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Research Article

A Half-Discrete Hilbert-Type Inequality in the Whole Plane with Multiparameters

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By the use of weight functions and technique of real analysis, a new half-discrete Hilbert-type inequality in the whole plane with multiparameters and the best possible constant factor is given. Furthermore, the equivalent forms, two kinds of particular inequalities, and the operator expressions with the norm are considered.

1. Introduction

If $p > 1$, $1/p + 1/q = 1$, $a_m, b_n > 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following discrete Hardy-Hilbert inequality (cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \quad (1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible one. Assuming that $f(x)$, $g(y) \geq 0$, satisfying $0 < \int_0^{\infty} f^p(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^q(y)\,dy < \infty$, we have the following Hardy-Hilbert integral inequality (cf. [2]):

$$\iint_0^{\infty}\frac{f(x)\,g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{1/p}\left(\int_0^{\infty} g^q(y)\,dy\right)^{1/q}, \quad (2)$$

with the best possible constant factor $\pi/\sin(\pi/p)$. Recently, half-discrete Hardy-Hilbert's inequality with the same best possible constant factor was given as follows [3]:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{b_n f(x)}{x+n}dx < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}. \quad (3)$$

Inequalities (1), (2), and (3) are important in analysis and its applications (cf. [2, 4, 5]).

Noticing that inequalities (1) and (2) are with homogenous kernels of degree −1, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree negative numbers and some parameters is given in [6]. Recently, some inequalities with the homogenous kernels of degree 0 and nonhomogenous kernels have been studied in [7, 8]. The other kinds of Hilbert-type inequalities are provided in [9–19]. All of the above inequalities are built in the quarter plane of the first quadrant.

In 2007, Yang [20] gave a new Hilbert-type integral inequality in the whole plane as follows:

$$\iint_{-\infty}^{\infty}\frac{f(x)\,g(y)}{(1+e^{x+y})^{\lambda}}dx\,dy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\cdot\left(\int_{-\infty}^{\infty} e^{-\lambda x} f^2(x)\,dx\int_{-\infty}^{\infty} e^{-\lambda y} g^2(y)\,dy\right)^{1/2}, \quad (4)$$

where the constant factor $B(\lambda/2, \lambda/2)$ $(\lambda > 0)$ is the best possible one ($B(u, v)$ is the beta function). And Zeng et al. [21, 22] also published some new Hilbert-type integral inequalities in the whole plane.

In this paper, by the use of weight functions and technique of real analysis, a new half-discrete Hilbert-type inequality in the whole plane with the best possible constant factor is built as follows: for $\rho > 0$, $0 < \sigma < \gamma$ $(\sigma \le 1)$,

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{f(x) b_n}{1+\rho|nx|^{\gamma}} dx < \frac{2\pi}{\gamma\rho^{\sigma/\gamma} \sin\pi(\sigma/\gamma)} \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)\, dx\right]^{1/p} \cdot \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q\right]^{1/q}. \quad (5)$$

Furthermore, an extension of (5) with multiparameters is given. The equivalent forms, two kinds of particular inequalities, and the operator expressions with the norm are considered.

2. Some Lemmas

In the following, we agree that $\delta \in \{-1, 1\}$, $\alpha, \beta \in (0, \pi)$, $\rho, \gamma > 0$:

$$k(x, y) := \frac{1}{1+\rho\left[(|y| + y\cos\beta)/(|x| + x\cos\alpha)^{\delta}\right]^{\gamma}} \quad (x, y \in \mathbf{R}\setminus\{0\}), \quad (6)$$

wherefrom

$$k(x, y) = \frac{1}{1+\rho\left[y(1+\cos\beta)/(|x| + x\cos\alpha)^{\delta}\right]^{\gamma}} \quad (y > 0),$$
$$k(x, y) = \frac{1}{1+\rho\left\{(|y| + y\cos\beta)/[x(1+\cos\alpha)]^{\delta}\right\}^{\gamma}} \quad (x > 0),$$
$$k(-x, y) = \frac{1}{1+\rho\left\{(|y| + y\cos\beta)/[x(1-\cos\alpha)]^{\delta}\right\}^{\gamma}} \quad (x > 0), \quad (7)$$
$$k(x, -y) = \frac{1}{1+\rho\left[y(1-\cos\beta)/(|x| + x\cos\alpha)^{\delta}\right]^{\gamma}} \quad (y > 0).$$

Lemma 1 (cf. [23]). *Suppose that $g(t)$ (> 0) is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty)$ $(n_0 \in \mathbf{N})$, satisfying $\int_0^{\infty} g(t)dt \in \mathbf{R}_+$. One has*

$$\int_1^{\infty} g(t)\, dt < \sum_{n=1}^{\infty} g(n) < \int_0^{\infty} g(t)\, dt. \quad (8)$$

Lemma 2. *One defines two weight functions $\omega(\sigma, n)$ and $\varpi(\sigma, x)$ as follows:*

$$\omega(\sigma, n) := \int_{-\infty}^{\infty} k(x, n) \frac{(|n| + n\cos\beta)^{\sigma}\, dx}{(|x| + x\cos\alpha)^{1+\delta\sigma}} \quad (|n| \in \mathbf{N}), \quad (9)$$

$$\varpi(\sigma, x) := \sum_{|n|=1}^{\infty} k(x, n) \frac{(|x| + x\cos\alpha)^{-\delta\sigma}}{(|n| + n\cos\beta)^{1-\sigma}} \quad (x \in \mathbf{R}\setminus\{0\}), \quad (10)$$

where $\sum_{|n|=1}^{\infty} \cdots = \sum_{n=-1}^{-\infty} \cdots + \sum_{n=1}^{\infty} \cdots$. Then, for $0 < \sigma < \gamma$, one has

$$\omega(\sigma, n) = k_{\alpha}(\sigma) := \frac{2\pi \csc^2\alpha}{\gamma\rho^{\sigma/\gamma} \sin\pi(\sigma/\gamma)} \in \mathbf{R}_+ \quad (|n| \in \mathbf{N}); \quad (11)$$

for $0 < \sigma < \gamma (\sigma \le 1)$, one has

$$k_{\beta}(\sigma)(1 - \theta(\sigma, x)) < \varpi(\sigma, x) < k_{\beta}(\sigma) \quad (x \in \mathbf{R}\setminus\{0\}), \quad (12)$$

where

$$\theta(\sigma, x) := \frac{\sin\pi(\sigma/\gamma)}{\pi} \int_0^{\rho[(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}} \frac{u^{\sigma/\gamma-1}}{1+u} du = O\left(\frac{1}{(|x| + x\cos\alpha)^{\delta\sigma}}\right) \in (0, 1). \quad (13)$$

Proof. (i) We have

$$\begin{aligned}\omega(\sigma, n) &= \int_{-\infty}^{0} k(x, n) \frac{(|n| + n\cos\beta)^{\sigma}}{[x(\cos\alpha - 1)]^{1+\delta\sigma}} dx + \int_0^{\infty} k(x, n) \frac{(|n| + n\cos\beta)^{\sigma}}{[x(\cos\alpha + 1)]^{1+\delta\sigma}} dx \\ &= \int_0^{\infty} k(-x, n) \frac{(|n| + n\cos\beta)^{\sigma}}{[x(1 - \cos\alpha)]^{1+\delta\sigma}} dx + \int_0^{\infty} k(x, n) \frac{(|n| + n\cos\beta)^{\sigma}}{[x(1 + \cos\alpha)]^{1+\delta\sigma}} dx.\end{aligned} \quad (14)$$

Setting $u=\rho\{(|n|+n\cos\beta)/[x(1-\cos\alpha)]^{\delta}\}^{\gamma}$ ($u=\rho\{(|n|+n\cos\beta)/[x(1+\cos\alpha)]^{\delta}\}^{\gamma}$) in the above first (second) integral, by simplifications, we find

$$\begin{aligned}\omega(\sigma,n) &= \frac{1}{(1-\cos\alpha)\gamma\rho^{\sigma/\gamma}}\int_0^{\infty}\frac{1}{1+u}u^{\sigma/\gamma-1}du \\ &\quad + \frac{1}{(1+\cos\alpha)\gamma\rho^{\sigma/\gamma}}\int_0^{\infty}\frac{1}{1+u}u^{\sigma/\gamma-1}du \\ &= \frac{2\pi\csc^2\alpha}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}.\end{aligned} \tag{15}$$

Hence, we have (11).

(ii) We have

$$\begin{aligned}\varpi(\sigma,x) &= \sum_{n=-1}^{-\infty}k(x,n)\frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(|n|+n\cos\beta)^{1-\sigma}} \\ &\quad + \sum_{n=1}^{\infty}k(x,n)\frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(|n|+n\cos\beta)^{1-\sigma}} \\ &= \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{k(x,-n)}{n^{1-\sigma}} \\ &\quad + \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{k(x,n)}{n^{1-\sigma}}.\end{aligned} \tag{16}$$

Since, for $0<\sigma<\gamma$ ($\sigma\le 1$), both $k(x,-y)/y^{1-\sigma}$ and $k(x,y)/y^{1-\sigma}$ are strictly decreasing in $(0,\infty)$, satisfying

$$\begin{aligned}\frac{d}{dy}\frac{k(x,-y)}{y^{1-\sigma}} &< 0, \\ \frac{d}{dy}\frac{k(x,y)}{y^{1-\sigma}} &< 0,\end{aligned} \tag{17}$$

by (16) and (8), we obtain

$$\begin{aligned}\varpi(\sigma,x) &< \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\int_0^{\infty}\frac{k(x,-y)}{y^{1-\sigma}}dy \\ &\quad + \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\int_0^{\infty}\frac{k(x,y)}{y^{1-\sigma}}dy.\end{aligned} \tag{18}$$

Setting $u=\rho[y(1-\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}$ ($u=\rho[y(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}$) in the above first (second) integral, by simplifications, we find

$$\varpi(\sigma,x) < \frac{2\pi\csc^2\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)} = k_{\beta}(\sigma). \tag{19}$$

By (16) and (8), we still have

$$\begin{aligned}\varpi(\sigma,x) &> \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\int_1^{\infty}\frac{k(x,-y)}{y^{1-\sigma}}dy \\ &\quad + \frac{(|x|+x\cos\alpha)^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\int_1^{\infty}\frac{k(x,y)}{y^{1-\sigma}}dy.\end{aligned} \tag{20}$$

Setting $u=\rho[y(1-\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}$ ($u=\rho[y(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}$) in the above first (second) integral, by simplifications, we obtain

$$\begin{aligned}\varpi(\sigma,x) &> \frac{1}{\gamma\rho^{\sigma/\gamma}(1-\cos\beta)} \\ &\quad \cdot \int_{\rho[(1-\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}}^{\infty}\frac{u^{\sigma/\gamma-1}}{1+u}du \\ &\quad + \frac{1}{\gamma\rho^{\sigma/\gamma}(1+\cos\beta)} \\ &\quad \cdot \int_{\rho[(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}}^{\infty}\frac{u^{\sigma/\gamma-1}}{1+u}du \ge \frac{2\csc^2\beta}{\gamma\rho^{\sigma/\gamma}} \\ &\quad \cdot \int_{\rho[(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}}^{\infty}\frac{u^{\sigma/\gamma-1}}{1+u}du = k_{\beta}(\sigma) \\ &\quad \cdot (1-\theta(\sigma,x)) > 0,\end{aligned} \tag{21}$$

where $\theta(\sigma,x)$ is indicated by (13). We find

$$\begin{aligned}0 &< \theta(\sigma,x) \\ &\le \frac{\sin\pi(\sigma/\gamma)}{\pi}\int_0^{\rho[(1+\cos\beta)/(|x|+x\cos\alpha)^{\delta}]^{\gamma}}u^{\sigma/\gamma-1}du \\ &= \frac{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}{\sigma\pi}\left[\frac{1+\cos\beta}{(|x|+x\cos\alpha)^{\delta}}\right]^{\sigma},\end{aligned} \tag{22}$$

and then (12) and (13) follow. □

Lemma 3. *For $\varepsilon>0$, setting $E_{\delta}:=\{x\in\mathbf{R}\setminus\{0\};1/(|x|+x\cos\alpha)^{\delta}\ge 1\}$, one has*

$$H_{\delta} := \int_{E_{\delta}}\frac{1}{(|x|+x\cos\alpha)^{1+\delta\varepsilon}}dx = \frac{2}{\varepsilon}\csc^2\alpha. \tag{23}$$

Proof. Setting

$$\begin{aligned}E_{\delta}^{+} &:= \left\{x>0;\frac{1}{[x(1+\cos\alpha)]^{\delta}}\ge 1\right\}, \\ E_{\delta}^{-} &:= \left\{x<0;\frac{1}{[-x(1-\cos\alpha)]^{\delta}}\ge 1\right\},\end{aligned} \tag{24}$$

it follows that $E_\delta = E_\delta^+ \cup E_\delta^-$. We find

$$H_\delta = \int_{E_\delta^+} \frac{dx}{[x(1+\cos\alpha)]^{1+\delta\varepsilon}} + \int_{E_\delta^-} \frac{dx}{[-x(1-\cos\alpha)]^{1+\delta\varepsilon}}. \quad (25)$$

Setting $u = [x(1+\cos\alpha)]^\delta$ ($u = [-x(1-\cos\alpha)]^\delta$) in the above first (second) integral, we obtain

$$H_\delta = \left(\frac{1}{1+\cos\alpha} + \frac{1}{1-\cos\alpha}\right)\int_1^\infty \frac{du}{u^{1+\varepsilon}} = \frac{2}{\varepsilon}\csc^2\alpha. \quad (26)$$

Hence we have (23). □

Lemma 4. *For $\varepsilon > 0$, setting*

$$H_\varepsilon(\beta) := \sum_{|n|=1}^{\infty} \frac{1}{(|n| + n\cos\beta)^{1+\varepsilon}}, \quad (27)$$

we have

$$H_\varepsilon(\beta) < \frac{1}{\varepsilon}\left(2\csc^2\beta + o(1)\right)(\varepsilon+1) \quad (\varepsilon \longrightarrow 0^+). \quad (28)$$

Proof. We have

$$H_\varepsilon(\beta) = \sum_{n=-1}^{-\infty} \frac{1}{[n(\cos\beta - 1)]^{1+\varepsilon}} + \sum_{n=1}^{\infty} \frac{1}{[n(\cos\beta + 1)]^{1+\varepsilon}} = \left[\frac{1}{(1-\cos\beta)^{1+\varepsilon}} + \frac{1}{(1+\cos\beta)^{1+\varepsilon}}\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \quad (29)$$

By (29) and (8), we find

$$H_\varepsilon(\beta) = \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right] \cdot \left(1 + \sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}}\right) < \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right] \cdot \left(1 + \int_1^\infty \frac{dy}{y^{1+\varepsilon}}\right) = \frac{1}{\varepsilon}\left(2\csc^2\beta + o(1)\right)(\varepsilon+1) \quad (\varepsilon \longrightarrow 0^+). \quad (30)$$

Hence, we have (28). □

3. Main Results

Theorem 5. *Suppose that $p > 1$, $1/p + 1/q = 1$, $0 < \sigma < \gamma$ ($\sigma \le 1$), and*

$$k_{\alpha,\beta}(\sigma) := k_\alpha^{1/q}(\sigma)\, k_\beta^{1/p}(\sigma) = \frac{2\pi\csc^{2/q}\alpha\csc^{2/p}\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}. \quad (31)$$

If $f(x), b_n \ge 0$, satisfying

$$0 < \int_{-\infty}^{\infty} (|x| + x\cos\alpha)^{p(1+\delta\sigma)-1} f^p(x)\,dx < \infty, \qquad 0 < \sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\sigma)-1} b_n^q < \infty, \quad (32)$$

then we have the following equivalent inequalities:

$$I := \sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)\,b_n}{1 + \rho\left(\left(|n| + n\cos\beta\right)/\left(|x| + x\cos\alpha\right)^\delta\right)^\gamma}dx < k_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty}(|x| + x\cos\alpha)^{p(1+\delta\sigma)-1} f^p(x)\,dx\right]^{1/p} \cdot \left[\sum_{|n|=1}^{\infty}(|n| + n\cos\beta)^{q(1-\sigma)-1} b_n^q\right]^{1/q}, \quad (33)$$

$$J_1 := \left\{\sum_{|n|=1}^{\infty}(|n| + n\cos\beta)^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{f(x)\,dx}{1 + \rho\left(\left(|n| + n\cos\beta\right)/\left(|x| + x\cos\alpha\right)^\delta\right)^\gamma}\right]^p\right\}^{1/p} < k_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty}(|x| + x\cos\alpha)^{p(1+\delta\sigma)-1} f^p(x)\,dx\right]^{1/p}, \quad (34)$$

$$J_2 := \left\{\int_{-\infty}^{\infty}\frac{1}{(|x| + x\cos\alpha)^{q\delta\sigma+1}}\left[\sum_{|n|=1}^{\infty}\frac{b_n}{1 + \rho\left(\left(|n| + n\cos\beta\right)/\left(|x| + x\cos\alpha\right)^\delta\right)^\gamma}\right]^q dx\right\}^{1/q} < k_{\alpha,\beta}(\sigma)\left[\sum_{|n|=1}^{\infty}(|n| + n\cos\beta)^{q(1-\sigma)-1} b_n^q\right]^{1/q}. \quad (35)$$

In particular, for $\alpha = \beta = \pi/2$, we have the following equivalent inequalities:

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{f(x) b_n}{1+\rho\left(|n| / |x|^{\delta}\right)^{\gamma}} dx < \frac{2\pi}{\gamma\rho^{\sigma/\gamma} \sin(\pi\sigma/\gamma)} \left[\int_{-\infty}^{\infty} |x|^{p(1+\delta\sigma)-1} \cdot f^p(x)\, dx\right]^{1/p} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q\right]^{1/q}, \tag{36}$$

$$\left\{\sum_{|n|=1}^{\infty} |n|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \frac{f(x)}{1+\rho\left(|n| / |x|^{\delta}\right)^{\gamma}} dx\right]^p\right\}^{1/p} < \frac{2\pi}{\gamma\rho^{\sigma/\gamma} \sin(\pi\sigma/\gamma)} \left[\int_{-\infty}^{\infty} |x|^{p(1+\delta\sigma)-1} \cdot f^p(x)\, dx\right]^{1/p}, \tag{37}$$

$$\left\{\int_{-\infty}^{\infty} \frac{1}{|x|^{q\delta\sigma+1}} \left[\sum_{|n|=1}^{\infty} \frac{b_n}{1+\rho\left(|n| / |x|^{\delta}\right)^{\gamma}}\right]^q dx\right\}^{1/q} < \frac{2\pi}{\gamma\rho^{\sigma/\gamma} \sin(\pi\sigma/\gamma)} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q\right]^{1/q}. \tag{38}$$

Proof. By Hölder's inequality (cf. [24]) and (9), we find

$$\begin{aligned}\left(\int_{-\infty}^{\infty} k(x,n) f(x)\, dx\right)^p &= \left[\int_{-\infty}^{\infty} k(x,n) \frac{(|x|+x\cos\alpha)^{(1+\delta\sigma)/q}}{(|n|+n\cos\beta)^{(1-\sigma)/p}} f(x) \cdot \frac{(|n|+n\cos\beta)^{(1-\sigma)/p}}{(|x|+x\cos\alpha)^{(1+\delta\sigma)/q}} dx\right]^p \leq \int_{-\infty}^{\infty} k(x,n) \\ &\cdot \frac{(|x|+x\cos\alpha)^{(1+\delta\sigma)(p-1)}}{(|n|+n\cos\beta)^{1-\sigma}} f^p(x)\, dx \cdot \left[\int_{-\infty}^{\infty} k(x,n) \frac{(|n|+n\cos\beta)^{(1-\sigma)(q-1)}}{(|x|+x\cos\alpha)^{1+\delta\sigma}} dx\right]^{p-1} \\ &= \frac{\omega^{p-1}(\sigma,n)}{(|n|+n\cos\beta)^{p\sigma-1}} \int_{-\infty}^{\infty} k(x,n) \cdot \frac{(|x|+x\cos\alpha)^{(1+\delta\sigma)(p-1)}}{(|n|+n\cos\beta)^{1-\sigma}} f^p(x)\, dx.\end{aligned} \tag{39}$$

Then by (11) and Lebesgue term-by-term integration theorem (cf. [25]), in view of (10), we find

$$\begin{aligned}J_1 &\leq k_{\alpha}^{1/q}(\sigma) \left[\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} k(x,n) \cdot \frac{(|x|+x\cos\alpha)^{(1+\delta\sigma)(p-1)}}{(|n|+n\cos\beta)^{1-\sigma}} f^p(x)\, dx\right]^{1/p} = k_{\alpha}^{1/q}(\sigma) \\ &\cdot \left[\int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} k(x,n) \cdot \frac{(|x|+x\cos\alpha)^{(1+\delta\sigma)(p-1)}}{(|n|+n\cos\beta)^{1-\sigma}} f^p(x)\, dx\right]^{1/p} = k_{\alpha}^{1/q}(\sigma) \\ &\cdot \left[\int_{-\infty}^{\infty} \varpi(\sigma,x) (|x|+x\cos\alpha)^{p(1+\delta\sigma)-1} f^p(x)\, dx\right]^{1/p}.\end{aligned} \tag{40}$$

Then, by (12), we have (34).

By Hölder's inequality (cf. [24]), we have

$$\begin{aligned}I &= \sum_{|n|=1}^{\infty} \left[(|n|+n\cos\beta)^{-1/p+\sigma} \int_{-\infty}^{\infty} k(x,n) f(x)\, dx\right] \cdot \left[(|n|+n\cos\beta)^{1/p-\sigma} b_n\right] \\ &\leq J_1 \left[\sum_{|n|=1}^{\infty} (|n|+n\cos\beta)^{q(1-\sigma)-1} b_n^q\right]^{1/q}.\end{aligned} \tag{41}$$

Then, by (34), we have (33). On the other hand, assuming that (33) is valid, we set

$$b_n := (|n|+n\cos\beta)^{p\sigma-1} \left[\int_{-\infty}^{\infty} k(x,n) f(x)\, dx\right]^{p-1} \quad (|n| \in \mathbf{N}). \tag{42}$$

Then we find

$$J_1 = \left[\sum_{|n|=1}^{\infty} (|n|+n\cos\beta)^{q(1-\sigma)-1} b_n^q\right]^{1/p}. \tag{43}$$

In view of (40), it follows that $J_1 < \infty$. If $J_1 = 0$, and then (34) is trivially valid; if $J_1 > 0$, then, by (33), we have

$$\begin{aligned}\sum_{|n|=1}^{\infty} (|n|+n\cos\beta)^{q(1-\sigma)-1} b_n^q &= J_1^p = I < k_{\alpha,\beta}(\sigma) \\ &\cdot \left[\int_{-\infty}^{\infty} (|x|+x\cos\alpha)^{p(1+\delta\sigma)-1} f^p(x)\, dx\right]^{1/p} \\ &\cdot \left[\sum_{|n|=1}^{\infty} (|n|+n\cos\beta)^{q(1-\sigma)-1} b_n^q\right]^{1/q},\end{aligned}$$

$$\left[\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}b_n^q\right]^{1/p}=J_1<k_{\alpha,\beta}(\sigma)\cdot\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx\right]^{1/p};\quad(44)$$

namely, (34) holds, which is equivalent to (33).

In the same way of obtaining (40), we have

$$J_2\le k_{\beta}^{1/p}(\sigma)\cdot\left[\sum_{|n|=1}^{\infty}\omega(\sigma,n)(|n|+n\cos\beta)^{q(1-\sigma)-1}b_n^q\right]^{1/q}.\quad(45)$$

We have proved that (33) is valid. Setting

$$f(x):=\int_{-\infty}^{\infty}\frac{1}{(|x|+x\cos\alpha)^{q\delta\sigma+1}}\left[\sum_{|n|=1}^{\infty}k(x,n)\,b_n\right]^{q-1}\quad(x\in\mathbf{R}\setminus\{0\}),\quad(46)$$

it follows that

$$J_2=\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx\right]^{1/q},\quad(47)$$

and, in view of (45), $J_2<\infty$. If $J_2=0$, then (35) is trivially valid; if $J_2>0$, then, by (33), we have

$$\begin{aligned}\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx&=J_2^q=I\\&<k_{\alpha,\beta}(\sigma)\cdot\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx\right]^{1/p}\\&\cdot\left[\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}b_n^q\right]^{1/q},\end{aligned}\quad(48)$$

$$\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx\right]^{1/q}=J_2<k_{\alpha,\beta}(\sigma)\left[\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}b_n^q\right]^{1/q};$$

namely, (35) follows.

On the other hand, assuming that (35) is valid, by Hölder's inequality (cf. [24]) and in the same way of obtaining (41), we have

$$I\le\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}f^p(x)\,dx\right]^{1/p}J_2.\quad(49)$$

Then, by (35), we have (33), which is equivalent to (35).

Therefore, inequalities (33), (34), and (35) are equivalent. □

Theorem 6. *As regards the assumptions of Theorem 5, the constant factor $k_{\alpha,\beta}(\sigma)$ in (33), (34), and (35) is the best possible one.*

Proof. For $0<\varepsilon<q\sigma$, we set $\widetilde{\sigma}=\sigma-\varepsilon/q\in(0,1)$ $(\widetilde{\sigma}<\gamma)$,

$$\widetilde{f}(x):=\begin{cases}\dfrac{1}{(|x|+x\cos\alpha)^{\delta(\sigma+\varepsilon/p)+1}}, & x\in E_{\delta},\\ 0, & x\in\mathbf{R}\setminus E_{\delta},\end{cases}\quad(50)$$

and $\widetilde{b}_n:=(|n|+n\cos\beta)^{(\sigma-\varepsilon/q)-1}$, $|n|\in\mathbf{N}$. Then, by (23) and (28), we find

$$\begin{aligned}\widetilde{I}_1&:=\left[\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\delta\sigma)-1}\widetilde{f}^p(x)\,dx\right]^{1/p}\cdot\left[\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}\widetilde{b}_n^q\right]^{1/q}\\&=\left[\int_{-\infty}^{\infty}\frac{dx}{(|x|+x\cos\alpha)^{\delta\varepsilon+1}}\right]^{1/p}\cdot\left[\sum_{|n|=1}^{\infty}\frac{1}{(|n|+n\cos\beta)^{\varepsilon+1}}\right]^{1/q}\le\frac{1}{\varepsilon}\left(2\csc^2\alpha\right)^{1/p}\\&\cdot\left[\left(2\csc^2\beta+o(1)\right)(\varepsilon+1)\right]^{1/q}.\end{aligned}\quad(51)$$

By (12) and (23), we still have

$$\begin{aligned}\widetilde{I}&:=\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}k(x,n)\,\widetilde{f}(x)\,\widetilde{b}_n dx\\&=\int_{E_{\delta}}\sum_{|n|=1}^{\infty}k(x,n)\frac{(|x|+x\cos\alpha)^{-\delta(\widetilde{\sigma}+\varepsilon)-1}}{(|n|+n\cos\beta)^{1-\widetilde{\sigma}}}dx\\&=\int_{E_{\delta}}\frac{\varpi(\widetilde{\sigma},x)\,dx}{(|x|+x\cos\alpha)^{\delta\varepsilon+1}}\ge k_{\beta}(\widetilde{\sigma})\end{aligned}$$

$$\cdot \int_{E_\delta} \frac{1-\theta(\tilde{\sigma},x)}{(|x|+x\cos\alpha)^{\delta\varepsilon+1}}dx = k_\beta(\tilde{\sigma})$$
$$\cdot \left[\int_{E_\delta} \frac{dx}{(|x|+x\cos\alpha)^{\delta\varepsilon+1}} - \int_{E_\delta} \frac{dx}{O\left((|x|+x\cos\alpha)^{\delta(\sigma+\varepsilon/p)+1}\right)}\right] = \frac{1}{\varepsilon}k_\beta\left(\sigma - \frac{\varepsilon}{q}\right)\left(2\csc^2\alpha - \varepsilon O(1)\right). \tag{52}$$

If the constant factor $k_{\alpha,\beta}(\sigma)$ in (33) is not the best possible one, then, there exists a positive number k with $k_{\alpha,\beta}(\sigma) > k$, such that (33) is still valid when replacing $k_{\alpha,\beta}(\sigma)$ by k. Then, in particular, we have $\varepsilon\widetilde{I} < \varepsilon k\widetilde{I}_1$; namely,

$$k_\beta\left(\sigma - \frac{\varepsilon}{q}\right)\left(2\csc^2\alpha - \varepsilon O(1)\right) < k\left(2\csc^2\alpha\right)^{1/p}\left[\left(2\csc^2\beta + o(1)\right)(\varepsilon+1)\right]^{1/q}. \tag{53}$$

It follows that $2k_\beta(\sigma)\csc^2\alpha \le 2k\csc^{2/p}\alpha\csc^{2/q}\beta$ $(\varepsilon \to 0^+)$, and then

$$k_{\alpha,\beta}(\sigma) = \frac{2\pi\csc^{2/q}\alpha\csc^{2/p}\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)} \le k, \tag{54}$$

which contradicts the fact that $k_{\alpha,\beta}(\sigma) > k$. Hence, the constant factor $k_{\alpha,\beta}(\sigma)$ in (33) is the best possible one.

The constant factor $k_{\alpha,\beta}(\sigma)$ in (34) ((35)) is still the best possible one. Otherwise, we would reach a contradiction by (41) ((49)) that the constant factor $k_{\alpha,\beta}(\sigma)$ in (33) is not the best possible one. □

4. Operator Expressions

Suppose that $p > 1$ and $1/p + 1/q = 1$. We set the following functions:

$$\Phi(x) := (|x| + x\cos\alpha)^{p(1+\delta\sigma)-1}, \quad \Psi(n) := (|n| + n\cos\beta)^{q(1-\sigma)-1}, \tag{55}$$

wherefrom $\Phi^{1-q}(x) = (|x| + x\cos\alpha)^{-q\delta\sigma-1}$ and

$$\Psi^{1-p}(n) = (|n| + n\cos\beta)^{p\sigma-1} \quad (x \in \mathbf{R}\setminus\{0\},\ |n| \in \mathbf{N}). \tag{56}$$

Define the following real weight normed linear spaces:

$$L_{p,\Phi}(\mathbf{R}) := \left\{f; \|f\|_{p,\Phi} := \left(\int_{-\infty}^{\infty}\Phi(x)|f(x)|^p dx\right)^{1/p} < \infty\right\},$$
$$L_{q,\Phi^{1-q}}(\mathbf{R}) := \left\{h; \|h\|_{q,\Phi^{1-q}} := \left(\int_{-\infty}^{\infty}\Phi^{1-q}(x)|h(x)|^q dx\right)^{1/q} < \infty\right\},$$
$$l_{q,\Psi} := \left\{b = \{b_n\}_{|n|=1}^{\infty}; \|b\|_{q,\Psi} := \left(\sum_{|n|=1}^{\infty}\Psi(n)|b_n|^q\right)^{1/q} < \infty\right\}, \tag{57}$$
$$l_{p,\Psi^{1-p}} := \left\{c = \{c_n\}_{|n|=1}^{\infty}; \|c\|_{p,\Psi^{1-p}} := \left(\sum_{|n|=1}^{\infty}\Psi^{1-p}(n)|c_n|^p\right)^{1/p} < \infty\right\}.$$

(a) In view of Theorem 5, for $f \in L_{p,\Phi}(\mathbf{R})$, setting

$$H^{(1)}(n) := \int_{-\infty}^{\infty}k(x,n)|f(x)|dx \quad (|n| \in \mathbf{N}), \tag{58}$$

by (34), we have

$$\left\|H^{(1)}\right\|_{p,\Psi^{1-p}} = \left[\sum_{|n|=1}^{\infty}\Psi^{1-p}(n)\left(H^{(1)}(n)\right)^p\right]^{1/p} < k_{\alpha,\beta}(\sigma)\|f\|_{p,\Phi} < \infty; \tag{59}$$

namely, $H^{(1)} \in l_{p,\Psi^{1-p}}$.

Definition 7. Define a half-discrete Hilbert-type operator in the whole plane $T^{(1)} : L_{p,\Phi}(\mathbf{R}) \to l_{p,\Psi^{1-p}}$ as follows: for any $f \in L_{p,\Phi}(\mathbf{R})$, there exists a unique representation $T^{(1)}f = H^{(1)} \in l_{p,\Psi^{1-p}}$, satisfying, for any $|n| \in \mathbf{N}$, $(T^{(1)}f)(n) = H^{(1)}(n)$.

In view of (59), it follows that $\|T^{(1)}f\|_{p,\Psi^{1-p}} = \|H^{(1)}\|_{p,\Psi^{1-p}} \le k_{\alpha,\beta}\|f\|_{p,\Phi}$, and then the operator $T^{(1)}$ is bounded satisfying

$$\left\|T^{(1)}\right\| := \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbf{R})}\frac{\left\|T^{(1)}f\right\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi}} \le k_{\alpha,\beta}(\sigma). \tag{60}$$

Since the constant factor $k_{\alpha,\beta}(\sigma)$ in (59) is the best possible one, we have

$$\left\|T^{(1)}\right\| = k_{\alpha,\beta}(\sigma) = \frac{2\pi\csc^{2/q}\alpha\csc^{2/p}\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}. \tag{61}$$

If we define the formal inner product of $T^{(1)}f$ and b ($\in l_{q,\Psi}$) as follows:

$$\left(T^{(1)}f,b\right) := \sum_{|n|=1}^{\infty}\left(\int_{-\infty}^{\infty}k(x,n)\,f(x)\,dx\right)b_n \tag{62}$$

then we can rewrite (33) and (34) as follows:

$$\begin{gathered}\left(T^{(1)}f,b\right) < \left\|T^{(1)}\right\|\cdot\|f\|_{p,\Psi}\|b\|_{q,\Phi},\\ \left\|T^{(1)}f\right\|_{p,\Psi^{1-p}} < \left\|T^{(1)}\right\|\cdot\|f\|_{p,\Phi}.\end{gathered} \tag{63}$$

(b) In view of Theorem 5, for $b \in l_{q,\Psi}$, setting

$$H^{(2)}(x) := \sum_{|n|=1}^{\infty}k(x,n)\,b_n \quad (x\in\mathbf{R}\setminus\{0\}), \tag{64}$$

then, by (35), we have

$$\begin{aligned}\left\|H^{(2)}\right\|_{q,\Phi^{1-q}} &= \left[\int_{-\infty}^{\infty}\Phi^{1-q}(x)\left(H^{(2)}(x)\right)^q dx\right]^{1/q}\\ &< k_{\alpha,\beta}(\sigma)\|b\|_{q,\Psi} < \infty;\end{aligned} \tag{65}$$

namely, $H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$.

Definition 8. Define a half-discrete Hilbert-type operator in the whole plane $T^{(2)}: l_{q,\Psi} \to L_{q,\Psi^{1-q}}(\mathbf{R})$ as follows: for any $b \in l_{q,\Psi}$, there exists a unique representation $T^{(2)}b = H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$, satisfying, for any $x \in \mathbf{R}\setminus\{0\}$, $(T^{(2)}b)(x) = H^{(2)}(x)$.

In view of (65), it follows that $\|T^{(2)}b\|_{q,\Phi^{1-q}} = \|H^{(2)}\|_{q,\Phi^{1-q}} \leq k_{\alpha,\beta}(\sigma)\|b\|_{q,\Psi}$, and then the operator $T^{(2)}$ is bounded, satisfying

$$\left\|T^{(2)}\right\| := \sup_{b(\neq\theta)\in l_{q,\Psi}}\frac{\left\|T^{(2)}b\right\|_{q,\Phi^{1-q}}}{\|b\|_{q,\Psi}} \leq k_{\alpha,\beta}(\sigma). \tag{66}$$

Since the constant factor $k_{\alpha,\beta}(\sigma)$ in (65) is the best possible one, we have

$$\left\|T^{(2)}\right\| = k_{\alpha,\beta}(\sigma) = \left\|T^{(1)}\right\|. \tag{67}$$

If we define the formal inner product of $T^{(2)}b$ and f ($\in L_{p,\Phi}(\mathbf{R})$) as follows:

$$\left(T^{(2)}b,f\right) := \int_{-\infty}^{\infty}\left(\sum_{|n|=1}^{\infty}k(x,n)\,b_n\right)f(x)\,dx, \tag{68}$$

then we can rewrite (33) and (35) as follows:

$$\begin{gathered}\left(T^{(2)}b,f\right) < \left\|T^{(2)}\right\|\cdot\|f\|_{p,\Psi}\|b\|_{q,\Phi},\\ \left\|T^{(2)}b\right\|_{q,\Phi^{1-q}} < \left\|T^{(2)}\right\|\cdot\|b\|_{q,\Psi}.\end{gathered} \tag{69}$$

Remark 9. (i) For $\delta = -1$, (36) reduces to (5). If $f(-x) = f(x)$ $(x > 0)$ and $b_{-n} = b_n$ $(n \in \mathbf{N})$, then (5) reduces to the following half-discrete Hilbert-type inequality:

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{1+\rho(nx)^{\gamma}}f(x)\,b_n dx\\ &\quad< \frac{\pi}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)\,dx\right]^{1/p}\\ &\quad\cdot\left[\sum_{n=1}^{\infty}n^{q(1-\sigma)-1}b_n^q\right]^{1/q}.\end{aligned} \tag{70}$$

(ii) For $\delta = 1$, (33) reduces to the following particular inequality with the homogeneous kernel of degree 0:

$$\begin{aligned}&\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)\,b_n}{1+\rho\left[(|n|+n\cos\beta)/(|x|+x\cos\alpha)\right]^{\gamma}}dx\\ &\quad< \frac{2\pi\csc^{2/q}\alpha\csc^{2/p}\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}\left\{\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1+\sigma)-1}\right.\\ &\quad\left.\cdot f^p(x)\,dx\right\}^{1/p}\cdot\left\{\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}\right.\\ &\quad\left.\cdot b_n^q\right\}^{1/q}.\end{aligned} \tag{71}$$

(iii) For $\delta = -1$, (33) reduces to the following particular inequality with the nonhomogeneous kernel:

$$\begin{aligned}&\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)\,b_n}{1+\rho\left[(|x|+x\cos\alpha)(|n|+n\cos\beta)\right]^{\gamma}}dx\\ &\quad< \frac{2\pi\csc^{2/q}\alpha\csc^{2/p}\beta}{\gamma\rho^{\sigma/\gamma}\sin\pi(\sigma/\gamma)}\left\{\int_{-\infty}^{\infty}(|x|+x\cos\alpha)^{p(1-\sigma)-1}\right.\\ &\quad\left.\cdot f^p(x)\,dx\right\}^{1/p}\cdot\left\{\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\sigma)-1}\right.\\ &\quad\left.\cdot b_n^q\right\}^{1/q}.\end{aligned} \tag{72}$$

The constant factors in (5) and the above inequalities are all the best possible ones.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Bicheng Yang carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. Qunwei Ma and Leping He participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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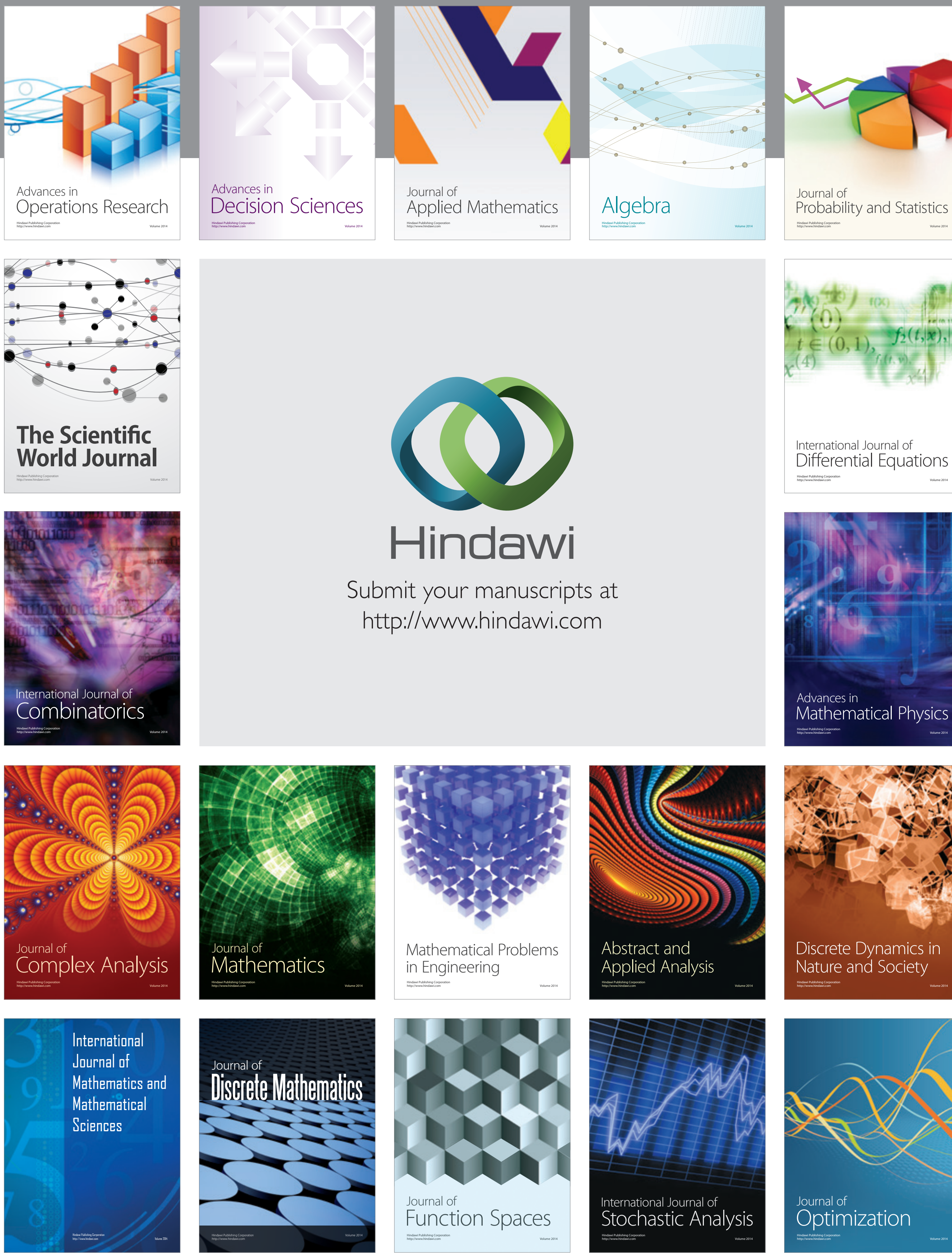
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