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Research Article

Existence of Equilibria and Fixed Points of Set-Valued Mappings on Epi-Lipschitz Sets with Weak Tangential Conditions

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We prove a new result of existence of equilibria for an u.s.c. set-valued mapping F on a compact set S of $\mathbb{R}^n$ which is epi-Lipschitz and satisfies a weak tangential condition. Equivalently this provides existence of fixed points of the set-valued mapping $x \rightrightarrows F(x) - x$. The main point of our result lies in the fact that we do not impose the usual tangential condition in terms of the Clarke tangent cone. Illustrative examples are stated showing the importance of our results and that the existence of such equilibria does not need necessarily such usual tangential condition.

1. Introduction

Let F be a set-valued mapping defined from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$. A familiar result on existence of equilibria on convex compact sets is formulated as follows (see [1]).

Theorem 1. *Let F be an u.s.c. set-valued mapping defined on a convex compact set $S \subset \mathbb{R}^n$ and suppose that $F(x)$ is nonempty convex compact, $\forall x \in S$. If*

$$F(x) \cap T(S; x) \neq \emptyset, \quad \forall x \in S, \tag{1}$$

then F has equilibria on S; that is, $\exists x^* \in S$ such that $0 \in F(x^*)$.

Here $T(S;x)$ is the tangent cone in the sense of convex analysis defined as $T(S;x) = \mathrm{cl}(\mathbb{R}_+(S-x))$, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers and cl denotes the closure in E. Assume now that S is not necessarily convex and assume that in the tangential condition (1) the tangent cone $T(S;x)$ is replaced by the Clarke tangent cone $T^C(S;x)$; that is,

$$F(x) \cap T^C(S; x) \neq \emptyset, \quad \forall x \in S. \tag{2}$$

In order to get the same conclusion of Theorem 1 in the nonconvex case, we need one more assumption on S, which is the epi-Lipschitzness of S. The following theorem is an extension of Theorem 1 to the nonconvex case (see [1]).

Theorem 2. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Assume that S is epi-Lipschitz and (2) holds. Then F has equilibria on S.*

It is very important to point out that for epi-Lipschitz sets the tangential condition (2) cannot be weakened to

$$F(x) \cap K(S; x) \neq \emptyset, \quad \forall x \in S, \tag{3}$$

where $K(S;x)$ is (generally greater) the contingent cone defined below in Section 2 (see Example 3.1 in [1]).

Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in [-1, 0]\}$ and $F(x_1, x_2) = (2x_1 + x_2; 2x_1 + 3x_2 - 1) + \mathbb{B}$. Clearly S is an epi-Lipschitz compact convex set in $\mathbb{R}^2$ and F is u.s.c. on $\mathbb{R}^2$ with $T^C(S; u_i) \cap F(u_i) = \emptyset$ for $u_1 = (-1, 0)$, $u_2 = (0, -1)$, and $u_3 = (-1, -1)$. However, the point $x^* = (0, 0) \in S$ satisfies $(0, 0) \in F(x^*)$. This shows that the tangential condition (2) in Theorem 2 is not necessary to get equilibria on S. Our main purpose in this work is to replace $T^C(S;x)$ in (2) by a new tangent set which is always larger than $T^C(S;x)$ and to prove the existence of equilibria under the new tangential condition.

The main result of the paper is read as follows.

Theorem 3. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ which is epi-Lipschitz and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Let Ω be a convex* *compact set in $\mathbb{R}^n$ with $0 \in \Omega$. Assume that there exists some $\beta \geq 0$ such that for any $x \in S$*

$$F(x) \cap T^{C}_{\beta,\Omega}(S; x) \neq \emptyset. \quad (4)$$

If S is tangentially regular, then there exists $x^* \in S$ such that

$$0 \in F(x^*). \quad (5)$$

We point out that the price that we pay for replacing (2) by (4) is the tangential regularity of the set S (i.e., $K(S; x) = T^C(S; x)$) and it is an open question to extend the result of Theorem 3 to the general case without the tangential regularity assumption. The first corollary of Theorem 3 is the following result.

Corollary 4. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ which is epi-Lipschitz and tangentially regular and let F be an u.s.c. set-valued mapping on S with nonempty closed convex values. Assume that (2) holds. Then F has equilibria on S.*

Proof. Take $\Omega = \{0\}$ in Theorem 3. All the assumptions are fulfilled and hence F has equilibria on S. □

The paper is organised as follows. The next section is devoted to some preliminary concepts and results needed in the development of our approach. In Section 3 we prove our main result stated in Theorem 3 and we present an illustrative example showing the importance of our main result in Theorem 3. An application to fixed point results is presented at the end of paper.

2. Preliminaries

Throughout this section, we assume that E is a Hausdorff topological vector space. We will denote by E^* the topological dual of E and by $\langle \cdot, \cdot \rangle$ the pairing between the spaces E and E^*.

Let S be a nonempty closed subset of E and let $\overline{x}$ be a point in S. Let us recall [2, 3] the following classical tangent cones $K(S; \overline{x})$ of Bouligand and $T^C(S; \overline{x})$ of Clarke.

(i) The Bouligand tangent cone (also called contingent cone) $K(S; \overline{x})$ to S at $\overline{x}$ is the set of all $h \in E$ such that, for every neighborhood H of h in E and for every $\epsilon > 0$, there exists $t \in (0, \epsilon)$ such that

$$(\overline{x} + tH) \cap S \neq \emptyset. \quad (6)$$

(ii) The Clarke tangent cone $T^C(S; \overline{x})$ to S at $\overline{x}$ is the set of all $h \in E$ such that for every neighborhood H of h in E there exist a neighborhood X of $\overline{x}$ in E and a real number $\epsilon > 0$ such that

$$(x + tH) \cap S \neq \emptyset \quad \forall x \in X \cap S, \ t \in (0, \epsilon). \quad (7)$$

(iii) The Rockefeller hypertangent cone $H(S; \overline{x})$ to S at $\overline{x}$ is the set of all $h \in E$ for which there exist a neighborhood H of h in E, a neighborhood X of $\overline{x}$ in E, and a real number $\delta > 0$ such that

$$X \cap S + tH \subset S \quad \forall t \in (0, \delta). \quad (8)$$

Clearly $H(S; \overline{x}) \subset T^C(S; \overline{x}) \subset K(S; \overline{x})$. Recall that (see [3, 4]) $T^C(S; \overline{x})$ is a closed convex cone and $K(S; \overline{x})$ is a closed cone (that may be nonconvex) while $H(S; \overline{x})$ is an open convex cone.

For a closed convex set S and $\overline{x} \in S$, both Clarke tangent cone and Bouligand tangent cone coincide and they are equal to the convex tangent cone $T(S; \overline{x})$. The class of nonempty closed sets satisfying the equality $T^C(S; \overline{x}) = K(S; \overline{x})$ is called the class of *tangentially regular* sets.

In [5], we introduced the concept of Ω-epi-Lipschitz sets as follows.

Definition 5. Let S be a closed subset of E and $\overline{x} \in S$, and let Ω be a bounded set in E. One will say that S is Ω-*epi-Lipschitz* at $\overline{x}$ in a direction $\overline{h}$ if and only if the minimal time function $T_{S,\Omega}$ associated with S and Ω is directionally Lipschitz at $\overline{x}$ in the direction $\overline{h}$ in the sense of Rockefeller [6], that is, if and only if there exists $\beta := \beta(\overline{x}, \overline{h}) \geq 0$ such that $T^{\sharp}_{S,\Omega}(\overline{x}, \overline{h}) \leq \beta$, where

$$f^{\sharp}(\overline{x}; \overline{h}) := \limsup_{\substack{(x,\alpha)\downarrow_f \overline{x} \\ (t,h')\to(0^+,\overline{h})}} t^{-1} \left[f(x + th') - \alpha \right] < +\infty \quad (9)$$

and $(x, \alpha) \downarrow_f \overline{x}$ means $(x, \alpha) \in \operatorname{epi} f := \{(z, \beta) \in E \times \mathbb{R};\ f(z) \leq \beta\}$ and $(x, \alpha) \to (\overline{x}, f(\overline{x}))$. Recall that the minimal time function $T_{S,\Omega}$ is defined as follows:

$$T_{S,\Omega}(x) := \inf \{t > 0 : S \cap (x + t\Omega) \neq \emptyset\}. \quad (10)$$

A geometric characterization of Ω-epi-Lipschitz sets has been established in [5] saying that a set S is Ω-epi-Lipschitz at $\overline{x}$ in a direction $\overline{h}$ if and only if there exist $\beta := \beta(\overline{x}, \overline{h}) \geq 0$, $\delta > 0$, $H \in \mathcal{N}(\overline{h})$, and $X \in \mathcal{N}(\overline{x})$ such that

$$X \cap S + tH \subset S - t\beta\Omega \quad \forall t \in (0, \delta). \quad (11)$$

In our analysis in this work we need the constant β to be uniform with respect to the directions; that is, $\beta := \beta(\overline{x})$ is only dependent on $\overline{x}$ and in this case we say that S is $(\beta_{\overline{x}}, \Omega)$-*epi-Lipschitz* at $\overline{x}$ in the direction $\overline{h} \in E$. Since β does not depend on the directions, we define $H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$ the set of all directions $h \in E$ satisfying $T^{\sharp}_{S,\Omega}(\overline{x}, h) \leq \beta_{\overline{x}}$; that is,

$$H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) := \left\{ h \in E : T^{\sharp}_{S,\Omega}(\overline{x}, h) \leq \beta_{\overline{x}} \right\}. \quad (12)$$

Following the same lines of the proof of Proposition 3.4 in [5] we can prove the following result.

Proposition 6. *Let S be a closed subset of E and $\overline{x} \in S$, $\overline{h} \in E$, and let Ω be a bounded set in E.*

(i) *If there exist $\beta_{\overline{x}} \geq 0$, $\delta > 0$, $H \in \mathcal{N}(\overline{h})$, and $X \in \mathcal{N}(\overline{x})$ such that*

$$X \cap S + tH \subset S - t\beta_{\overline{x}}\Omega \quad \forall t \in (0, \delta), \quad (13)$$

then S is $(\beta_{\overline{x}}, \Omega)$-epi-Lipschitz at $\overline{x}$ in the direction $\overline{h}$.

(ii) *Conversely, if there exists $\beta_{\overline{x}} \geq 0$ for which S is $(\beta_{\overline{x}}, \Omega)$-epi-Lipschitz at $\overline{x}$ in the direction $\overline{h}$, then there exist $\delta > 0$, $H \in \mathcal{N}(\overline{h})$, and $X \in \mathcal{N}(\overline{x})$ such that*

$$X \cap S + tH \subset S - t\left(\beta_{\overline{x}} + 1\right)\Omega \quad \forall t \in (0, \delta). \tag{14}$$

Remark 7. Clearly, if S is (β_x, Ω)-epi-Lipschitz at $x \in S$, then S is (β, Ω)-epi-Lipschitz at x for any $\beta \geq \beta_x$. Consequently, any epi-Lipschitz set in the sense of Rockefeller [4] is (β, Ω)-epi-Lipschitz for any $\beta \geq 0$ and for any Ω with $0 \in \Omega$ and the constant β is uniform for any $x \in S$. We note that the notion of (β, Ω)-epi-Lipschitz sets recovers some well known concepts in variational analysis.

(i) Obviously a closed set in E is epi-Lipschitz in the sense of Rockefeller [4] if and only if it is $(\beta, \{0\})$-epi-Lipschitz in the sense of Definition 5 and if and only if it is $(0, \{\Omega\})$-epi-Lipschitz and if and only if it is $(\beta, \{\Omega\})$-epi-Lipschitz for any $\beta > 0$.

(ii) In normed spaces, any compactly epi-Lipschitz set S at $\overline{x} \in S$ with respect to a convex compact set K in the sense of Borwein and Strójwas [7] is $(1, \Omega)$-epi-Lipschitz in the direction $\overline{k}$ with $\Omega := K - \overline{k}$, for any $\overline{k} \in K$. We recall that S is compactly epi-Lipschitz at $\overline{x} \in S$ with respect to a convex compact set K in the sense of [7] provided that there exists $r > 0$ such that

$$(\overline{x} + r\mathbb{B}) \cap S + tr\mathbb{B} \subset S - tK \quad \forall t \in (0, r). \tag{15}$$

(iii) Assume that E is a normed space and K and S are closed sets in E. If S is K-directionally Lipschitz in the sense of [8], then S is $(1, \Omega)$-epi-Lipschitz with $\Omega = K - \overline{k}$, for any $\overline{k} \in K$.

Our main tools in the present work are two tangent sets associated with (β, Ω)-epi-Lipschitz sets. The first tangent set is $H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$ defined above and will be called the (β, Ω)-hypertangent set and it characterizes the class of (β, Ω)-epi-Lipschitz sets by its nonemptiness. The second tangent set will be called the (β, Ω)-Clarke tangent set and is defined as the set of all $\overline{h} \in E$ satisfying that for any neighborhood H of $\overline{h}$ there exist $\delta > 0$ and $X \in \mathcal{N}(\overline{x})$ such that

$$[x + t(H + \beta\Omega)] \cap S \neq \emptyset \quad \forall t \in (0, \delta),\ x \in S \cap X. \tag{16}$$

Observe that for $\Omega = \{0\}$ both sets $H_{\beta,\Omega}(S; \overline{x})$ and $T^C_{\beta,\Omega}(S; \overline{x})$ coincide, respectively, with the hypertangent cone $H(S; \overline{x})$ and the Clarke tangent cone $T^C(S; \overline{x})$.

Obviously, we always have the following inclusions: $H(S; \overline{x}) \subset H_{\beta,\Omega}(S; \overline{x})$ and $T^C(S; \overline{x}) \subset T^C_{\beta,\Omega}(S; \overline{x})$. Consequently, any epi-Lipschitz set is (β, Ω)-epi-Lipschitz. In our analysis we need to prove many properties for the tangent sets $H_{\beta,\Omega}(S; \overline{x})$ and $T^C_{\beta,\Omega}(S; \overline{x})$. We notice that $H_{\beta,\Omega}(S; \overline{x})$ is an open set in E and $T^C_{\beta,\Omega}(S; \overline{x})$ is a closed set in E and both are not necessarily convex.

Proposition 8. *Let E be a Hausdorff topological vector space, let S be a nonempty closed subset of E, and let Ω be a convex bounded set in E with $0 \in \Omega$. Let $\overline{x} \in S$.*

(1) *S is $(\beta_{\overline{x}}, \Omega)$-epi-Lipschitz at $\overline{x}$ if and only if $H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) \neq \emptyset$.*

(2) $T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x}) + H(S; \overline{x}) \subset H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.

(3) $H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) + H(S; \overline{x}) \subset H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.

(4) *If S is epi-Lipschitz at $\overline{x} \in S$, then $H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) = \operatorname{int}(T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})) \neq \emptyset$.*

(5) *If S is epi-Lipschitz at $\overline{x} \in S$, then $\operatorname{cl}(H_{\beta_{\overline{x}},\Omega}(S; \overline{x})) = T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.*

Proof. (1) It follows directly from the definition of $H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.

(2) Let $h_1 \in H(S; \overline{x})$ and $h_2 \in T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})$. By definition of $H(S; \overline{x})$ there exist $V \in \mathcal{N}(0)$, $X_1 \in \mathcal{N}(\overline{x})$, and $\delta_1 > 0$ such that

$$X_1 \cap S + t(h_1 + V) \subset S, \quad \forall t \in (0, \delta_1). \tag{17}$$

Choose a symmetric neighborhood W of 0 in E such that $W + W \subset V$. By definition of $T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})$ there exist $X_2 \in \mathcal{N}(\overline{x})$ and $\delta_2 > 0$ such that

$$[X_2 \cap S + t(h_2 + W) + t\beta_{\overline{x}}\Omega] \cap S \neq \emptyset, \quad \forall t \in (0, \delta_2). \tag{18}$$

Choose now $X' \in \mathcal{N}(\overline{x})$ and $\delta' > 0$ such that

$$X' + (0, \delta')(h_2 + W + \beta_{\overline{x}}\Omega) \subset X_1. \tag{19}$$

Put $X := X' \cap X_1 \cap X_2$ and $\delta := \min\{\delta', \delta_1, \delta_2\}$. Fix any $t \in (0, \delta)$, any $w \in W$, and any $x \in X \cap S$. Then by (18) there exist $\overline{w} \in W$ and $\overline{y} \in \Omega$ such that

$$x + t(h_2 + \overline{w} + \beta_{\overline{x}}\overline{y}) \in S. \tag{20}$$

Put $x' := x + t(h_2 + \overline{w} + \beta_{\overline{x}}\overline{y})$. Then the previous inclusion and (19) ensure that $x' \in X_1 \cap S$ and hence we can use (17) to get

$$x' + t(h_1 + w - \overline{w}) \in S. \tag{21}$$

Therefore,

$$x + t(h_1 + h_2 + w) \in S - t\beta_{\overline{x}}\Omega. \tag{22}$$

Thus, for any $x \in X \cap S$, any $t \in (0, \delta)$, and any $h \in H$ we have

$$x + th \in S - t\beta_{\overline{x}}\Omega. \tag{23}$$

This ensures by definition of the $(\beta_{\overline{x}}, \Omega)$-hypertangent cone that $h_1 + h_2 \in H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.

(3) It is a direct consequence of Part (2) and the inclusions $H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) \subset T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})$.

(4) First observe that since S is epi-Lipschitz at $\overline{x}$ we have that $H(S; \overline{x})$ is nonempty. Hence, there exists some $h_0 \in H(S; \overline{x})$. Now, since $H_{\beta_{\overline{x}},\Omega}(S; \overline{x}) \subset T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})$ and as $H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$ is open, it is enough to show the inclusion $\operatorname{int}[T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})] \subset H_{\beta_{\overline{x}},\Omega}(S; \overline{x})$. Consider $h \in \operatorname{int}[T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})]$. Then there exists a positive number $\lambda > 0$ such that $h - \lambda h_0 \in \operatorname{int}[T^C_{\beta_{\overline{x}},\Omega}(S; \overline{x})]$. Since $H(S; \overline{x})$ is a cone, we have $\lambda h_0 \in H(S; \overline{x})$ and hence by Part (2) we obtain

$h = (h - \lambda h_0) + \lambda h_0 \in T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x}) + H(S;\bar{x}) \subset H_{\beta_{\bar{x}},\Omega}(S;\bar{x})$ and so the proof of Part (4) is finished.

(5) Taking into account the inclusion $H_{\beta_{\bar{x}},\Omega}(S;\bar{x}) \subset T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x})$ and the fact that $T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x})$ is closed, it is sufficient to show the inclusion $T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x}) \subset \mathrm{cl}[H_{\beta_{\bar{x}},\Omega}(S;\bar{x})]$.

Fix again some element $h_0 \in H(S;\bar{x})$. For any $h \in T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x})$ we have by Part (2)

$$h + \frac{1}{n}h_0 \in T^C_{\beta_{\bar{x}},\Omega}(S;\bar{x}) + H(S;\bar{x}) \subset H_{\beta_{\bar{x}},\Omega}(S;\bar{x}), \quad \forall n \geq 1. \tag{24}$$

This ensures that $h \in \mathrm{cl}[H_{\beta_{\bar{x}},\Omega}(S;\bar{x})]$ because $h = \lim_{n\to\infty}(h + (1/n)h_0)$. So the proof is complete. □

We require the notion of lower semicontinuity for set-valued mappings. Let F be a set-valued mapping on $S \subset E$, with values that are subsets of another Hausdorff topological vector space G. We say that $F : S \rightrightarrows G$ is lower semicontinuous (l.s.c.) at $(\bar{x},\bar{y}) \in \mathrm{gph}\, F := \{(x,y) \in E \times G : y \in F(x)\}$ provided that, for every $V \in \mathcal{N}(0)$, there exists $W \in \mathcal{N}(0)$ such that

$$x \in S \cap (\bar{x} + W) \Longrightarrow \bar{y} \in F(x) + V. \tag{25}$$

We say that $F : S \rightrightarrows G$ is upper semicontinuous (u.s.c.) at $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ provided that, for every $V \in \mathcal{N}(0)$, there exists $W \in \mathcal{N}(0)$ such that

$$x \in S \cap (\bar{x} + W) \Longrightarrow F(x) \subset F(\bar{x}) + V. \tag{26}$$

In [6], the author proved the l.s.c. of the set-valued mapping $x \mapsto T^C(S;x)$ whenever S is epi-Lipschitz. In the following proposition we extend Theorem 3 in [6] to the set-valued mapping $x \mapsto T^C_{\beta,\Omega}(S;x)$ with $\beta \geq 0$. This l.s.c. property is very important in the proof of our main result in Theorem 3.

Proposition 9. *Let E be a Hausdorff locally convex topological vector space, let S be a nonempty closed subset of E, and let $\beta \geq 0$, Ω be a convex bounded in E with $0 \in \Omega$. If S is epi-Lipschitz at $\bar{x} \in S$, then the set-valued mapping $x \mapsto T^C_{\beta,\Omega}(S;x)$ is lower semicontinuous at $\bar{x}$.*

Proof. Let $\bar{x} \in S$ and $\bar{u} \in T^C_{\beta,\Omega}(S;\bar{x})$. Assume that S is epi-Lipschitz at $\bar{x}$. Then there exists $\bar{h} \in H(S;\bar{x})$. Let U be any neighborhood of zero in G. Since G is locally convex we can choose a balanced neighborhood U_0 such that $U_0 \subset U$. Set $V := U_0 \cap (-U_0)$. Clearly $V \subset U$ is a symmetric convex neighborhood of zero in G. Fix any $v_0 := \bar{u} + \bar{h}/(\rho_V(\bar{h}) + 1)$ where ρ_V is the Minkowski function associated with V. Since $H(S;\bar{x})$ is a cone, we have $\bar{h}/(\rho_V(\bar{h}) + 1) \in H(S;\bar{x})$ and hence by Part (2) in Proposition 8 we obtain

$$v_0 = \bar{u} + \frac{\bar{h}}{\rho_V(\bar{h}) + 1} \in T^C_{\beta,\Omega}(S;\bar{x}) + H(S;\bar{x}) \subset H_{\beta,\Omega}(S;\bar{x}). \tag{27}$$

Thus, there exists $\delta > 0$, $V_1 \in \mathcal{N}(0)$, and $V_2 \in \mathcal{N}(0)$ such that

$$(\bar{x} + V_1) \cap S + t(v_0 + V_2) \subset S - t\beta\Omega, \quad \forall t \in (0,\delta). \tag{28}$$

Let $W \in \mathcal{N}(0)$ such that $W + W \subset V_1$. Fix any $x \in (\bar{x} + W) \cap S \neq \emptyset$. Let any $x' \in (x + W) \cap S$. Then $x' \in (\bar{x} + W + W) \cap S \subset (\bar{x} + V_1) \cap S$ and so by (28) we get

$$x' + t(v_0 + V_2) \subset S - t\beta\Omega, \quad \forall t \in (0,\delta). \tag{29}$$

Therefore, for any $x \in (\bar{x} + W) \cap S$ we have $v_0 \in H_{\beta,\Omega}(S;x)$. Hence, $\bar{u} \in H_{\beta,\Omega}(S;x) - \bar{h}/(\rho_V(\bar{h}) + 1)$. Observe that $\rho_V(\bar{h}/(\rho_V(\bar{h}) + 1)) = \rho_V(\bar{h})/(\rho_V(\bar{h}) + 1) < 1$ which implies that $\bar{h}/(\rho_V(\bar{h}) + 1) \in V$ and hence $-(\bar{h}/(\rho_V(\bar{h}) + 1)) \in -V = V$. Thus, $\bar{u} \in H_{\beta,\Omega}(S;x) + V \subset T^C_{\beta,\Omega}(S;x) + V \subset T^C_{\beta,\Omega}(S;x) + U$. Consequently, for any $\bar{u} \in T^C_{\beta,\Omega}(S;\bar{x})$ and any $U \in \mathcal{N}(0)$ there exists $W \in \mathcal{N}(0)$ such that

$$x \in S \cap (\bar{x} + W) \Longrightarrow \bar{u} \in T^C_{\beta,\Omega}(S;x) + U. \tag{30}$$

This proves that the set-valued mapping $x \mapsto T^C_{\beta,\Omega}(S;x)$ is lower semicontinuous at $\bar{x}$ and the proof is complete. □

Remark 10. Following the proof of the previous proposition, we can prove the lower semicontinuity of the set-valued mapping $x \mapsto H_{\beta,\Omega}(S;x)$ at $\bar{x}$ whenever S is epi-Lipschitz set at $\bar{x}$.

Lemma 11. *For any $\bar{x} \in S$ and any $\beta \geq 0$ one has*

$$H(S;\bar{x}) - \beta\Omega \subset H_{\beta,\Omega}(S;\bar{x}). \tag{31}$$

Proof. Let $\bar{x} \in S$. Without loss of generality we assume that $H(S;\bar{x}) - \beta\Omega \neq \emptyset$. Then, for any $h_0 \in H(S;\bar{x}) - \beta\Omega$, there exists $\bar{h} \in H(S;\bar{x})$ and $w \in \Omega$ such that $h_0 = \bar{h} - \beta w$. By definition of the hypertangent $H(S;\bar{x})$ there exist $V \in \mathcal{N}(0)$, $X \in \mathcal{N}(x)$, and $\delta > 0$ such that

$$X \cap S + t(\bar{h} + V) \subset S, \quad \forall t \in (0,\delta). \tag{32}$$

By adding $-t\beta w$ to both sides we obtain

$$X \cap S + t(\bar{h} - \beta w + V) \subset S - t\beta w \subset S - t\beta\Omega, \quad \forall t \in (0,\delta). \tag{33}$$

This ensures by definition of $H_{\beta,\Omega}(S;\bar{x})$ that

$$h_0 = \bar{h} - \beta w \in H_{\beta,\Omega}(S;\bar{x}). \tag{34}$$

Therefore, $H(S;\bar{x}) - \beta\Omega \subset H_{\beta,\Omega}(S;\bar{x})$ and hence the proof is complete. □

The next lemma establishes an analogue result for the (β,Ω)-Clarke tangent set.

Lemma 12. *Assume that Ω is a convex bounded set with $0 \in \Omega$ and that S is epi-Lipschitz at $\bar{x}$. Then*

$$T^C(S;\bar{x}) - \beta\Omega \subset T^C_\Omega(S;\bar{x}). \tag{35}$$

Proof. Let $\bar{x} \in S$. Since S is epi-Lipschitz at $\bar{x}$, we have by Part (5) in Proposition 8 the equalities $T^C(S;x) = \mathrm{cl}(H(S;x))$ and $T^C_{\beta,\Omega}(S;x) = \mathrm{cl}(H_\Omega(S;x))$. Therefore, the conclusion follows directly from the previous lemma. □

It is a natural question to ask whether the inclusion in the previous lemma becomes an equality. The next lemma establishes a positive answer whenever the set is assumed to be tangentially regular. Its proof needs the following proposition which is also needed in the proof of Theorem 19. It has been proved in Proposition 5.1 in [5] for Ω-epi-Lipschitz sets. The proof stated below is a direct adaptation of the proof in [5] that we give for the sake of completeness of the paper. To do that, we need the following characterization of $K(S;x)$ in terms of nets (see, e.g., [9]). A vector $v \in K(S;\overline{x})$ if and only if there exist a net $(t_j)_{j\in J}$ of positive real numbers converging to zero and a net $(v_j)_{j\in J}$ in E converging to v such that

$$\overline{x} + t_j v_j \in S, \quad \text{for each } j \in J. \tag{36}$$

Proposition 13. *Let E be a Hausdorff topological vector space, let S be a nonempty closed subset of E, let Ω be a convex compact set in E with $0 \in \Omega$, and let $\overline{x} \in S$. Assume that for some $\beta_{\overline{x}} \geq 0$ one has $\overline{h} \in H_{\beta_{\overline{x}},\Omega}(S;\overline{x})$; then*

$$\overline{h} \in K(S;\overline{x}) - \beta_{\overline{x}}\Omega. \tag{37}$$

Proof. By definition of $H_{\beta_{\overline{x}},\Omega}(S;\overline{x})$, there exist $\delta > 0$, $W \in \mathcal{N}(0)$, and $X \in \mathcal{N}(\overline{x})$ such that

$$\left[\overline{x} + t\left(\overline{h} + v\right) + t\beta_{\overline{x}}\Omega\right] \cap S \neq \emptyset, \qquad \forall t \in (0,\delta), \ \forall v \in W. \tag{38}$$

Choose $V \in \mathcal{N}(0)$ such that $V + V \subset W$ and let $(t_j, v_j)_{j\in J}$ be a net converging to $(0,0)$ in $(0,\delta)\times V$. Applying (38) we get, for any $j \in J$, the existence of $w_j \in \Omega$ such that

$$\overline{x} + t_j\left(\overline{h} + v_j\right) + t_j\beta_{\overline{x}}w_j \in S. \tag{39}$$

Since Ω is compact, we may extract a subnet $(w_{s(j)})_j$ of $(w_j)_j$ converging to some point $\overline{w} \in \Omega$. Put $\tau_j := t_{s(j)}$, $\widetilde{w}_j := w_{s(j)}$, and $h_j := \overline{h} + v_{s(j)} + \beta_{\overline{x}}\widetilde{w}_j$. Then (39) ensures

$$\overline{x} + \tau_j h_j \in S, \quad \forall j \in J. \tag{40}$$

Since $h_j \to \overline{h} + \beta_{\overline{x}}\overline{w}$ and $\tau_j \to 0$, we deduce from the characterization of the Bouligand cone that $\overline{h} + \beta_{\overline{x}}\overline{w} \in K(S;\overline{x})$. Thus,

$$\overline{h} \in K(S;\overline{x}) - \beta_{\overline{x}}\Omega. \tag{41}$$

This completes the proof. □

Lemma 14. *Let $\beta \geq 0$. Assume that Ω is convex compact with $0 \in \Omega$ and that S is epi-Lipschitz at $\overline{x}$. If, in addition, S is assumed to be tangentially regular at $\overline{x}$, then one has*

$$T^C_{\beta,\Omega}(S;\overline{x}) = T^C(S;\overline{x}) - \beta\Omega. \tag{42}$$

Proof. Let $\overline{x} \in S$. Since S is epi-Lipschitz at $\overline{x}$, then S is (β,Ω)-epi-Lipschitz at $\overline{x}$ and hence $H_{\beta,\Omega}(S;\overline{x}) \neq \emptyset$. Fix any $\overline{h} \in H_{\beta,\Omega}(S;\overline{x})$. By Proposition 13, we have $\overline{h} \in K(S;\overline{x}) - \beta\Omega$. Consequently, the tangential regularity of S at $\overline{x}$ implies that

$$\overline{h} \in K(S;\overline{x}) - \beta\Omega = T^C(S;\overline{x}) - \beta\Omega. \tag{43}$$

Since $\overline{h}$ is taken to be arbitrary in $H_{\beta,\Omega}(S;\overline{x})$, then

$$H_{\beta,\Omega}(S;\overline{x}) \subset T^C(S;\overline{x}) - \beta\Omega. \tag{44}$$

Taking the closure of both sides of the previous inclusion and taking into account the fact that $T^C_{\beta,\Omega}(S;\overline{x}) = \operatorname{cl}(H_{\beta,\Omega}(S;\overline{x}))$, we obtain

$$\begin{aligned} T^C_{\beta,\Omega}(S;\overline{x}) &= \operatorname{cl}\left(H_{\beta,\Omega}(S;\overline{x})\right) \subset \operatorname{cl}\left(T^C(S;\overline{x}) - \beta\Omega\right) \\ &= T^C(S;\overline{x}) - \beta\Omega. \end{aligned} \tag{45}$$

Therefore, the proof of this lemma is finished since the reverse inclusion is always true by Lemma 12. □

A direct and very important result on the convexity of the set $H_{\beta,\Omega}(S;\overline{x})$ can be deduced from the previous lemma.

Corollary 15. *Let $\beta \geq 0$, let Ω be convex compact with $0 \in \Omega$, and let S be epi-Lipschitz at $\overline{x}$. If S is tangentially regular at $\overline{x} \in S$, then $H_{\beta,\Omega}(S;\overline{x})$ and $T^C_{\beta,\Omega}(S;\overline{x})$ are both convex.*

Using Lemma 14 we can easily construct many examples of closed sets S and set-valued mappings F for which the tangential condition (2) is not satisfied and the new tangential condition (4) is satisfied.

Example 16. Let $E = \mathbb{R}^2$, $S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1, x_2 \in [-1,0]\}$, $\Omega = -S$ (see Figure 1), and $F(x_1,x_2) = (2x_1 + x_2, 2x_2 + 3x_1 - 1) + \mathbb{B}$. Let $u_1 := (-1,0)$, $u_2 := (0,-1)$, and $u_3 := (-1,-1)$. Simple computations yield the following:

$$\begin{aligned} F(u_1) &= (-1,-4) + \mathbb{B}, \\ T^C(S;u_1) &= \mathbb{R}_+ \times \mathbb{R}_-, \\ T^C_{\beta,\Omega}(S;u_1) &= [-\beta,\infty) \times \mathbb{R}_-, \\ F(u_2) &= (-1,-3) + \mathbb{B}, \\ T^C(S;u_2) &= \mathbb{R}_- \times \mathbb{R}_+, \\ T^C_{\beta,\Omega}(S;u_2) &= \mathbb{R}_- \times [-\beta,\infty), \\ F(u_3) &= (-2,-6) + \mathbb{B}, \\ T^C(S;u_3) &= \mathbb{R}_+ \times \mathbb{R}_+, \\ T^C_{\beta,\Omega}(S;u_3) &= [-\beta,\infty)^2. \end{aligned} \tag{46}$$

Hence, $F(u_i) \cap T^C(S;u_i) = \emptyset$, for $i = 1,2,3$. And $F(x_1,x_2) \cap T^C_{\beta,\Omega}(S;(x_1,x_2)) \neq \emptyset$, for $\beta = 5$ and for any $(x_1,x_2) \in S$. Although the set S is closed convex, the tangential condition (2) is not satisfied while the tangential condition (4) is satisfied.

Consider now another example with S that is a nonconvex tangentially regular set.

Example 17. Let $E = \mathbb{R}^2$ and $S = \{(x_1,x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$ (see Figure 2). Clearly S is a closed nonconvex set in

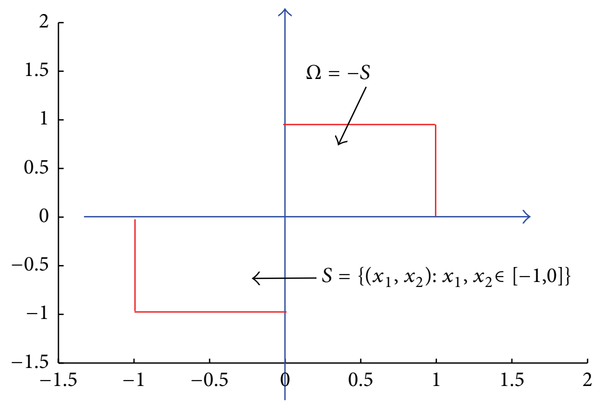

FIGURE 1: The convex set S and the target set Ω in Example 16.

$\mathbb{R}^2$. Also the set S is epi-Lipschitz and tangentially regular. Let $\overline{x} = (\overline{x}_1, \overline{x}_2) \in S$. We can check the following:

$$T^C (S; (\overline{x}_1, \overline{x}_2)) = \begin{cases} \mathbb{R}^2; & 1 < \overline{x}_1^2 + \overline{x}_2^2 < 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \le -\overline{x}_1 u\right\}; & \overline{x}_1^2 + \overline{x}_2^2 = 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \ge -\overline{x}_1 u\right\}; & \overline{x}_1^2 + \overline{x}_2^2 = 1. \end{cases} \tag{47}$$

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in [1,0]\}$ and $\beta \ge 0$. Then using Lemma 14 we obtain

$$T^C_{\beta,\Omega} (S; (\overline{x}_1, \overline{x}_2)) = \begin{cases} \mathbb{R}^2; & 1 < \overline{x}_1^2 + \overline{x}_2^2 < 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \le -\overline{x}_1 u + \beta (\overline{x}_1 + \overline{x}_2)\right\}; & \overline{x}_1^2 + \overline{x}_2^2 = 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \ge -\overline{x}_1 u + \beta (\overline{x}_1 + \overline{x}_2)\right\}; & \overline{x}_1^2 + \overline{x}_2^2 = 1. \end{cases} \tag{48}$$

Also, we can verify that for $\beta = 1$ and for some points in S we have

$$T^C_{\beta,\Omega} (S; (\overline{x}_1, \overline{x}_2)) = T^C (S; (\overline{x}_1, \overline{x}_2)) = \begin{cases} \mathbb{R}^2; & 1 < \overline{x}_1^2 + \overline{x}_2^2 < 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \ge -\overline{x}_1 u\right\}; & \overline{x}_1 \le 0,\ \overline{x}_2 \le 0,\ \overline{x}_1^2 + \overline{x}_2^2 = 4; \\ \left\{(u,v) \in \mathbb{R}^2 : \overline{x}_2 v \ge -\overline{x}_1 u\right\}; & \overline{x}_1 \ge 0,\ \overline{x}_2 \ge 0,\ \overline{x}_1^2 + \overline{x}_2^2 = 1. \end{cases} \tag{49}$$

For the rest of points in S the Clarke tangent cone $T^C(S; (\overline{x}_1, \overline{x}_2))$ is strictly included in $T^C_{\beta,\Omega}(S; (\overline{x}_1, \overline{x}_2))$. Define now the set-valued mapping F as follows:

$$F (x_1, x_2) = (|x_1|, |x_2|) + \frac{3}{2}\mathbb{B}. \tag{50}$$

Clearly F is an u.s.c. set-valued mapping with closed convex values. For this couple F and S we have the following facts:

(i)

$$F (\overline{x}_1, \overline{x}_2) \cap T^C (S; (\overline{x}_1, \overline{x}_2)) \begin{cases} = \emptyset; & \text{for any } \overline{x}_1 \ge 0,\ \overline{x}_2 \ge 0,\ \overline{x}_1^2 + \overline{x}_2^2 = 4; \\ \ne \emptyset; & \text{otherwise}, \end{cases} \tag{51}$$

(ii)

$$F (\overline{x}_1, \overline{x}_2) \cap T^C_{\beta,\Omega} (S; (\overline{x}_1, \overline{x}_2)) \ne \emptyset, \quad \text{for any } (\overline{x}_1, \overline{x}_2) \in S. \tag{52}$$

Thus, for the nonconvex epi-Lipschitz tangentially regular set S, the tangential condition (2) is not satisfied and the tangential condition (4) is satisfied.

3. Existence Results

Throughout this section $E = \mathbb{R}^n$. Let us recall the following important facts needed in our next proofs.

Fact 1 (see [10]). For any finite covering $\cup_{i=1}^k C_i$ of a compact set $S \subset E$, with each C_i being open and bounded, there exists a Lipschitz partition of unity $\{p_1, p_2, \dots, p_k\}$; that is, each p_i is a Lipschitz function and the following hold:

(i) $0 \le p_i(x) \le 1$, $\forall x \in S$, $\forall i = 1, 2, \dots, k$.

(ii) $x \notin C_i \Rightarrow p_i(x) = 0$, $\forall i = 1, 2, \dots, k$.

(iii) $\sum_{i=1}^k p_i(x) = 1$, $\forall x \in S$.

Fact 2 (see [11]). Let S be a compact subset of E and suppose that F is an u.s.c. set-valued mapping on S with images that are nonempty closed convex sets in E. Let $\gamma_i \downarrow 0$ be given. Then there exists a sequence of set-valued mappings $F_j : S \rightrightarrows E$ with closed convex values such that the following hold:

(a) $F(x) \subset F_{j+1}(x) \subset F_j(x) \subset F(\{x + \gamma_j \operatorname{int}(\mathbb{B})\})$, $\forall x \in S$, $\forall j = 1, 2, \dots$;

(b) F_j is l.s.c.

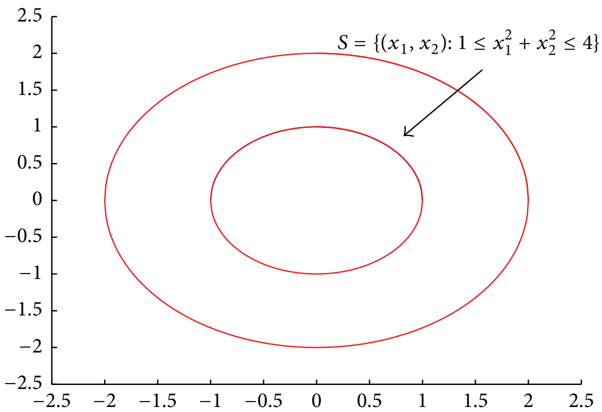

FIGURE 2: The nonconvex set S in Example 17.

Fact 3 (see [1]). Let $S \subset \mathbb{R}^n$ be homeomorphic to a convex compact set and let F be an upper semicontinuous set-valued mapping on S with nonempty closed convex values in E. Assume that S is epi-Lipschitz and suppose that (2) holds. Then F has a zero on S.

In order to prove the first main result we need to prove the following lemma.

Lemma 18. *Let S be a compact set. Assume that S is an epi-Lipschitz compact set, $\bar{\epsilon} > 0$, F is l.s.c. on S, and for any $x \in S$ there exists $\beta_x \ge 0$ such that*

$$F(x) \cap \operatorname{int}\left(T^C_{\beta_x,\Omega}(S;x)\right) \neq \emptyset. \tag{53}$$

Then there exists $\beta \ge 0$ (not depending on x nor on $\bar{\epsilon}$) and a Lipschitz function $f_{\bar{\epsilon}}$ such that

$$f_{\bar{\epsilon}}(x) \in (F(x) + \bar{\epsilon} \operatorname{int} \mathbb{B}) \cap H_{\beta,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{54}$$

Proof. Since S is epi-Lipschitz, we have that by Proposition 9 the set-valued mapping $T^C_{\beta_x,\Omega}(S;\cdot)$ is l.s.c. at any $x \in S$. Let $\epsilon > 0$ be too small so that $\epsilon < \bar{\epsilon}$. Thus, for any $x \in S$ and any $y \in F(x) \cap \operatorname{int}(T^C_{\beta_x,\Omega}(S;x)) \neq \emptyset$, the l.s.c. of both F and $T^C_{\beta_x,\Omega}(S;\cdot)$ imply the existence of $\delta(x,y,\epsilon) > 0$ such that

$$\text{we have } y \in \left(F\left(x'\right) + \epsilon \operatorname{int}(\mathbb{B})\right) \cap \left(\operatorname{int} T^C_{\beta_x,\Omega}\left(S;x'\right)\right) \quad \forall x' \in \operatorname{int}(\mathbb{B}(x,\delta(x,y,\epsilon))). \tag{55}$$

Hence, the family of open balls

$$\operatorname{int}(\mathbb{B}(x,\delta(x,y,\epsilon))) := \left\{x + \delta(x,y,\epsilon) \operatorname{int} \mathbb{B}; \epsilon \in (0,\bar{\epsilon}), x \in S, \ y \in F(x) \cap \operatorname{int}\left(T^C_{\beta_x,\Omega}(S;x)\right)\right\} \tag{56}$$

forms an open covering of S. By compactness of S there exists a finite subcover $\cup_{i=1}^k \operatorname{int} \mathbb{B}(x_i;\delta(x_i,y_i,\epsilon_i))$ of S and let $\{p_1,p_2,\ldots,p_k\}$ be a Lipschitz partition of unity subordinate to this subcover (by Fact 1). Define now the Lipschitz function

$$f_{\bar{\epsilon}}(z) := \sum_{i=1}^k p_i(z)\, y_i. \tag{57}$$

Then

$$\begin{aligned} &p_i(z) \neq 0 \Longrightarrow \\ &\quad z \in \operatorname{int} \mathbb{B}(x_i;\delta(x_i,y_i,\epsilon_i)) \Longrightarrow \\ &\quad y_i \in (F(z) + \epsilon_i \operatorname{int}(\mathbb{B})) \cap \operatorname{int}\left(T^C_{\beta_{x_i},\Omega}(S;z)\right). \end{aligned} \tag{58}$$

Therefore, the convexity of both (β_{x_i},Ω)-Clarke tangent sets (by Corollary 15) and images of the set-valued mappings F imply that the selection f satisfies

$$f_{\bar{\epsilon}}(z) \in (F(z) + \epsilon_i \operatorname{int}(\mathbb{B})) \cap \operatorname{int}\left(T^C_{\beta_{x_i},\Omega}(S;z)\right) \quad \forall z \in S, \ \exists i \in \{1,2,\ldots,k\}. \tag{59}$$

Set $\beta := \max\{\beta_{x_i};\ i \in \{1,\ldots,k\}\}$. Then the fact that $\epsilon_i < \bar{\epsilon}$ and $\beta \ge \beta_{x_i}$ for any i and the fact that Ω is balanced ensure the inclusion

$$\begin{aligned} T^C_{\beta_{x_i},\Omega}(S;z) &\subset T^C_{\beta,\Omega}(S;z), \\ F(z) + \epsilon_i \operatorname{int}(\mathbb{B}) &\subset F(z) + \bar{\epsilon} \operatorname{int}(\mathbb{B}). \end{aligned} \tag{60}$$

By Proposition 8 we have $\operatorname{int}(T^C_{\beta,\Omega}(S;z)) = H_{\beta,\Omega}(S;z)$, and consequently we obtain

$$f_{\bar{\epsilon}}(z) \in (F(z) + \bar{\epsilon} \operatorname{int}(\mathbb{B})) \cap H_{\beta,\Omega}(S;z), \quad \forall z \in S, \tag{61}$$

which completes the proof. □

The following result can be seen as an approximate equilibria result. It will be used to prove the main result of the paper.

Theorem 19. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ which is epi-Lipschitz and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Let Ω be a convex compact set in $\mathbb{R}^n$ with $0 \in \Omega$. Assume that for any $x \in S$ there exist some $\beta_x \ge 0$ such that*

$$F(x) \cap T^C_{\beta_x,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{62}$$

If S is tangentially regular, then there exist $\beta \ge 0$ and $x^* \in S$ such that

$$0 \in F(x^*) + \beta\Omega. \tag{63}$$

Proof.

Case 1. First assume that F is l.s.c. on S and

$$F(x) \cap \operatorname{int}\left(T^C_{\beta_x,\Omega}(S;x)\right) \neq \emptyset, \quad \forall x \in S. \tag{64}$$

Then for any $\epsilon > 0$ there exists, by the previous lemma, a Lipschitz function f_ϵ and a constant $\beta \ge 0$ such that

$$f_\epsilon(z) \in (F(z) + \epsilon \operatorname{int}(\mathbb{B})) \cap H_{\beta,\Omega}(S;z) \quad \forall z \in S. \tag{65}$$

Let us introduce the ordinary differential equation

$$\dot{x}(t) = f_\epsilon(x(t)). \tag{66}$$

In view of (65) we have for any $z \in S$

$$f_\epsilon(z) \in H_{\beta,\Omega}(S;z); \tag{67}$$

that is, for any $z \in S$ the set S is (β,Ω)-epi-Lipschitz at z in the direction $f_\epsilon(z)$. Use now Proposition 13 to deduce that

$$f_\epsilon(z) \in K(S;z) - \beta\Omega. \tag{68}$$

Since S is tangentially regular, we get

$$T^C(S;z) \cap (f_\epsilon(z) + \beta\Omega) \neq \emptyset, \quad \forall z \in S. \tag{69}$$

Define the set-valued mapping $F_\epsilon(x) := f_\epsilon(x) + \beta\Omega$. Clearly this set-valued mapping satisfies the hypothesis of Fact 3 and so there exists $x_\epsilon^* \in S$ such that $0 \in F_\epsilon(x_\epsilon^*) := f_\epsilon(x_\epsilon^*) + \beta\Omega$. Clearly,

$$0 \in f_\epsilon(x_\epsilon^*) + \beta\Omega \subset F(x_\epsilon^*) + \epsilon \operatorname{int}(\mathbb{B}) + \beta\Omega. \tag{70}$$

Case 2. Assume now that F is u.s.c. Let $\gamma_j \downarrow 0$ as $j \to \infty$ and let F_j be a sequence of l.s.c. approximations of F as in Fact 2. Let $\epsilon_j \downarrow 0$ as $j \to \infty$ and let $\widetilde{F}_j(x) := F_j(x) + \gamma_j\mathbb{B}$. Clearly $\widetilde{F}_j$ is l.s.c. and $\widetilde{F}_j(x) \cap \operatorname{int} T^C_{\beta_x,\Omega}(S;x) \neq \emptyset$, $\forall x \in S$. Indeed, by Part (b) in Fact 2 we have

$$\emptyset \neq F(x) \cap T^C_{\beta_x,\Omega}(S;x) \subset F_j(x) \cap T^C_{\beta_x,\Omega}(S;x), \tag{71}$$

and so obviously we obtain

$$\left(F_j(x) + \gamma_j\mathbb{B}\right) \cap \operatorname{int} T^C_{\beta_x,\Omega}(S;x) \neq \emptyset. \tag{72}$$

And hence

$$\widetilde{F}_j(x) \cap \operatorname{int} T^C_{\beta_x,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{73}$$

Applying Case 1 for all $\widetilde{F}_j$ we obtain a constant $\beta \geq 0$ and an element $x_j^* \in S$ such that

$$0 \in \widetilde{F}_j(x_j^*) + \epsilon_j \operatorname{int}(\mathbb{B}) + \beta\Omega. \tag{74}$$

Consequently, from the monotonicity of F_j (by Part (a) in Fact 1), we have

$$0 \in \widetilde{F}_j(x_i^*) + \epsilon_i \operatorname{int}(\mathbb{B}) + \beta\Omega, \quad \forall i > j. \tag{75}$$

Since S is assumed to be compact, we can extract a subsequence of x_j^* (still denoted by x_j^*) converging to a limit $x^* \in S$. Let $x_{i_j}^*$ be such that $i_j > j$. Then

$$\begin{aligned} 0 &\in \widetilde{F}_j\left(x_{i_j}^*\right) + \epsilon_{i_j} \operatorname{int}(\mathbb{B}) + \beta\Omega \\ &\subset F_j\left(x_{i_j}^*\right) + \gamma_j\mathbb{B} + \epsilon_{i_j} \operatorname{int}(\mathbb{B}) + \beta\Omega \\ &\subset F_j\left(x_{i_j}^*\right) + \left(2\gamma_j + \epsilon_{i_j}\right) \operatorname{int}(\mathbb{B}) + \beta\Omega \end{aligned} \tag{76}$$

and therefore by Part (a) in Fact 1 we get

$$0 \in F\left(\left\{x_{i_j}^* + \gamma_j \operatorname{int}(\mathbb{B})\right\}\right) + \left(2\gamma_j + \epsilon_{i_j}\right) \operatorname{int}(\mathbb{B}) + \beta\Omega. \tag{77}$$

Upon letting $j \to \infty$ we obtain $0 \in F(x^*) + \beta\Omega$ and hence the proof is complete. □

Observe that β in the previous theorem cannot be controlled since it depends on the pointwise constants β_x in (4). However, if we assume that the tangential condition (4) is satisfied with a uniform β (i.e., β does not depend on x and it is the same for any $x \in S$), then we get the following first corollary in which the constant is the same satisfying (4).

Corollary 20. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Let Ω be a convex compact set in $\mathbb{R}^n$ with $0 \in \Omega$. Assume that S is epi-Lipschitz and for some $\beta \geq 0$ one has*

$$F(x) \cap T^C_{\beta,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{78}$$

If S is tangentially regular, then there exists $x^* \in S$ such that

$$0 \in F(x^*) + \beta\Omega. \tag{79}$$

Using this corollary we prove our main result establishing an existence result of exact equilibria of F on S.

Theorem 21. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Let Ω be a convex compact set in $\mathbb{R}^n$ with $0 \in \Omega$. Assume that S is epi-Lipschitz and for some $\beta \geq 0$ one has*

$$F(x) \cap T^C_{\beta,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{80}$$

If S is tangentially regular, then there exists $x^* \in S$ such that

$$0 \in F(x^*). \tag{81}$$

Proof. We proceed by approximation. Fix any $\epsilon > 0$. First observe the following fact due to the balanced property of Ω:

$$\begin{aligned} T^C_{\epsilon\beta,\Omega}(S;x) &= T^C(S;x) - \epsilon\beta\Omega = T^C(S;x) - \beta\Omega \\ &= T^C_{\beta,\Omega}(S;x). \end{aligned} \tag{82}$$

Then (80) ensures

$$F(x) \cap T^C_{\epsilon\beta,\Omega}(S;x) \neq \emptyset, \quad \forall x \in S. \tag{83}$$

Applying now Theorem 19 we obtain for any $\epsilon > 0$ the existence of some point $x_\epsilon^* \in S$ with $0 \in F(x_\epsilon^*) + \epsilon\beta\Omega$. Using the fact that S is compact we can extract a subsequence of $\{x_\epsilon^*\}_\epsilon$ converging to some limit $x^* \in S$ and since obviously our assumptions on F ensure that the limit satisfies $0 \in F(x^*)$, then the proof is complete. □

We apply this existence result to the following two examples for which we cannot apply the results in [1] because the tangential condition (2) is not satisfied.

Example 22. Let $E = \mathbb{R}^2$, $S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1,x_2 \in [-1,0]\}$, $\Omega = -S$, and $F(x_1,x_2) = (2x_1 + x_2, 2x_2 + 3x_1 - 1) + \mathbb{B}$. Since $F(x_1,x_2) \cap T^C_{\beta,\Omega}(S;(x_1,x_2)) \neq \emptyset$ for any $(x_1,x_2) \in S$ (by Example 16) and since all the assumptions on F and S in Theorem 21 are fulfilled, then there exists some $x^* \in S$ with $(0,0) \in F(x^*)$. In this example the equilibrium is $x^* = (0,0) \in S$ and it is unique.

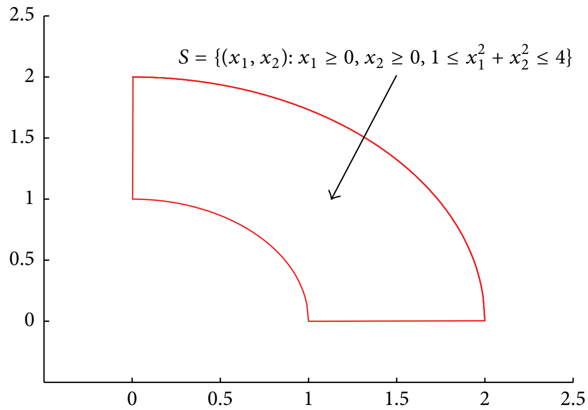

Figure 3: The nonconvex set S in Example 23.

Example 23. Let $E = \mathbb{R}^2$, $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, 1 \leq x_1^2 + x_2^2 \leq 4\}$, and $F(x_1, x_2) = (x_1, x_2) + (3/2)\mathbb{B}$. The set S is the intersection of the set given in Example 17 with the first quarter in $\mathbb{R}^2$ (see Figure 3). Using the computations presented in Example 17 we can check that, for any $(x_1, x_2) \in S$, $F(x_1, x_2) \cap T^C_{\beta,\Omega}(S; (x_1, x_2)) \neq \emptyset$. Also, the set S is homeomorphic to a convex compact set. Indeed, S can be mapped continuously and with continuous inverse to the segment $\{(0, x_2) : x_2 \in [1, 2]\}$ by projection. Thus, all the assumptions of Theorem 21 are satisfied and so there exists $x^* \in S$ such that $(0, 0) \in F(x^*)$. In this example we do not have the uniqueness of the equilibria x^* and there is an infinity of them. Indeed, for any $(x_1, x_2) \in S$ with $x_1^2 + x_2^2 = 1$ we have $(0, 0) \in F(x_1, x_2)$.

Let us now apply our main result in Theorem 21 for the existence of fixed points for set-valued mappings on nonconvex sets. It extends Theorems 1.7 and 1.8 in [1] from the case of S and F satisfying the tangential condition (80) with $\beta = 0$ to the general case with any $\beta \geq 0$.

Theorem 24. *Let S be homeomorphic to a convex compact set in $\mathbb{R}^n$ and let F be an u.s.c. set-valued mapping with nonempty closed convex values. Let Ω be a convex compact set in $\mathbb{R}^n$ with $0 \in \Omega$. Assume that S is epi-Lipschitz and tangentially regular and for some $\beta \geq 0$ one has*

$$G(x) \cap \left[x + T^C_{\beta,\Omega}(S; x)\right] \neq \emptyset, \quad \forall x \in S. \tag{84}$$

Then G has a fixed point in S; that is, there exists $x^* \in S$ such that $x^* \in G(x^*)$.

Proof. It follows directly from Theorem 21 by taking $F(x) := G(x) - x$. □

Conflict of Interests

The author, Professor Messaoud Bounkhel, declares that there is no conflict of interests regarding the publication of this paper.

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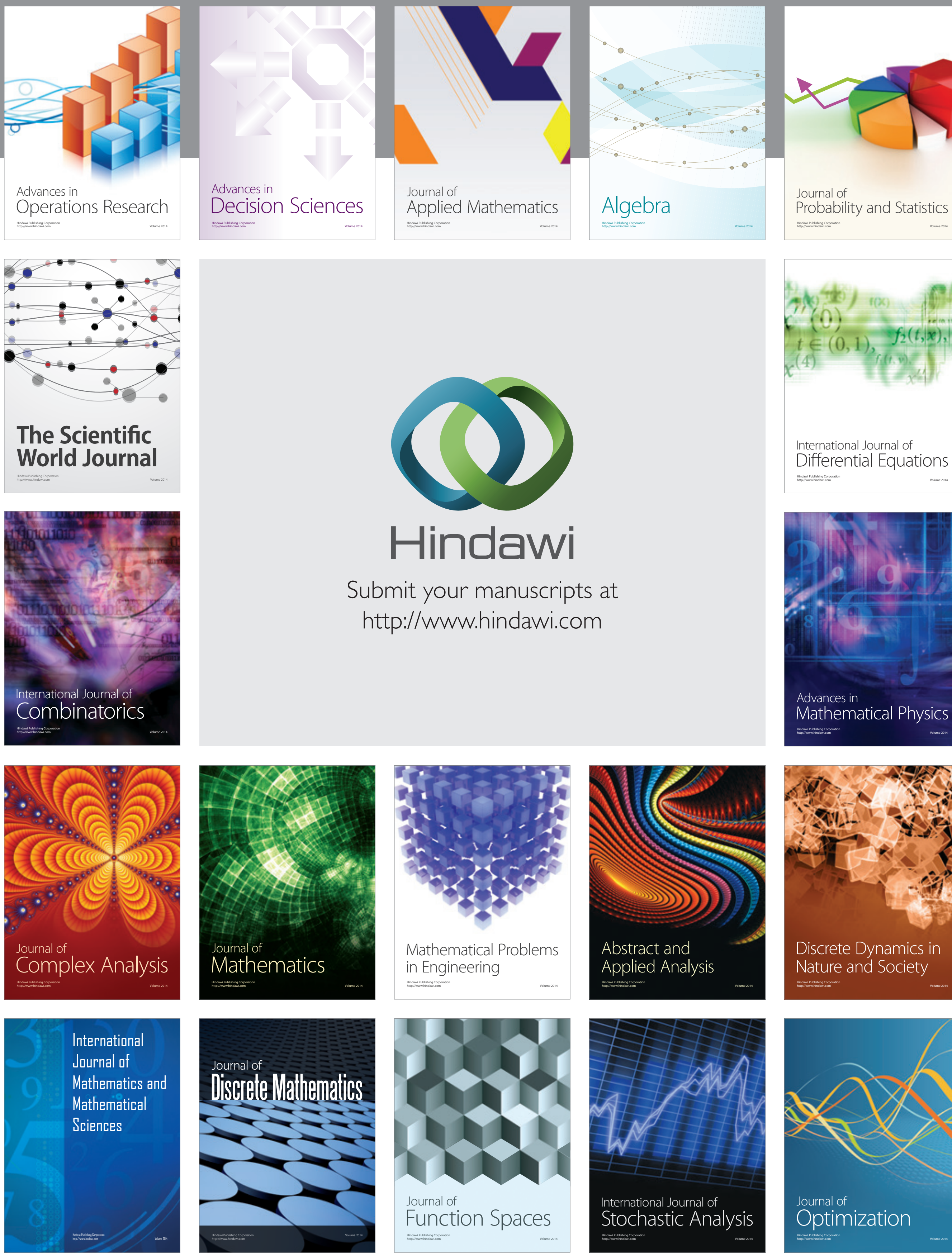
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