for some $x \in M$, then $\bigcap_{x \in M} H(x) \neq \emptyset$.*

Lemma 6 (see [32]). *If $x \in \operatorname{dom} f$, then the following statements are equivalent:*

(i) *the function f is totally convex at x;*

(ii) *for any sequence $\{y_n\} \subset \operatorname{dom} f$*

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \quad \textit{implies} \lim_{n\to\infty} \|y_n - x\| = 0. \quad (25)$$

Recall that a function $f : E \to (-\infty, +\infty]$ is called sequentially consistent [29] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \text{implies} \lim_{n\to\infty} \|y_n - x_n\| = 0. \quad (26)$$

Lemma 7 (see [24]). *The function $f : E \to (-\infty, +\infty]$ is totally convex on bounded sets if and only if f is sequentially consistent.*

Lemma 8 (see [26]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of E^*. Let $x \in E$. If the sequence $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 9 (see [33]). *Let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then for any $x, y \in B_r$ and $\alpha \in (0,1)$,*

$$f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y) - \alpha(1-\alpha)\rho_r(\|x-y\|), \tag{27}$$

where ρ_r is the gauge of the uniform convexity of f.

Lemma 10 (see [34]). *Let $f : E \to (-\infty, +\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to strong topology of E^*.*

Lemma 11 (see [28]). *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive function. If ∇f is uniformly continuous on bounded subsets of E, then $f^* : E^* \to (-\infty, +\infty]$ is uniformly convex on bounded subsets of E^*.*

Lemma 12 (see [28]). *Let $f : E \to (-\infty, +\infty]$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(i) *f is strongly coercive and uniformly convex on bounded subsets of E;*

(ii) *f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of* $\operatorname{dom} f^* = E^*$.

Lemma 13 (see [29]). *Let C be a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$. Then*

(i) *$z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(y), y - z\rangle \le 0$ for all $y \in C$;*

(ii) *$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y, x)$ for all $y \in C$.*

Lemma 14 (see [35]). *Let E be a reflexive Banach space and C be a nonempty closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex which is bounded on bounded subsets of E. Let $T : C \to C$ be a closed and Bregman quasiasymptotically nonexpansive mapping. If $F(T) \ne \emptyset$, then it is closed and convex.*

Lemma 15. *Let E be a reflexive Banach space and C be a nonempty closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre, uniformly Fréchet differentiable, strongly coercive, and totally convex function on bounded subsets of E. Then the Bregman projection $P_C^f : E \to C$ is continuous.*

Proof. Let $\{x_n\}$ be a sequence in E such that $x_n \to x$ as $n \to \infty$. Let $x_n' = P_C^f x_n$ and $x' = P_C^f x$. By Lemma 13(ii), we have

$$D_f(y, x_n') + D_f(x_n', x_n) \le D_f(y, x_n), \quad \forall y \in C. \tag{28}$$

From inequality (28), we have

$$D_f(y, x_n') \le D_f(y, x_n), \quad \forall y \in C. \tag{29}$$

Since $\{D_f(y, x_n)\}$ converges, it is bounded and using the above inequality it follows that $D_f(y, x_n')$ is bounded. The function f is strongly coercive and totally convex which is bounded on bounded subsets of E; therefore in view of Lemma 12 ∇f^* is uniformly norm-to-norm continuous on bounded subsets of $\operatorname{dom} f^* = E^*$ and consequently ∇f^* is bounded. Hence by Lemma 8 we obtain that $\{x_n'\}$ is bounded.

Since $x' = P_C^f x$ we have $D_f(x', x) \le D_f(x_n', x)$. Therefore

$$\begin{aligned} D_f(x_n', x_n) - D_f(x', x) &\ge D_f(x_n', x_n) - D_f(x_n', x) \\ &= f(x) - f(x_n) + \langle \nabla f(x_n) - \nabla f(x), x_n - x_n'\rangle \\ &\quad + \langle \nabla f(x), x_n - x\rangle. \end{aligned} \tag{30}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, it follows that f is uniformly continuous on bounded subsets of E (see, e.g., [18]). Thus, taking lim inf as $n \to \infty$ of both sides of the above inequality, we obtain

$$\liminf_{n\to\infty} D_f(x_n', x_n) - D_f(x', x) \ge 0. \tag{31}$$

Now let $\epsilon > 0$. Using (28), we have

$$\begin{aligned} D_f(x', x_n') &\le D_f(x', x_n) - D_f(x_n', x_n) \\ &\le f(x) - f(x_n) \\ &\quad + \langle \nabla f(x) - \nabla f(x_n), x' - x\rangle \\ &\quad + \langle \nabla f(x_n), x_n - x\rangle + \epsilon \quad \forall n \ge N_0, \end{aligned} \tag{32}$$

where N_0 is some natural number. As $n \to \infty$ and $\epsilon \to 0$, we obtain

$$D_f(x', x_n') \longrightarrow 0. \tag{33}$$

By total convexity of f, we get $x_n' \to x'$ as $n \to \infty$. This completes the proof. □

3. Main Results

Lemma 16. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E with the dual E^*. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Let $A : C \to E^*$ be η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1) and (C4). Let $\psi : C \to \mathbb{R}$ be proper, convex, and lower semicontinuous. For $r > 0$ and $x \in E$, suppose the following conditions hold:*

(i) *$\eta(z, z) = 0$ $\forall z \in C$;*

(ii) *$\langle Av, \eta(\cdot, u)\rangle$ is convex for fixed $u, v \in C$.*

Then problems (34) and (35) are equivalent.

Find $z \in C$ such that

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \quad \forall y \in C. \tag{34}$$

Find $z \in C$ such that

$$g(z,y) + \langle Ay, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq \alpha(y - z) \quad \forall y \in C. \tag{35}$$

Proof. Suppose (34) holds. Let $z \in C$ be a solution of (34); then

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \quad \forall y \in C. \tag{36}$$

Since A is relaxed $\eta - \alpha$ monotone, we obtain

$$\begin{aligned} &g(z,y) + \langle Ay, \eta(y,z)\rangle + \psi(y) - \psi(z) \\ &\quad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq g(z,y) \\ &\quad + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) \\ &\quad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle + \alpha(y - z) \\ &\qquad \forall y \in C \\ &\geq \alpha(y - z) \quad \forall y \in C. \end{aligned} \tag{37}$$

This shows that $z \in C$ is a solution of (35).

Conversely, suppose (35) holds; that is, $z \in C$ is a solution of (35). Let $y \in C$ such that $\psi(y) < \infty$; then $\psi(z) < \infty$. Let $y_t = (1-t)z + ty$, $t \in (0,1)$. Since $z \in C$ is a solution of (35), we have

$$\begin{aligned} &g(z,y_t) + \langle Ay_t, \eta(y_t,z)\rangle + \psi(y_t) - \psi(z) \\ &\quad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y_t - z\rangle \geq \alpha(y_t - z) \\ &\quad = \alpha(t(y - z)) \geq t^p \alpha(y - z). \end{aligned} \tag{38}$$

Hence by (C1), (C4), (i), and (ii), we obtain

$$\begin{aligned} &t^p \alpha(y - z) \\ &\quad \leq g(z,(1-t)z + ty) \\ &\qquad + \langle Ay_t, \eta((1-t)z + ty), z\rangle \\ &\qquad + \psi((1-t)z + ty) - \psi(z) \\ &\qquad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), ((1-t)z + ty) - z\rangle \\ &\quad \leq (1-t)g(z,z) + tg(z,y) \\ &\qquad + (1-t)\langle Ay_t, \eta(z,z)\rangle + t\langle Ay_t, \eta(y,z)\rangle \\ &\qquad + t\psi(y) - t\psi(z) \\ &\qquad + t\left(\frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle\right). \end{aligned} \tag{39}$$

Thus,

$$\begin{aligned} t^{p-1}\alpha(y - z) &\leq g(z,y) + \langle Ay_t, \eta(y,z)\rangle + \psi(y) \\ &\quad - \psi(z) \\ &\quad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \\ &= g(z,y) \\ &\quad + \langle A((1-t)z + ty), \eta(y,z)\rangle \\ &\quad + \psi(y) - \psi(z) \\ &\quad + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle. \end{aligned} \tag{40}$$

Since A is η-hemicontinuous and $p > 1$, by allowing $t \to 0^+$ we obtain

$$0 \leq g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle. \tag{41}$$

This shows that $z \in C$ is a solution of (34) □

Lemma 17. *Let E be a reflexive Banach space with the dual E^* and let C be a nonempty closed, convex, and bounded subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function and uniformly Fréchet differentiable on bounded subsets of E. Let $A : C \to E^*$ be η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1) and (C4). Let $\psi : C \to \mathbb{R}$ be proper, convex lower semicontinuous. For $r > 0$ and $x \in E$, suppose*

(i) *$\eta(x,x) = 0$ for all $x \in C$,*

(ii) *$\eta(z,y) + \eta(y,z) = 0$ $\forall z, y \in C$,*

(iii) *$\langle Au, \eta(\cdot, v)\rangle$ is convex and lower semicontinuous for fixed $u, v \in C$,*

(iv) *$\alpha : E \to \mathbb{R}$ is weakly lower semicontinuous.*

Then there exists $z \in C$ such that

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \quad \forall y \in C. \tag{42}$$

Proof. Define two set-valued mappings $H, W : C \to 2^E$ as follows:

$$\begin{aligned} H(y) &= \left\{ z \in C : g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \right\} \quad \forall y \in C, \\ W(y) &= \left\{ z \in C : g(z,y) + \langle Ay, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq \alpha(y - z) \right\} \quad \forall y \in C. \end{aligned} \tag{43}$$

We claim H is a KKM mapping. By contradiction suppose then there do not exist $\{y_1, y_2, \ldots, y_n\} \subset C$ and $t_i > 0$ ($i = 1, 2, \ldots, n$) such that $\sum_{i=1}^n t_i = 1$ and $y = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n H(y_i)$. This implies

$$g(y, y_i) + \langle Ay, \eta(y_i, y)\rangle + \psi(y_i) - \psi(y) + \frac{1}{r}\langle \nabla f(y) - \nabla f(x), y - y_i\rangle < 0 \quad \forall i = 1, 2, \ldots, n. \tag{44}$$

It follows that

$$\begin{aligned} 0 &= g(y,y) + \langle Ay, \eta(y,y)\rangle = g\left(y, \sum_{i=1}^n t_i y_i\right) + \left\langle Ay, \eta\left(\sum_{i=1}^n t_i y_i, y\right)\right\rangle \leq \sum_{i=1}^n t_i g(y, y_i) \\ &+ \sum_{i=1}^n t_i \langle Ay, \eta(y_i, y)\rangle < \sum_{i=1}^n t_i \left(\psi(y) - \psi(y_i) + \frac{1}{r}\langle \nabla f(y) - \nabla f(x), y_i - y\rangle\right) = \psi(y) \\ &- \sum_{i=1}^n t_i \psi(y_i) \leq \psi(y) - \psi\left(\sum_{i=1}^n t_i y_i\right) = 0, \end{aligned} \tag{45}$$

which is a contradiction. Thus H is a KKM mapping.

Next we show that $\forall y \in C$ $H(y) \subset W(y)$.

Let $z \in H(y)$. Then

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0. \tag{46}$$

Since A is relaxed $\eta - \alpha$ monotone, we have

$$\begin{aligned} g(z,y) &+ \langle Ay, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq g(z,y) \\ &+ \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle + \alpha(y - z) \\ &\geq \alpha(y - z). \end{aligned} \tag{47}$$

Showing that $z \in W(y)$ for all $y \in C$. By Remark 4 it follows that W is a KKM mapping.

We claim also that $W(y)$ is closed in the weak topology of E. Let $y \in C$ and $\overline{z}$ be the weak closure point of $W(y)$. Since E is reflexive, there exists a sequence $\{z_n\} \subset W(y)$ such that $z_n \rightharpoonup \overline{z} \in C$ as $n \to \infty$. Observe that

$$g(z_n, y) + \langle Ay, \eta(y, z_n)\rangle + \psi(y) - \psi(z_n) + \frac{1}{r}\langle \nabla f(z_n) - \nabla f(x), y - z_n\rangle \geq \alpha(y - z_n) \tag{48}$$

is equivalent to

$$\begin{aligned} &\frac{1}{r}\langle \nabla f(z_n) - \nabla f(x), y - z_n\rangle \\ &\quad \geq g(y, z_n) + \langle Ay, \eta(z_n, y)\rangle + \psi(z_n) - \psi(y) + \alpha(y - z_n). \end{aligned} \tag{49}$$

By (iii) and (iv) and taking lim inf as $n \to \infty$ of both sides of (49) we obtain

$$\begin{aligned} &\frac{1}{r}\langle \nabla f(\overline{z}) - \nabla f(x), y - \overline{z}\rangle \\ &\quad \geq g(y, \overline{z}) + \langle Ay, \eta(\overline{z}, y)\rangle + \psi(\overline{z}) - \psi(y) + \alpha(y - \overline{z}). \end{aligned} \tag{50}$$

That is, $\overline{z} \in W(y)$ $\forall y \in C$. This implies $W(y)$ is weakly closed for all $y \in C$. Since C is weakly compact, then $W(y)$ is weakly compact in C for all $y \in C$.

It is clear that the solution sets of problem (34) and (35) are $\bigcap_{y \in C} H(y)$ and $\bigcap_{y \in C} W(y)$. Using Lemmas 16 and 5 we obtain

$$\bigcap_{y \in C} H(y) = \bigcap_{y \in C} W(y) \neq \emptyset. \tag{51}$$

Hence there exists $z \in C$ such that

$$g(z,y) + \langle Az, \eta(y,z)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \quad \forall y \in C. \tag{52}$$

This completes the proof. □

Lemma 18. *Let E be a reflexive Banach space with the dual E^* and let C be a nonempty closed, convex, and bounded subset of E. Let $f : E \to (-\infty, +\infty]$ be Legendre and Gâteaux differentiable function. Let $A : C \to E^*$ be η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1), (C2), and (C4). Let $\psi : C \to \mathbb{R}$ be proper, convex, and lower semicontinuous. For $r > 0$ and $x \in E$, define a map $T_r : E \to 2^C$ by*

$$T_r(x) = \Big\{z \in C : g(z, y) + \langle Ay, \eta(yz)\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \forall y \in C\Big\}. \tag{53}$$

Assume that

(i) $\eta(z, y) + \eta(y, z) = 0 \ \forall z, y \in C$;

(ii) $\langle Au, \eta(\cdot, v)\rangle$ *is convex and lower semicontinuous for fixed* $u, v \in C$;

(iii) $\alpha : E \to \mathbb{R}$ *is weakly lower semicontinuous;*

(iv) $\alpha(x - y) + \alpha(y - z) \geq 0 \ \forall x, y \in C$.

Then

(1) T_r *is single-valued;*

(2) T_r *is a Bregman firmly nonexpansive type mapping; that is,*

$$\langle \nabla f(T_r x) - \nabla f(T_r y), T_r x - T_r y\rangle \leq \langle \nabla f(x) - \nabla f(y), T_r x - T_r y\rangle \quad \forall x, y \in C; \tag{54}$$

(3) $F(T_r) = EP(g, A)$;

(4) T_r *is a Bregman quasi nonexpansive satisfying*

$$D_f(u, T_r x) + D_f(T_r x, x) \leq D_f(u, x); \tag{55}$$

(5) $EP(g, A)$ *is closed and convex.*

Proof. First we show that T_r is single-valued. Let $z_1, z_2 \in T_r x$; then

$$\begin{aligned} &g(z_1, z_2) + \langle Az_1, \eta(z_2, z_1)\rangle + \psi(z_2) - \psi(z_1) + \frac{1}{r}\langle \nabla f(z_1) - \nabla f(x), z_2 - z_1\rangle \geq 0, \\ &g(z_2, z_1) + \langle Az_2, \eta(z_1, z_2)\rangle + \psi(z_1) - \psi(z_2) + \frac{1}{r}\langle \nabla f(z_2) - \nabla f(x), z_1 - z_2\rangle \geq 0. \end{aligned} \tag{56}$$

By using (C2), adding (56) yields

$$\langle Az_1, \eta(z_2, z_1)\rangle + \langle Az_2, \eta(z_1, z_2)\rangle + \frac{1}{r}\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq 0. \tag{57}$$

By (i) we have

$$\langle Az_1 - Az_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq 0. \tag{58}$$

Since A is relaxed $\eta - \alpha$ monotone, we obtain

$$\frac{1}{r}\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq \langle Az_2 - Az_1, \eta(z_2, z_1)\rangle \geq \alpha(z_2 - z_1). \tag{59}$$

Thus

$$\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq r\alpha(z_2 - z_1). \tag{60}$$

Interchanging z_1 and z_2 in (60), we have

$$\langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2\rangle \geq r\alpha(z_1 - z_2). \tag{61}$$

Adding (60) and (61), we have

$$\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle + \langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2\rangle \geq r(\alpha(z_2 - z_1) + \alpha(z_1 - z_2)). \tag{62}$$

Hence

$$2\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq r(\alpha(z_2 - z_1) + \alpha(z_1 - z_2)). \tag{63}$$

By (iv), we have

$$\langle \nabla f(z_1) - \nabla f(z_2), z_2 - z_1\rangle \geq 0. \tag{64}$$

Thus,

$$\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle \leq 0. \tag{65}$$

Since f is convex and Gâteaux differentiable we have

$$\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle \geq 0. \tag{66}$$

By (65) and (66) we obtain

$$\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle = 0. \tag{67}$$

Since f is Legendre function, then $z_1 = z_2$.

Next we show that T_r is Bregman firmly nonexpansive type. Let $x, y \in C$; then

$$\begin{aligned} &g(T_r x, T_r y) + \langle AT_r x, \eta(T_r y, T_r x)\rangle + \psi(T_r y) - \psi(T_r x) + \frac{1}{r}\langle \nabla f(T_r x) - \nabla f(x), T_r y - T_r x\rangle \geq 0, \\ &g(T_r y, T_r x) + \langle AT_r y, \eta(T_r x, T_r y)\rangle + \psi(T_r x) - \psi(T_r y) + \frac{1}{r}\langle \nabla f(T_r y) - \nabla f(y), T_r x - T_r y\rangle \geq 0. \end{aligned} \tag{68}$$

Adding (68), using (i) and (C2), we obtain

$$\begin{aligned}&\langle AT_r x - AT_r y, \eta(T_r y, T_r x)\rangle + \frac{1}{r}\langle \nabla f(T_r x)\\&\quad - \nabla f(T_r y) - \nabla f(x) + \nabla f(y), T_r y - T_r x\rangle \geq 0,\end{aligned} \tag{69}$$

so that

$$\begin{aligned}&\frac{1}{r}\langle \nabla f(T_r x) - \nabla f(T_r y) - \nabla f(x) + \nabla f(y), T_r y\\&\quad - T_r x\rangle \geq \langle AT_r y - AT_r x, \eta(T_r y, T_r x)\rangle.\end{aligned} \tag{70}$$

Since A is relaxed $\eta - \alpha$ monotone and $r > 0$, we have

$$\begin{aligned}&\langle \nabla f(T_r x) - \nabla f(T_r y) - \nabla f(x) + \nabla f(y), T_r y - T_r x\rangle\\&\quad \geq r\alpha(T_r y - T_r x).\end{aligned} \tag{71}$$

Also interchanging the roles of x and y in (71) and applying (iv), we have

$$\begin{aligned}&2\langle \nabla f(T_r x) - \nabla f(T_r y) - \nabla f(x) + \nabla f(y), T_r y\\&\quad - T_r x\rangle \geq 0.\end{aligned} \tag{72}$$

Hence,

$$\begin{aligned}&\langle \nabla f(T_r x) - \nabla f(T_r y), T_r x - T_r y\rangle\\&\quad \leq \langle \nabla f(x) - \nabla f(y), T_r x - T_r y\rangle,\end{aligned} \tag{73}$$

showing that T_r is Bregman firmly nonexpansive type.

We now show that $F(T_r) = \mathrm{EP}(g, A)$. Indeed

$$\begin{aligned}&u \in F(T_r) \Longleftrightarrow u \in T_r u \Longleftrightarrow\\&g(u, y) + \langle Au, \eta(y, u)\rangle + \psi(y) - \psi(u)\\&\quad + \frac{1}{r}\langle \nabla f(u) - \nabla f(u), y - u\rangle \geq 0 \quad \forall y \in C \Longleftrightarrow\\&g(u, y) + \langle Au, \eta(y, u)\rangle + \psi(y) - \psi(u) \geq 0\\&\qquad\qquad \forall y \in C \Longleftrightarrow\\&u \in \mathrm{EP}(g, A).\end{aligned} \tag{74}$$

Next we prove that T_r is Bregman quasi nonexpansive mapping.

Since

$$\begin{aligned}&D_f(T_r x, T_r y) + D_f(T_r y, T_r x)\\&\quad \leq D_f(T_r x, y) + D_f(T_r y, x) - D_f(T_r x, x)\\&\quad - D_f(T_r y, y),\end{aligned} \tag{75}$$

we have

$$\begin{aligned}&D_f(T_r x, T_r y) + D_f(T_r y, T_r x)\\&\quad \leq D_f(T_r x, y) + D_f(T_r y, x).\end{aligned} \tag{76}$$

Let $u = y \in F(T_r)$; then from (76) we obtain

$$D_f(u, T_r x) \leq D_f(u, x). \tag{77}$$

This shows that T_r is Bregman quasi nonexpansive mapping which is Bregman quasiasymptotically nonexpansive. Also from (75), we have

$$\begin{aligned}&D_f(T_r x, T_r y) + D_f(T_r y, T_r x) + D_f(T_r x, x)\\&\quad + D_f(T_r y, y) \leq D_f(T_r x, y) + D_f(T_r y, x).\end{aligned} \tag{78}$$

As $u = y \in F(T_r)$, we obtain

$$D_f(u, T_r x) + D_f(T_r x, x) \leq D_f(u, x). \tag{79}$$

Lastly, using (3), (4), and Lemma 14 we obtain that $\mathrm{EP}(g, A)$ is closed and convex. □

Definition 19. Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $\{T_i\}_{i=1}^N$ be a finite family of Bregman quasiasymptotically nonexpansive mappings. For any $n \in \mathbb{N}$, define a mapping $K_n : C \to C$ as follows:

$$\begin{aligned}&S_{n,0}x = x\\&S_{n,1}x = P_C^f(\nabla f^*(\alpha_{n,1}\nabla f(T_1^n x) + (1 - \alpha_{n,1})\nabla f(x)))\\&S_{n,2}x = P_C^f(\nabla f^*(\alpha_{n,2}\nabla f(T_2^n S_{n,1}x)\\&\quad + (1 - \alpha_{n,2})\nabla f(S_{n,1}x)))\\&S_{n,3}x = P_C^f(\nabla f^*(\alpha_{n,3}\nabla f(T_3^n S_{n,2}x)\\&\quad + (1 - \alpha_{n,3})\nabla f(S_{n,2}x)))\\&\qquad\vdots\\&S_{n,N-1}x = P_C^f(\nabla f^*(\alpha_{n,N-1}\nabla f(T_{N-1}^n S_{n,N-2}x)\\&\quad + (1 - \alpha_{n,N-1})\nabla f(S_{n,N-2}x)))\\&K_n x = S_{n,N}x = P_C^f(\nabla f^*(\alpha_{n,N}\nabla f(T_N^n S_{n,N-1}x)\\&\quad + (1 - \alpha_{n,N})\nabla f(S_{n,N-1}x))).\end{aligned} \tag{80}$$

Such a mapping K_n is called the Bregman K-mapping generated by $T_1, T_2, T_3, \ldots, T_N$ and $\alpha_{n,i} \in (0, 1)$, $i = 1, 2, 3, \ldots, N$.

Using the above definition, we have the following Lemma.

Lemma 20. *Let E be a reflexive Banach space with the dual E^* and let C be a nonempty, closed, convex, and bounded subset of E. Let $f : E \to (-\infty, +\infty]$ be strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of E. Let $\{T_i\}_{i=1}^N$ be a finite*

family of continuous Bregman quasiasymptotically nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_{n,i}\}$ be a real sequence in $(0,1)$ such that $\liminf_{n\to\infty}\alpha_{n,i} > 0$ $\forall i \in \{1,2,3,\ldots,N\}$. Let K_n be Bregman K-mapping generated by $T_1, T_2, T_3, \ldots, T_N$ in (80). Then

(i) *$D_f(x^*, K_n x) \leq (1+t_n) D_f(x^*, x)$ for all $x^* \in F(K_n)$ and $x \in C$, where $t_n \to 0$ as $n \to \infty$;*

(ii) *$F(K_n) = \bigcap_{i=1}^{N} F(T_i)$;*

(iii) *K_n is a closed mapping.*

Proof. Let $x^* \in F(K_n)$ and $x \in C$. Then by Lemma 13(ii) and inequality (16) we have

$$\begin{aligned}
&D_f(x^*, K_n x)\\
&= D_f\left(x^*, P_C^f\left(\nabla f^*\left(\alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}x\right)\right.\right.\right.\\
&\quad \left.\left.\left. + (1-\alpha_{n,N})\nabla f\left(S_{n,N-1}x\right)\right)\right)\right)\\
&\leq D_f\left(x^*, \nabla f^*\left(\alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}x\right) + (1-\alpha_{n,N})\nabla f\right.\right.\\
&\quad \left.\left. \cdot \left(S_{n,N-1}x\right)\right)\right)\\
&\leq \alpha_{n,N} D_f\left(x^*, T_N^n S_{n,N-1}x\right) + (1-\alpha_{n,N}) D_f\left(x^*,\right.\\
&\quad \left. S_{n,N-1}x\right)\\
&\leq \alpha_{n,N}(1+\nu_{n,N}) D_f\left(x^*, S_{n,N-1}x\right) + (1-\alpha_{n,N})\\
&\quad \cdot D_f\left(x^*, S_{n,N-1}x\right)\\
&= (1+\alpha_{n,N}\nu_{n,N}) D_f\left(x^*, S_{n,N-1}x\right)\\
&= (1+\alpha_{n,N}\nu_{n,N}) D_f\left(x^*, \nabla f^*\left(\alpha_{n,N-1}\nabla f\right.\right.\\
&\quad \left.\left. \cdot \left(T_{N-1}^n S_{n,N-2}x\right) + (1-\alpha_{n,N-1})\nabla f\left(S_{n,N-2}x\right)\right)\right)\\
&\leq (1+\alpha_{n,N}\nu_{n,N})\left[\alpha_{n,N-1} D_f\left(x^*, T_{N-1}^n S_{n,N-2}x\right) + (1\right.\\
&\quad \left. - \alpha_{n,N-1}) D_f\left(x^*, S_{n,N-2}x\right)\right]\\
&\leq (1+\alpha_{n,N}\nu_{n,N})\left[\alpha_{n,N-1}(1+\nu_{n,N-1}) D_f\left(x^*, S_{n,N-2}x\right)\right.\\
&\quad \left. + (1-\alpha_{n,N-1}) D_f\left(x^*, S_{n,N-2}x\right)\right]\\
&= (1+\alpha_{n,N}\nu_{n,N})(1+\alpha_{n,N-1}\nu_{n,N-1}) D_f\left(x^*, S_{n,N-2}x\right)\\
&\leq (1+\alpha_{n,N}\nu_{n,N})(1+\alpha_{n,N-1}\nu_{n,N-1})(1+\alpha_{n,N-2}\nu_{n,N-2})\\
&\quad \cdot D_f\left(x^*, S_{n,N-3}x\right)\\
&\qquad \vdots\\
&\leq (1+\alpha_{n,N}\nu_{n,N})(1+\alpha_{n,N-1}\nu_{n,N-1})\cdots(1+\alpha_{n,2}\nu_{n,2})\\
&\quad \cdot D_f\left(x^*, S_{n,1}x\right).
\end{aligned} \tag{81}$$

Hence,

$$\begin{aligned}
&D_f(x^*, K_n x) \leq (1+\alpha_{n,N}\nu_{n,N})\\
&\quad \cdot (1+\alpha_{n,N-1}\nu_{n,N-1})\cdots(1+\alpha_{n,2}\nu_{n,2})\\
&\quad \cdot D_f\left(x^*, \nabla f^*\left(\alpha_{n,1}\nabla f\left(T_1^n x\right) + (1-\alpha_{n,1})\nabla f(x)\right)\right)\\
&\quad \leq (1+\alpha_{n,N}\nu_{n,N})\\
&\quad \cdot (1+\alpha_{n,N-1}\nu_{n,N-1})\cdots(1+\alpha_{n,1}\nu_{n,1}) D_f(x^*, x)\\
&\quad = \prod_{i=1}^{N}(1+\alpha_{n,i}\nu_{n,i}) D_f(x^*, x).
\end{aligned} \tag{82}$$

Observe that $\alpha_{n,i}\nu_{n,i} \to 0$ as $n \to \infty$ for all $i \in \{1,2,3,\ldots,N\}$. Let $\prod_{i=1}^{N}(1+\alpha_{n,i}\nu_{n,i}) = (1+t_n)$; then $t_n \to 0$ as $n \to \infty$ and

$$D_f(x^*, K_n x) \leq (1+t_n) D_f(x^*, x). \tag{83}$$

Next we show that $F(K_n) = \bigcap_{i=1}^{N} F(T_i)$. It is obvious that $\bigcap_{i=1}^{N} F(T_i) \subset F(K_n)$. Now let $z \in F(K_n)$ and x^* be any point in $\bigcap_{i=1}^{N} F(T_i)$.

Let $r_1 = \sup\{\|\nabla f(z)\|, \|\nabla f(S_{n,i}z)\|, \|\nabla f(T_i^n S_{n,i}z)\|\}$ for $1 \leq i \leq N$. Then

$$\begin{aligned}
&D_f(x^*, z) = D_f(x^*, K_n z) = D_f\left(x^*,\right.\\
&\quad \left. P_C^f\left(\nabla f^*\left(\alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}z\right)\right.\right.\right.\\
&\quad \left.\left.\left. + (1-\alpha_{n,N})\nabla f\left(S_{n,N-1}z\right)\right)\right)\right) \leq D_f\left(x^*,\right.\\
&\quad \left.\nabla f^*\left(\alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}z\right) + (1-\alpha_{n,N})\nabla f\right.\right.\\
&\quad \left.\left. \cdot \left(S_{n,N-1}z\right)\right)\right) = V_f\left(x^*, \alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}z\right)\right.\\
&\quad \left. + (1-\alpha_{n,N})\nabla f\left(S_{n,N-1}z\right)\right) = f(x^*) - \left\langle x^*, \alpha_{n,N}\nabla f\right.\\
&\quad \left. \cdot \left(T_N^n S_{n,N-1}z\right) + (1-\alpha_{n,N})\nabla f\left(S_{n,N-1}z\right)\right\rangle\\
&\quad + f^*\left(\alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}z\right) + (1-\alpha_{n,N})\nabla f\right.\\
&\quad \left. \cdot \left(S_{n,N-1}z\right)\right).
\end{aligned} \tag{84}$$

Since f is uniformly Fréchet differentiable function which is bounded on bounded subsets of E, then by Lemma 10 ∇f is uniformly continuous on bounded subsets and consequently from Lemma 11 f^* is uniformly convex. Therefore in view of Lemma 9 we have

$$\begin{aligned}
&D_f(x^*, z) \leq f(x^*) - \left\langle x^*, \alpha_{n,N}\nabla f\left(T_N^n S_{n,N-1}z\right)\right.\\
&\quad \left. + (1-\alpha_{n,N})\nabla f\left(S_{n,N-1}z\right)\right\rangle\\
&\quad + \alpha_{n,N} f^*\left(\nabla f\left(T_N^n S_{n,N-1}z\right)\right) + (1-\alpha_{n,N})\\
&\quad \cdot f^*\left(\nabla f\left(S_{n,N-1}z\right)\right) - \alpha_{n,N}(1-\alpha_{n,N})\\
&\quad \cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1}z\right) - \nabla f\left(S_{n,N-1}z\right)\right\|\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n,N}\left(f\left(x^*\right) - \left\langle x^*, \nabla f\left(T_N^n S_{n,N-1} z\right)\right\rangle\right. \\
&+ f^*\left(\nabla f\left(T_N^n S_{n,N-1} z\right)\right)) + \left(1-\alpha_{n,N}\right)\left(f\left(x^*\right)\right. \\
&- \left\langle x^*, \nabla f\left(S_{n,N-1} z\right)\right\rangle + f^*\left(\nabla f\left(S_{n,N-1} z\right)\right)) \\
&- \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right) \\
&= \alpha_{n,N} V_f\left(x^*, \nabla f\left(T_N^n S_{n,N-1} z\right)\right) + \left(1-\alpha_{n,N}\right) \\
&\cdot V_f\left(x^*, \nabla f\left(S_{n,N-1} z\right)\right) - \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right) \\
&= \alpha_{n,N} D_f\left(x^*, T_N^n S_{n,N-1} z\right) + \left(1-\alpha_{n,N}\right) \\
&\cdot D_f\left(x^*, S_{n,N-1} z\right) - \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right) \le \alpha_{n,N}\left(1\right. \\
&\left.+ \nu_{n,N}\right) D_f\left(x^*, S_{n,N-1} z\right) + \left(1-\alpha_{n,N}\right) \\
&\cdot D_f\left(x^*, S_{n,N-1} z\right) - \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right) = \left(1\right. \\
&\left.+ \alpha_{n,N}\nu_{n,N}\right) D_f\left(x^*, S_{n,N-1} z\right) - \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right).
\end{aligned} \tag{85}
$$

Continuing in this fashion we obtain

$$
\begin{aligned}
&D_f\left(x^*, z\right) \le \left(1+\alpha_{n,N}\nu_{n,N}\right) \\
&\cdot \left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,1}\nu_{n,1}\right) D_f\left(x^*, z\right) \\
&- \left(1+\alpha_{n,N}\nu_{n,N}\right) \\
&\cdot \left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,2}\nu_{n,2}\right)\alpha_{n,1}\left(1-\alpha_{n,1}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_1^n z\right) - \nabla f\left(z\right)\right\|\right) - \left(1+\alpha_{n,N}\nu_{n,N}\right) \\
&\cdot \left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,3}\nu_{n,3}\right)\alpha_{n,2}\left(1-\alpha_{n,2}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_2^n S_{n,1} z\right) - \nabla f\left(S_{n,1} z\right)\right\|\right) \\
&- \left(1+\alpha_{n,N}\nu_{n,N}\right) \\
&\cdot \left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,4}\nu_{n,4}\right)\alpha_{n,3}\left(1-\alpha_{n,3}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_3^n S_{n,2} z\right) - \nabla f\left(S_{n,2} z\right)\right\|\right) \\
&\qquad\vdots \\
&- \left(1+\alpha_{n,N}\nu_{n,N}\right)\alpha_{n,N-1}\left(1-\alpha_{n,N-1}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_{N-1}^n S_{n,N-2} z\right) - \nabla f\left(S_{n,N-2} z\right)\right\|\right) \\
&- \alpha_{n,N}\left(1-\alpha_{n,N}\right) \\
&\cdot \rho_{r_1}^*\left(\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\|\right).
\end{aligned} \tag{86}
$$

From (86), we have

$$
\begin{aligned}
&\left(1+\alpha_{n,N}\nu_{n,N}\right)\left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,2}\nu_{n,2}\right) \\
&\cdot \alpha_{n,1}\left(1-\alpha_{n,1}\right)\rho_{r_1}^*\left(\left\|\nabla f\left(T_1^n z\right) - \nabla f\left(z\right)\right\|\right) \\
&\le \left(1+\alpha_{n,N}\nu_{n,N}\right) \\
&\cdot \left(1+\alpha_{n,N-1}\nu_{n,N-1}\right)\cdots\left(1+\alpha_{n,1}\nu_{n,1}\right) D_f\left(x^*, z\right) \\
&- D_f\left(x^*, z\right).
\end{aligned} \tag{87}
$$

Since $\liminf \alpha_{n,1}(1-\alpha_{n,1}) > 0$, we have

$$
\lim_{n\to\infty} \rho_{r_1}^*\left(\left\|\nabla f\left(T_1^n z\right) - \nabla f\left(z\right)\right\|\right) = 0. \tag{88}
$$

By the property of $\rho_{r_1}^*$, we obtain

$$
\lim_{n\to\infty}\left(\left\|\nabla f\left(T_1^n z\right) - \nabla f\left(z\right)\right\|\right) = 0. \tag{89}
$$

In similar fashion and assuming $\liminf \alpha_{n,i}(1-\alpha_{n,i}) > 0$ $\forall i \in \{2, 3, \ldots, N\}$ we have

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\nabla f\left(T_2^n S_{n,1} z\right) - \nabla f\left(S_{n,1} z\right)\right\| \\
&\quad = \lim_{n\to\infty}\left\|\nabla f\left(T_3^n S_{n,2} z\right) - \nabla f\left(S_{n,2} z\right)\right\| \\
&\quad = \lim_{n\to\infty}\left\|\nabla f\left(T_4^n S_{n,3} z\right) - \nabla f\left(S_{n,3} z\right)\right\| \\
&\qquad\vdots \\
&\quad = \lim_{n\to\infty}\left\|\nabla f\left(T_N^n S_{n,N-1} z\right) - \nabla f\left(S_{n,N-1} z\right)\right\| = 0.
\end{aligned} \tag{90}
$$

Since by Lemma 12 ∇f^* is uniformly continuous, we obtain from (89) that $T_1^n z \to z$ as $n \to \infty$. By our assumption, T_1 is continuous and so $T_1 z = z$, that is, $z \in F(T_1)$. Also from (90), we obtain

$$
\begin{aligned}
T_2^n S_{n,1} z &= S_{n,1} z \\
T_3^n S_{n,2} z &= S_{n,2} z \\
&\vdots \\
T_N^n S_{n,N-1} z &= S_{n,N-1} z.
\end{aligned} \tag{91}
$$

On the other hand

$$
\begin{aligned}
&D_f\left(z, S_{n,1} z\right) \\
&\quad \le D_f\left(z, \nabla f^*\left(\alpha_{n,1}\nabla f\left(T_1^n x\right) + \left(1-\alpha_{n,1}\right)\nabla f\left(x\right)\right)\right) \\
&\quad \le \alpha_{n,1} D_f\left(z, T_1^n z\right) + \left(1-\alpha_{n,1}\right) D_f\left(z, z\right).
\end{aligned} \tag{92}
$$

Since $T_1^n z = z$ we get

$$D_f(z, S_{n,1}z) = 0 \quad \text{and so } S_{n,1}z = z. \tag{93}$$

From (91) and (93), we have $z \in F(S_{n,1})$ and $z \in F(T_2)$. Applying the same argument we can conclude that $z \in F(T_i)$ for $i = 3, 4, \ldots, N$. Thus, it follows that $z \in \bigcap_{i=1}^N F(T_i)$.

Next we show that K_n is closed.

From (80)

$$S_{n,1}x = P_C^f\left(\nabla f^*\left(\alpha_{n,1}\nabla f\left(T_1^n x\right) + \left(1 - \alpha_{n,1}\right)\nabla f(x)\right)\right). \tag{94}$$

Let $\{x_m\}$ be a sequence in C such that $x_m \to \overline{x}$ and $S_{n,1}x_m \to y$ as $m \to \infty$. Since T_i is continuous for each $i = 1, 2, 3, \ldots, N$, $\nabla f, \nabla f^*$ are uniformly continuous and applying Lemma 15 we have

$$S_{n,1}x_m \longrightarrow S_{n,1}\overline{x} \quad \text{as } m \longrightarrow \infty. \tag{95}$$

By uniqueness of limit, we get $S_{n,1}\overline{x} = y$ showing that $S_{n,1}$ is closed. Using (95), we have that $S_{n,2}$ is closed. Continuing in this way we obtain that K_n is closed. □

Now we prove the strong convergence theorems.

Theorem 21. *Let E be a reflexive Banach space with the dual E^* and let C be a nonempty, closed, convex, and bounded subset of E. Let $f : E \to (-\infty, +\infty]$ be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of E. For each $j = 1, 2, 3, \ldots, m$, let $A_j : C \to E^*$ be η-hemicontinuous and relaxed $\eta - \alpha$ monotone mappings and $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying (C1)–(C4). Let $\psi_j : C \to \mathbb{R}$ be proper, convex, and lower semicontinuous mappings. Let $\{T_i\}_{i=1}^N$ be a finite family of continuous Bregman quasiasymptotically nonexpansive mappings of C into itself such that $\mathcal{F} = [\bigcap_{i=1}^N F(T_i)] \cap [EP(g_j, A_j)] \neq \emptyset$. Let K_n be the Bregman K-mapping generated by $T_1, T_2, T_3, \ldots, T_N$ in (80). Assume that the conditions of Lemma 18 and the following condition are satisfied:*

(v) *For all $x, y, z, w \in C$ one has*

$$\limsup_{t \to 0^+} \langle Az, \eta(x, ty + (1-t)w)\rangle \le \langle Az, \eta(x, w)\rangle. \tag{96}$$

Let $\{x_n\}$ be iteratively defined as follows:

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrarily},\\
&C_{1,j} = C = C_0\\
&y_n = \nabla f^*\left(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right),\\
&u_{n,j} \in C \quad \text{such that}\\
&g_j\left(u_{n,j}, y\right) + \left\langle A_j u_{n,j}, \eta\left(y, u_{n,j}\right)\right\rangle + \psi_j(y)\\
&\quad - \psi_j\left(u_{n,j}\right) + \frac{1}{r_n}\left\langle \nabla f\left(u_{n,j}\right) - \nabla f(y_n), y - u_{n,j}\right\rangle\\
&\quad \ge 0 \quad \forall y \in C,\\
&C_{n+1,j} = \left\{z \in C_n : D_f\left(z, u_{n,j}\right) \le D_f(z, x_n) + \theta_n\right\},\\
&C_{n+1} = \bigcap_{j=1}^m C_{n+1,j},\\
&x_{n+1} = P_{C_{n+1}}^f x_0, \quad n \ge 0,
\end{aligned} \tag{97}$$

where $\{\beta_n\}$ is a real sequence in $(0, 1)$ satisfying $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $\theta_n = (1-\beta_n)t_n \sup_{p\in\mathcal{F}} D_f(p, x_n)$. Then $\{x_n\}$ converges strongly to $u = P_{\mathcal{F}}^f x_0$.

Proof. It follows from Lemmas 20(ii) and 14 that $F(K_n)$ is closed and convex. On the other hand by Lemma 18(5) $EP(g_j, A_j)$ is closed and convex for each $j \in \{1, 2, 3, \ldots, m\}$; consequently $\mathcal{F}$ is closed and convex.

Next we prove C_n is closed and convex. The proof is by induction.

For any fixed j, $C_{1,j} = C$ is closed and convex which implies C_1 is closed and convex.

Now suppose $C_{k,j}$ is closed and convex for some $k \in \mathbb{N}$. Observe that

$$\begin{aligned}
&D_f\left(z, u_{n,j}\right) \le D_f(z, x_n) + \theta_n \Longleftrightarrow\\
&\left\langle \nabla f(x_n) - \nabla f\left(u_{n,j}\right), z\right\rangle\\
&\quad \le f\left(u_{n,j}\right) - f(x_n) + \left\langle \nabla f\left(u_{n,j}\right) - \nabla f(x_n), u_{n,j}\right\rangle\\
&\qquad + \left\langle \nabla f(x_n), x_n - u_{n,j}\right\rangle + \theta_n.
\end{aligned} \tag{98}$$

It follows that $C_{k+1,j}$ is closed and convex and so C_{k+1} is closed and convex. Hence C_n is closed and convex for all $n \ge 0$. This implies that the iterative sequence is well defined.

Next we show that $\mathcal{F} \subset C_n$ $\forall n \ge 0$. Obviously $\mathcal{F} \subset C_{1,j} = C$. Assume $\mathcal{F} \subset C_{k,j}$ for some $k \in \mathbb{N}$ and $\forall j = 1, 2, \ldots, m$. Let $x^* \in \mathcal{F}$; then $x^* \in C_{k,j}$ $\forall j = 1, 2, \ldots, m$ and so $x^* \in C_k$. Observe from scheme (97) that $u_{n,j} = T_{r_n}^j y_n$. Therefore by Lemma 18(4) we have

$$D_f\left(x^*, u_{n,j}\right) = D_f\left(x^*, T_{r_n}^j y_n\right) \le D_f\left(x^*, y_n\right). \tag{99}$$

But

$$\begin{aligned}
&D_f(x^*, y_n)\\
&\quad = D_f\left(x^*, \nabla f^*\left(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right)\right)\\
&\quad = V_f\left(x^*, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right)\\
&\quad = f(x^*) - \left\langle x^*, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right\rangle\\
&\qquad + f^*\left(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq f(x^*) - \beta_n \langle x^*, \nabla f(x_n)\rangle \\
&\quad - (1-\beta_n)\langle x^*, \nabla f(K_n x_n)\rangle + \beta_n f^*(\nabla f(x_n)) \\
&\quad + (1-\beta_n) f^*(\nabla f(K_n x_n)) \\
&= \beta_n V_f(x^*, \nabla f(x_n)) \\
&\quad + (1-\beta_n) V_f(x^*, \nabla f(K_n x_n)) \\
&= \beta_n D_f(x^*, x_n) + (1-\beta_n) D_f(x^*, K_n x_n) \\
&\leq \beta_n D_f(x^*, x_n) + (1-\beta_n)(1+t_n) D_f(x^*, x_n) \\
&= D_f(x^*, x_n) + (1-\beta_n) t_n D_f(x^*, x_n) \\
&\leq D_f(x^*, x_n) + (1-\beta_n) t_n \sup_{p\in\mathcal{F}} D_f(p, x_n) \\
&= D_f(x^*, x_n) + \theta_n.
\end{aligned} \tag{100}$$

Thus,

$$D_f(x^*, u_{n,j}) \leq D_f(x^*, x_n) + \theta_n. \tag{101}$$

This shows that $x^* \in C_{k+1,j}$ which implies $\mathcal{F} \subset C_{n,j}$ $\forall n \geq 0$, $\forall j \in \{1, 2, \ldots, m\}$. Therefore we obtain $\mathcal{F} \subset C_n$.

Next we show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Since $C_{n+1} \subset C_n$ $\forall n \geq 0$, we have $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ $\forall n \geq 0$. Therefore

$$D_f(x_n, x_0) = D_f\left(P^f_{C_n} x_0, x_0\right) \leq D_f(x_{n+1}, x_0), \tag{102}$$

showing that $\{D_f(x_n, x_0)\}$ is nondecreasing sequence of real numbers. On the other hand we have from Lemma 13(ii) that

$$\begin{aligned}
D_f(x_n, x_0) &\leq D_f(x^*, x_0) - D_f(x^*, x_n) \\
&\leq D_f(x^*, x_0).
\end{aligned} \tag{103}$$

Using (102) and (103) we have that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Now

$$\begin{aligned}
D_f(x_{n+1}, x_n) &= D_f\left(x_{n+1}, P^f_{C_n} x_0\right) \\
&\leq D_f(x_{n+1}, x_0) - D_f(x_n, x_0).
\end{aligned} \tag{104}$$

Since $\lim_{n\to\infty} D_f(x_n, x_0)$ exists, we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{105}$$

Since f is totally convex on bounded set, by Lemma 7 f is sequentially consistent and so we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{106}$$

Let $m > n$ where $n, m \in \mathbb{N}$. Using (104) we have

$$\begin{aligned}
D_f(x_m, x_n) &= D_f\left(x_m, P^f_{C_n} x_0\right) \\
&\leq D_f(x_m, x_0) - D_f(x_n, x_0).
\end{aligned} \tag{107}$$

Hence

$$D_f(x_m, x_n) \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty. \tag{108}$$

It follows that

$$\|x_m - x_n\| \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty. \tag{109}$$

This shows that $\{x_n\}$ is Cauchy sequence in E. Since E is reflexive and C is weakly closed, then there exists $u \in C$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0. \tag{110}$$

Observe that $x_{n+1} \in C_{n+1} = \bigcap_{j=1}^m C_{n+1,j}$, $\forall j \in \{1, 2, \ldots, m\}$. Hence we obtain

$$D_f\left(x_{n+1}, u_{n,j}\right) \leq D_f(x_{n+1}, x_n) + \theta_n. \tag{111}$$

By (105) and the fact that $\theta_n \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty} D_f\left(x_{n+1}, u_{n,j}\right) = 0 \quad \forall j \in \{1, 2, \ldots, m\}, \tag{112}$$

and so

$$\lim_{n\to\infty} \left\|x_{n+1} - u_{n,j}\right\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \tag{113}$$

Now

$$\left\|u_{n,j} - x_n\right\| \leq \left\|u_{n,j} - x_{n+1}\right\| + \left\|x_{n+1} - x_n\right\|. \tag{114}$$

Therefore using (106) and (113) we obtain

$$\lim_{n\to\infty} \left\|u_{n,j} - x_n\right\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \tag{115}$$

Also by (110) and (115), it follows that

$$\lim_{n\to\infty} \left\|u_{n,j} - u\right\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \tag{116}$$

Let $r_2 = \sup\{\|\nabla f(x_n)\|, \|\nabla f(K_n x_n)\|\}$. In view of Lemma 9, we have

$$\begin{aligned}
&D_f(x^*, y_n) \\
&\quad = D_f(x^*, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n))) \\
&\quad = V_f(x^*, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)) \\
&\quad = f(x^*) - \langle x^*, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\rangle \\
&\qquad + f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)) \\
&\quad \leq \beta_n V_f(x^*, \nabla f(x_n)) \\
&\qquad + (1-\beta_n) V_f(x^*, \nabla f(K_n x_n)) \\
&\qquad - \beta_n(1-\beta_n)\rho^*_{r_2}(\|\nabla f(x_n) - \nabla f(K_n x_n)\|) \\
&\quad = \beta_n D_f(x^*, x_n) + (1-\beta_n) D_f(x^*, K_n x_n) \\
&\qquad - \beta_n(1-\beta_n)\rho^*_{r_2}(\|\nabla f(x_n) - \nabla f(K_n x_n)\|).
\end{aligned} \tag{117}$$

From (99), we have

$$\begin{aligned} &D_f\left(x^*, u_{n,j}\right) \\ &\quad \le \beta_n D_f\left(x^*, x_n\right) + \left(1-\beta_n\right) D_f\left(x^*, K_n x_n\right) \\ &\qquad - \beta_n\left(1-\beta_n\right)\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right). \end{aligned} \tag{118}$$

Therefore by Lemma 20(i), we have

$$\begin{aligned} &D_f\left(x^*, u_{n,j}\right) \\ &\quad \le \beta_n D_f\left(x^*, x_n\right) + \left(1-\beta_n\right)\left(1+t_n\right) D_f\left(x^*, x_n\right) \\ &\qquad - \beta_n\left(1-\beta_n\right)\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right) \\ &\quad = D_f\left(x^*, x_n\right) + \left(1-\beta_n\right) t_n D_f\left(x^*, x_n\right) \\ &\qquad - \beta_n\left(1-\beta_n\right)\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right) \\ &\quad \le D_f\left(x^*, x_n\right) + \left(1-\beta_n\right) t_n \sup_{p\in\mathcal{F}} D_f\left(p, x_n\right) \\ &\qquad - \beta_n\left(1-\beta_n\right)\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right) \\ &\quad = D_f\left(x^*, x_n\right) + \theta_n \\ &\qquad - \beta_n\left(1-\beta_n\right)\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right). \end{aligned} \tag{119}$$

Observe that

$$\begin{aligned} &D_f\left(x^*, x_n\right) - D_f\left(x^*, u_{n,j}\right) \\ &\quad = f\left(u_{n,j}\right) - f\left(x_n\right) + \left\langle \nabla f\left(u_{n,j}\right), x^* - u_{n,j}\right\rangle \\ &\qquad - \left\langle \nabla f\left(x_n\right), x^* - x_n\right\rangle \\ &\quad = f\left(u_{n,j}\right) - f\left(x_n\right) + \left\langle \nabla f\left(u_{n,j}\right), x^* - x_n\right\rangle \\ &\qquad + \left\langle \nabla f\left(u_{n,j}\right), x_n - u_{n,j}\right\rangle - \left\langle \nabla f\left(x_n\right), x^* - x_n\right\rangle \\ &\quad = f\left(u_{n,j}\right) - f\left(x_n\right) \\ &\qquad + \left\langle \nabla f\left(u_{n,j}\right) - \nabla f\left(x_n\right), x^* - x_n\right\rangle \\ &\qquad + \left\langle \nabla f\left(u_{n,j}\right), x_n - u_{n,j}\right\rangle. \end{aligned} \tag{120}$$

Therefore

$$\begin{aligned} &\left|D_f\left(x^*, x_n\right) - D_f\left(x^*, u_{n,j}\right)\right| \\ &\quad \le \left|f\left(u_{n,j}\right) - f\left(x_n\right)\right| \\ &\qquad + \left|\left\langle \nabla f\left(u_{n,j}\right) - \nabla f\left(x_n\right), x^* - x_n\right\rangle\right| \\ &\qquad + \left|\left\langle \nabla f\left(u_{n,j}\right), x_n - u_{n,j}\right\rangle\right| \\ &\quad \le \left|f\left(u_{n,j}\right) - f\left(x_n\right)\right| \\ &\qquad + \left\|\nabla f\left(u_{n,j}\right) - \nabla f\left(x_n\right)\right\| \left\|x^* - x_n\right\| \\ &\qquad + \left\|f\left(u_{n,j}\right)\right\| \left\|x_n - u_{n,j}\right\|. \end{aligned} \tag{121}$$

From (115), we obtain

$$\begin{aligned} &\left|D_f\left(x^*, x_n\right) - D_f\left(x^*, u_{n,j}\right)\right| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty \ \forall j \in \{1, 2, \ldots, m\}. \end{aligned} \tag{122}$$

By (122) and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, it follows from (119) that

$$\lim_{n\to\infty}\rho_{r_2}^*\left(\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|\right) = 0. \tag{123}$$

From the property of $\rho_{r_2}^*$, we deduce $\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(K_n x_n)\| = 0$. By uniform continuity of ∇f^* on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\left\|x_n - K_n x_n\right\| = 0. \tag{124}$$

On the other hand

$$\left\|K_n x_n - u\right\| \le \left\|K_n x_n - x_n\right\| + \left\|x_n - u\right\|. \tag{125}$$

From (110) and (124), we have

$$\lim_{n\to\infty}\left\|K_n x_n - u\right\| = 0. \tag{126}$$

Now using (110), (126), and Lemma 20(iii), we obtain $K_n u = u$ $\forall n \in \mathbb{N}$. This implies $u \in F(K_n) = \bigcap_{i=1}^N F(T_i)$.

Next we show $u \in \mathrm{EP}(g_j, A_j)$.

From scheme (97)

$$\begin{aligned} &\left\|\nabla f\left(x_n\right) - \nabla f\left(y_n\right)\right\| \\ &\quad = \left(1-\beta_n\right)\left\|\nabla f\left(x_n\right) - \nabla f\left(K_n x_n\right)\right\|. \end{aligned} \tag{127}$$

Therefore by (124) and uniform continuity of ∇f, we have

$$\lim_{n\to\infty}\left\|\nabla f\left(x_n\right) - \nabla f\left(y_n\right)\right\| = 0. \tag{128}$$

As ∇f^* is uniformly continuous, we obtain

$$\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0. \tag{129}$$

From (115) and (129), we have

$$\lim_{n\to\infty}\left\|u_{n,j} - y_n\right\| = 0 \quad \forall j = 1, 2, \ldots, m. \tag{130}$$

Again since ∇f is uniformly continuous and $r_n > a$, we get

$$\lim_{n\to\infty}\frac{\left\|\nabla f\left(u_{n,j}\right) - \nabla f\left(y_n\right)\right\|}{r_n} = 0 \quad \forall j = 1, 2, \ldots, m. \tag{131}$$

From scheme (97)

$$\begin{aligned} &g_j\left(u_{n,j}, y\right) + \left\langle A_j u_{n,j}, \eta\left(y, u_{n,j}\right)\right\rangle + \psi_j\left(y\right) \\ &\quad - \psi_j\left(u_{n,j}\right) \\ &\quad + \frac{1}{r_n}\left\langle \nabla f\left(u_{n,j}\right) - \nabla f\left(y_n\right), y - u_{n,j}\right\rangle \ge 0 \\ &\qquad \forall y \in C. \end{aligned} \tag{132}$$

Using (C2) and Lemma 17(ii), it follows that

$$\begin{aligned}
&\frac{1}{r_n}\left\|\nabla f\left(u_{n,j}\right)-\nabla f\left(y_n\right)\right\|\left\|u_{n,j}-y\right\| \\
&\quad\geq\left\langle A_j u_{n,j},\eta\left(u_{n,j},y\right)\right\rangle+\psi_j\left(u_{n,j}\right)-\psi_j\left(y\right) \\
&\qquad-g_j\left(u_{n,j},y\right)\quad\forall y\in C \\
&\quad\geq\left\langle A_j u_{n,j},\eta\left(u_{n,j},y\right)\right\rangle+\psi_j\left(u_{n,j}\right)-\psi_j\left(y\right) \\
&\qquad+g_j\left(y,u_{n,j}\right)\quad\forall y\in C.
\end{aligned}\tag{133}$$

Using (116) and (131) and taking lim inf as $n\to\infty$ of the above inequality, we get

$$\begin{aligned}
0\geq\left\langle A_j u,\eta\left(u,y\right)\right\rangle+\psi_j\left(u\right)-\psi_j\left(y\right)+g_j\left(y,u\right) \\
\forall y\in C\ \forall j=1,2,\ldots,m.
\end{aligned}\tag{134}$$

Now for any $t\in(0,1)$ and $y\in C$, let $y_t=ty+(1-t)u$. Then $y_t\in C$ and so

$$\begin{aligned}
0\geq\left\langle A_j u,\eta\left(u,y_t\right)\right\rangle+\psi_j\left(u\right)-\psi_j\left(y_t\right)+g_j\left(y_t,u\right) \\
\forall j=1,2,\ldots,m.
\end{aligned}\tag{135}$$

Therefore by (C1), (C4), Lemma 18(i), (ii), and (135), we have

$$\begin{aligned}
0&=g_j\left(y_t,y_t\right)+\left\langle A_j u,\eta\left(y_t,y_t\right)\right\rangle+\psi_j\left(y_t\right) \\
&\quad-\psi_j\left(y_t\right)=g_j\left(y_t,ty+(1-t)u\right) \\
&\quad+\left\langle A_j u,\eta\left(ty+(1-t)u,y_t\right)\right\rangle+\psi_j\left(ty\right. \\
&\quad\left.+(1-t)u\right)-\psi_j\left(y_t\right)\leq t\left[g_j\left(y_t,y\right)\right. \\
&\quad\left.+\left\langle A_j u,\eta\left(y,y_t\right)\right\rangle+\psi_j\left(y\right)-\psi_j\left(y_t\right)\right]+(1-t) \\
&\quad\cdot\left[g_j\left(y_t,u\right)+\left\langle A_j u,\eta\left(u,y_t\right)\right\rangle+\psi_j\left(u\right)\right. \\
&\quad\left.-\psi_j\left(y_t\right)\right]\leq t\left[g_j\left(y_t,y\right)+\left\langle A_j u,\eta\left(y,y_t\right)\right\rangle\right. \\
&\quad\left.+\psi_j\left(y\right)-\psi_j\left(y_t\right)\right].
\end{aligned}\tag{136}$$

That is,

$$g_j\left(y_t,y\right)+\left\langle A_j u,\eta\left(y,y_t\right)\right\rangle+\psi_j\left(y\right)-\psi_j\left(y_t\right)\geq 0.\tag{137}$$

Since $y_t=ty+(1-t)u$, we have

$$\begin{aligned}
&g_j\left(ty+(1-t)u,y\right)+\left\langle A_j u,\eta\left(y,ty+(1-t)u\right)\right\rangle \\
&\quad+\psi_j\left(y\right)-\psi_j\left(ty+(1-t)u\right)\geq 0.
\end{aligned}\tag{138}$$

From (C3), (v), and lower semicontinuity of ψ, we have by allowing $t\to 0^+$

$$\begin{aligned}
g_j\left(u,y\right)+\left\langle A_j u,\eta\left(y,u\right)\right\rangle+\psi_j\left(y\right)-\psi_j\left(u\right)\geq 0 \\
\forall y\in C.
\end{aligned}\tag{139}$$

This shows that $u\in \mathrm{EP}(g_j,A_j)$.

Lastly we show $u=P_{\mathcal{F}}^f x_0$.

From $x_n=P_{C_n}^f x_0$ and Lemma 13(i) we have

$$\left\langle\nabla f\left(x_0\right)-\nabla f\left(x_n\right),x_n-z\right\rangle\geq 0\quad\forall z\in C_n.\tag{140}$$

Since $\mathcal{F}\subset C_n$, this implies that

$$\left\langle\nabla f\left(x_0\right)-\nabla f\left(x_n\right),x_n-h\right\rangle\geq 0\quad\forall h\in\mathcal{F}.\tag{141}$$

Letting $n\to\infty$ in (141), we obtain

$$\left\langle\nabla f\left(x_0\right)-\nabla f\left(x_n\right),x_n-h\right\rangle\geq 0\quad\forall h\in\mathcal{F}.\tag{142}$$

Again by Lemma 13(i) we have $u=P_{\mathcal{F}}^f x_0$. This completes the proof. □

If $N=1$ and $m=1$, in Theorem 21 then we have the following corollary.

Corollary 22. *Let E be a reflexive Banach space with the dual E^* and let C be a nonempty, closed, convex, and bounded subset of E. Let $f:E\to(-\infty,+\infty]$ be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of E. Let $A:C\to E^*$ be η-hemicontinuous and relaxed η–α monotone mapping and $g:C\times C\to\mathbb{R}$ be bifunctions satisfying (C1)–(C4). Let $\psi:C\to\mathbb{R}$ be a proper, convex, and lower semicontinuous mapping. Let T be a continuous Bregman quasiasymptotically nonexpansive mapping of C into itself such that $\mathcal{F}=F(T)\cap EP(g,A)\neq\emptyset$. Assume that the conditions of Lemma 18 and the following condition are satisfied:*

(v) *For all $x,y,z,w\in C$ one has*

$$\limsup_{t\to 0^+}\left\langle Az,\eta\left(x,ty+(1-t)w\right)\right\rangle\leq\left\langle Az,\eta\left(x,w\right)\right\rangle.\tag{143}$$

Let $\{x_n\}$ be iteratively defined as follows:

$$\begin{aligned}
&x_0=x\in C\quad\text{chosen arbitrarily}, \\
&y_n=\nabla f^*\left(\beta_n\nabla f\left(x_n\right)+\left(1-\beta_n\right)\nabla f\left(T^n x_n\right)\right), \\
&u_n\in C\quad\text{such that} \\
&g\left(u_n,y\right)+\left\langle Au_n,\eta\left(y,u_n\right)\right\rangle+\psi\left(y\right)-\psi\left(u_n\right) \\
&\quad+\frac{1}{r_n}\left\langle\nabla f\left(u_n\right)-\nabla f\left(y_n\right),y-u_n\right\rangle\geq 0\quad\forall y\in C, \\
&C_{n+1}=\left\{z\in C_n:D_f\left(z,u_n\right)\leq D_f\left(z,x_n\right)+\theta_n\right\}, \\
&x_{n+1}=P_{C_{n+1}}^f x_0,\quad n\geq 0,
\end{aligned}\tag{144}$$

where $\{\beta_n\}$ *is a real sequence in* $(0,1)$ *satisfying* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$ *and* $\theta_n = (1-\beta_n)t_n \sup_{p\in\mathcal{F}} D_f(p,x_n)$. *Then* $\{x_n\}$ *converges strongly to* $u = P^f_{\mathcal{F}} x_0$.

Setting $A_j = 0\ \forall j = 1,2,\ldots,m$ in Theorem 21 then we obtain the following result.

Corollary 23. *Let E be a reflexive Banach space with the dual* E^* *and let C be a nonempty, closed, convex, and bounded subset of E. Let* $f : E \to (-\infty,+\infty]$ *be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of E. For each* $j = 1,2,3,\ldots,m$, *let* $g_j : C\times C \to \mathbb{R}$ *be bifunctions satisfying (C1)–(C4) and* $\psi_j : C \to \mathbb{R}$ *be proper, convex, and lower semicontinuous mappings. Let* $\{T_i\}_{i=1}^N$ *be a finite family of continuous Bregman quasiasymptotically nonexpansive mapping of C into itself such that* $\mathcal{F} = [\bigcap_{i=1}^N F(T_i)] \cap [EP(g_j,\psi_j)] \neq \emptyset$. *Let* K_n *be the Bregman K-mapping generated by* $T_1,T_2,T_3,\ldots,T_N$ *in (80). Let* $\{x_n\}$ *be iteratively defined as follows:*

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrarily},\\
&C_{1,j} = C = C_0\\
&y_n = \nabla f^*\left(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right),\\
&u_{n,j} \in C \quad \text{such that}\\
&g_j(u_{n,j},y) + \psi_j(y) - \psi_j(u_{n,j})\\
&\quad + \frac{1}{r_n}\left\langle \nabla f(u_{n,j}) - \nabla f(y_n), y - u_{n,j}\right\rangle \geq 0 \qquad \forall y \in C,\\
&C_{n+1,j} = \left\{z \in C_n : D_f(z,u_{n,j}) \leq D_f(z,x_n) + \theta_n\right\},\\
&C_{n+1} = \bigcap_{j=1}^m C_{n+1,j},\\
&x_{n+1} = P^f_{C_{n+1}} x_0, \quad n \geq 0,
\end{aligned} \tag{145}$$

where $\{\beta_n\}$ *is a real sequence in* $(0,1)$ *satisfying* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$ *and* $\theta_n = (1-\beta_n)t_n \sup_{p\in\mathcal{F}} D_f(p,x_n)$. *Then* $\{x_n\}$ *converges strongly to* $u = P^f_{\mathcal{F}} x_0$.

If $A_j = 0$ and $\psi_j = 0\ \forall j = 1,2,\ldots,m$ in Theorem 21 then we obtain the following result.

Corollary 24. *Let E be a reflexive Banach space with the dual* E^* *and let C be a nonempty, closed, convex, and bounded subset of E. Let* $f : E \to (-\infty,+\infty]$ *be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of E. For each* $j = 1,2,3,\ldots,m$, *let* $g_j : C\times C \to \mathbb{R}$ *be bifunctions satisfying (C1)–(C4). Let* $\{T_i\}_{i=1}^N$ *be a finite family of continuous Bregman quasiasymptotically nonexpansive mappings of C into itself such that* $\mathcal{F} = [\bigcap_{i=1}^N F(T_i)] \cap [EP(g_j)] \neq \emptyset$. *Let* K_n *be the Bregman K-mapping generated by* $T_1,T_2,T_3,\ldots,T_N$ *in (80). Let* $\{x_n\}$ *be iteratively defined as follows:*

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrarily},\\
&C_{1,j} = C = C_0\\
&y_n = \nabla f^*\left(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(K_n x_n)\right),\\
&u_{n,j} \in C \quad \text{such that}\\
&g_j(u_{n,j},y) + \frac{1}{r_n}\left\langle \nabla f(u_{n,j}) - \nabla f(y_n), y - u_{n,j}\right\rangle \geq 0 \qquad \forall y \in C,\\
&C_{n+1,j} = \left\{z \in C_n : D_f(z,u_{n,j}) \leq D_f(z,x_n) + \theta_n\right\},\\
&C_{n+1} = \bigcap_{j=1}^m C_{n+1,j},\\
&x_{n+1} = P^f_{C_{n+1}} x_0, \quad n \geq 0,
\end{aligned} \tag{146}$$

where $\{\beta_n\}$ *is a real sequence in* $(0,1)$ *satisfying* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$ *and* $\theta_n = (1-\beta_n)t_n \sup_{p\in\mathcal{F}} D_f(p,x_n)$. *Then* $\{x_n\}$ *converges strongly to* $u = P^f_{\mathcal{F}} x_0$.

As a direct consequence of Theorem 21 and Remark 1, we obtain the convergence result for system of the mixed equilibrium problems and finite family of quasi-ϕ-asymptotically nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces.

Corollary 25. *Let E be a uniformly convex and uniformly smooth Banach space with the dual* E^* *and let C be a nonempty, closed, convex, and bounded subset of E. Let* $J : E \to E^*$ *be a normalised duality mapping. For each* $j = 1,2,3,\ldots,m$, *let* $A_j : C \to E^*$ *be* η*-hemicontinuous and relaxed* $\eta-\alpha$ *monotone mappings and* $g_j : C\times C \to \mathbb{R}$ *be bifunctions satisfying (C1)–(C4). Let* $\psi_j : C \to \mathbb{R}$ *be proper, convex, and lower semicontinuous mappings. Let* $\{T_i\}_{i=1}^N$ *be a finite family of continuous Bregman quasiasymptotically nonexpansive mappings of C into itself such that* $\mathcal{F} = [\bigcap_{i=1}^N F(T_i)] \cap [EP(g_j,A_j)] \neq \emptyset$. *Let* K_n *be the Bregman K-mapping generated by* $T_1,T_2,T_3,\ldots,T_N$ *in (80). Assume that the conditions of Lemma 18 and the following condition are satisfied:*

(v) *For all* $x,y,z,w \in C$ *one has*

$$\limsup_{t\to 0^+} \left\langle Az, \eta(x, ty + (1-t)w)\right\rangle \leq \left\langle Az, \eta(x,w)\right\rangle. \tag{147}$$

Let $\{x_n\}$ be iteratively defined as follows:

$$x_0 = x \in C \quad \text{chosen arbitrarily},$$

$$C_{1,j} = C = C_0$$

$$y_n = J^{-1}\left(\beta_n J(x_n) + (1-\beta_n) J(K_n x_n)\right),$$

$$u_{n,j} \in C \quad \text{such that}$$

$$g_j(u_{n,j}, y) + \left\langle A_j u_{n,j}, \eta(y, u_{n,j})\right\rangle + \psi_j(y) - \psi_j(u_{n,j}) + \frac{1}{r_n}\left\langle J(u_{n,j}) - J(y_n), y - u_{n,j}\right\rangle \geq 0 \quad \forall y \in C, \tag{148}$$

$$C_{n+1,j} = \left\{z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \theta_n\right\},$$

$$C_{n+1} = \bigcap_{j=1}^{m} C_{n+1,j},$$

$$x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0,$$

where $\{\beta_n\}$ is a real sequence in $(0,1)$ satisfying $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\{r_n\} \subset [a,\infty)$ for some $a > 0$ and $\theta_n = (1-\beta_n)t_n \sup_{p\in\mathcal{F}} D_f(p, x_n)$. Then $\{x_n\}$ converges strongly to $u = P_{\mathcal{F}}^f x_0$.

Remark 26. Our theorems and corollaries generalise the main theorem of Chen et al. [27] in the following senses:

(1) For the structure of Banach spaces, we extend the duality mapping to more general case: that is, a Legendre, strongly coercive, uniformly Fréchet differentiable, and totally convex function.

(2) For the mapping, we consider Bregman quasiasymptotically nonexpansive mappings which contain Bregman quasi nonexpansive mappings as a special case which itself is generalisation of quasi-ϕ-nonexpansive mappings.

(3) In Chen et al. [27] the authors considered mixed equilibrium problems while in this paper a system of equilibrium problems is considered.

Competing Interests

The authors declare that they have no competing interests.

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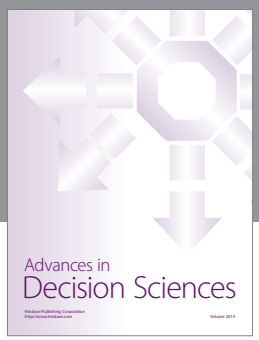

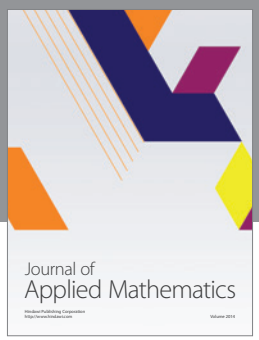

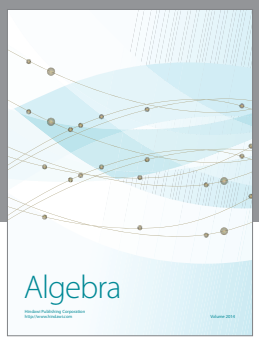

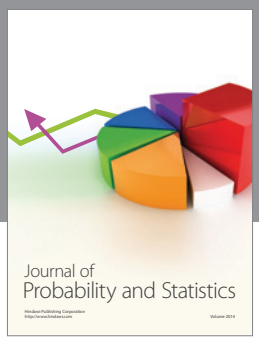

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