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Research Article

(m, λ)-Berezin Transform on the Weighted Bergman Spaces over the Polydisk

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We prove that every bounded linear operator on weighted Bergman space over the polydisk can be approximated by Toeplitz operators under some conditions. The main tool here is the so-called (m, λ)-Berezin transform. In particular, our results generalized the results of K. Nam and D. C. Zheng to the case of operators acting on $\mathscr{A}_\lambda^2(\mathbb{D}^n)$.

1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and $dA_\mu(z) = c_\mu(1-|z|^2)^\mu dA(z)$ be a positive standard weighted probability measure on $\mathbb{D}$, where the weighted parameter fulfills $\mu > -1$ and the normalized constant $c_\mu = \mu + 1$. For a fixed positive integer n, the polydisk $\mathbb{D}^n$ is the Cartesian product of n copies of $\mathbb{D}$ and

$$dA_\lambda(z) = dA_{\lambda_1}(z_1)\cdots dA_{\lambda_n}(z_n) \tag{1}$$

is the normalized weighted Lebesgue volume measure on the polydisk $\mathbb{D}^n$. The Bergman space $\mathscr{A}_\lambda^2(\mathbb{D}^n) = \mathscr{A}_\lambda^2(\mathbb{D}^n, dA_\lambda)$ is the set of all analytic functions on $\mathbb{D}^n$ in $L_\lambda^2(\mathbb{D}^n, dA_\lambda) = L_\lambda^2(\mathbb{D}^n)$. As is well known $\mathscr{A}_\lambda^2(\mathbb{D}^n)$ forms a closed subspace of $L_\lambda^2(\mathbb{D}^n)$ and has the structure of reproducing kernel Hilbert space. We denote by B_λ the Bergman projection of $L_\lambda^2(\mathbb{D}^n)$ onto $\mathscr{A}_\lambda^2(\mathbb{D}^n)$. In case of $\lambda = 0$, $\mathscr{A}_0^2(\mathbb{D}^n)$ is the unweighted Bergman space denoted by $\mathscr{A}^2(\mathbb{D}^n)$. Given an essentially bounded measurable function $a \in L^\infty(\mathbb{D}^n)$, we write T_a for the Toeplitz operator with the symbol a, which acts on $\mathscr{A}_\lambda^2(\mathbb{D}^n)$ as $T_a f = B_\lambda(af)$. That is, the Toeplitz operator is defined as the compression of a multiplication operator on $L_\lambda^2(\mathbb{D}^n)$ onto the Bergman space. The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the closed subalgebra of $\mathscr{L}(\mathscr{A}_\lambda^2)$ generated by $\{T_a : a \in L^\infty(\mathbb{D}^n)\}$, where $\mathscr{L}(\mathscr{A}_\lambda^2)$ denotes the algebra of all bounded linear operators on $\mathscr{A}_\lambda^2(\mathbb{D}^n)$.

Due to their simple structure Toeplitz operators form an important, tractable, and intensively studied subclass in the algebra $\mathscr{L}(\mathscr{A}_\lambda^2)$ of all bounded linear operators on $\mathscr{A}_\lambda^2(\mathbb{D}^n)$. The natural question is whether the Toeplitz algebra is dense in the algebra of all bounded linear operators on the Bergman space. On unweighted Bergman space over the unit disk and even more general domain in $\mathbb{C}$, it is proved in [1] that the Toeplitz algebra is dense in the algebra of all bounded linear operators in the sense of strong operator topology (SOT). In general, it is not true if the SOT is replaced by the norm topology.

Nam and Zheng give a criterion for bounded operators approximated by Toeplitz operators on $\mathscr{A}^2(\mathbb{D}^n)$. Since the Berezin transform is a useful tool to study operators on any reproducing kernel Hilbert space, the m-Berezin transform for any bounded linear operators acting on $\mathscr{A}^2(\mathbb{D}^n)$ was defined in [2]. The operator $S \in \mathscr{L}(L_a^2)$ can be approximated in the norm by Toeplitz operators on the unit ball (see [3]) by using the m-Berezin transform. In [4], the (k, α)-Berezin transform for complex-valued regular measures on the weighted p-Bergman space over the unit ball was defined and studied. Using it, they show that every $S \in \mathfrak{T}(L^\infty)$ can be approximated by certain localized operators and introduce a way to connect the behavior of these localized operators with the Berezin transform. The (m, λ)-Berezin transform for general bounded operators acting on the weighted Bergman space $\mathscr{A}_\lambda^2(\mathbb{B}^n)$ was defined in [5] and the authors establish various results on norm approximations via the (m, λ)-Berezin transform and describe conditions under which a bounded linear operator S can be approximated in norm by Toeplitz operators whose symbols are bounded functions.

In this paper, we will define and study the (m,λ)-Berezin transform for general bounded operators acting on the weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ in the third section. The (m,λ)-Berezin transform of a Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ coincides with $(0,\lambda+m)$-Berezin transform for T_a considered on the weighted Bergman space $\mathcal{A}^2_{\lambda+m}(\mathbb{D}^n)$. We will show that the (m,λ)-Berezin transforms are commuting with each other. In Section 4, we will establish various results on norm approximation by the (m,λ)-Berezin transform. More precisely, we describe how to approximate a bounded linear operator S on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ in norm by Toeplitz operators whose symbols are bounded functions which are given as the (m,λ)-Berezin transform of the initial operator S under some conditions. We would like to point out that these results generalize ideas and theorems in [2] to the case of operators acting on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$.

2. Preliminaries

Let $\mathbb{D}^n=\{z=(z_1,\dots,z_n):|z_i|<1,\ \text{for } i=1,\dots,n\}$ be the polydisk in $\mathbb{C}^n$ equipped with the standard weighted measure (1), where $\lambda=(\lambda_1,\dots,\lambda_n)$ is fixed and $\lambda_i>-1$ for any $i=1,\dots,n$. For a vector $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$ and a positive integer m we will employ the notations

$$\lambda+m=(\lambda_1+m,\dots,\lambda_n+m),$$
$$|\lambda+m|=\sum_{i=1}^n(\lambda_i+m)=nm+\sum_{i=1}^n\lambda_i, \tag{2}$$
$$[\lambda+m]=\prod_{i=1}^n(\lambda_i+m).$$

In addition, if λ_i is a positive integer for any $i=1,\dots,n$ and $m>0$, λ and $\lambda+m$ are multi-index. Let $\mathbb{Z}_+:=\{0,1,\dots\}$ be the set of nonnegative integers. With $\alpha\in\mathbb{Z}^n_+$, we use the standard notations $z^\alpha:=z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, $\alpha!:=\alpha_1!\cdots\alpha_n!$ and $|\alpha|:=\alpha_1+\cdots+\alpha_n$.

As we all know, for all $\alpha\in\mathbb{Z}^n_+$ and $\lambda=(\lambda_1,\dots,\lambda_n)$, where $\lambda_i>-1$, for $i=1,\dots,n$, we have

$$\|w^\alpha\|_{2,\lambda}=\sqrt{[\lambda+1]\prod_{i=1}^n\frac{\Gamma(\alpha_i+1)\Gamma(\lambda_i+1)}{\Gamma(\alpha_i+\lambda_i+2)}} \tag{3}$$

and then $\{e_\alpha=w^\alpha\|w^\alpha\|^{-1}_{2,\lambda}:\alpha\in\mathbb{Z}^n_+\}$ is the standard orthonormal basis of $\mathcal{A}^2_\lambda(\mathbb{D}^n)$. The reproducing kernel in $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ is given by

$$K^\lambda_z(w)=\prod_{i=1}^n\frac{1}{(1-\overline{z_i}w_i)^{2+\lambda_i}} \tag{4}$$

for $z,w\in\mathbb{D}^n$, and the normalized reproducing kernel $k^\lambda_z(w)=K^\lambda_z(w)/\|K^\lambda_z\|=K^\lambda_z(w)/\sqrt{K^\lambda_z(z)}=\prod_{i=1}^n((1-|z_i|^2)^{(2+\lambda_i)/2}/(1-\overline{z_i}w_i)^{2+\lambda_i})$. For $z\in\mathbb{D}^n$, let $\phi_z(w)=(\phi_{z_1}(w_1),\dots,\phi_{z_n}(w_n))$, where $\phi_{z_i}(w_i)=(z_i-w_i)/(1-w_i\overline{z_i})$, for $i=1,\dots,n$; then $\phi_z(w)$ is an automorphism on $\mathbb{D}^n$ that interchanges 0 and z. Let $\phi'_z(w)=(\phi'_{z_1}(w_1),\dots,\phi'_{z_n}(w_n))$; then

$$[\phi'_z(w)]=\prod_{i=1}^n(\phi'_{z_i}(w_i))=\prod_{i=1}^n\frac{|z_i|^2-1}{(1-\overline{z_i}w_i)^2}. \tag{5}$$

Given $z\in\mathbb{D}^n$, introduce the unitary operator U_z on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ given by $U_zf=f\circ\phi_z\cdot[(\phi'_z)^{(2+\lambda)/2}]$, where $[(\phi'_z)^{(2+\lambda)/2}]=\prod_{i=1}^n(\phi'_{z_i})^{(2+\lambda_i)/2}$. It is easy to see that U_z is self-adjoint and so $U_z^2=I$. We have $U_0f(w)=(-1)^{|2+\lambda|/2}f(-w)$.

For a fixed $z\in\mathbb{D}^n$ we define an automorphism on the algebra $\mathcal{L}(\mathcal{A}^2_\lambda)$ of all bounded operator on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ by $S\mapsto S_z:=U_zSU_z\in\mathcal{L}(\mathcal{A}^2_\lambda)$. In particular, if $S=T_a$ is a Toeplitz operator, then $(T_a)_z=T_{a\circ\phi_z}$.

The principle difference between the unit ball $\mathbb{B}^n$ and the polydisk $\mathbb{D}^n$ is that the later domain is reducible, which involves the tensor product structure of various objects introduced and studied in the paper. In particular, $L^2_\lambda(\mathbb{D}^n,dA_\lambda(z))=L^2_{\lambda_1}(\mathbb{D},dA_{\lambda_1}(z_1))\otimes\cdots\otimes L^2_{\lambda_n}(\mathbb{D},dA_{\lambda_n}(z_n))$ and $\mathcal{A}^2_\lambda(\mathbb{D}^n,dA_\lambda(z))=\mathcal{A}^2_{\lambda_1}(\mathbb{D},dA_{\lambda_1}(z_1))\otimes\cdots\otimes\mathcal{A}^2_{\lambda_n}(\mathbb{D},dA_{\lambda_n}(z_n))$. Therefore, for the orthonormal basis of $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ and the reproducing kernel in $\mathcal{A}^2_\lambda(\mathbb{D}^n)$, we have $w^\alpha=w^{\alpha_1}\cdots w^{\alpha_n}=w^{\alpha_1}\otimes\cdots\otimes w^{\alpha_n}$ and $K^\lambda_z(w)=K^{\lambda_1}_{z_1}(w_1)\otimes\cdots\otimes K^{\lambda_n}_{z_n}(w_n)$.

The unitary operator U_z on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ can be written by $U_z=U_{z_1}\otimes\cdots\otimes U_{z_n}$. In fact, $U_zw^\alpha=\phi^\alpha_z(w)\cdot(\phi'_z(w))^{(2+\lambda)/2}=\prod_{i=1}^n(\phi^{\alpha_i}_{z_i}(w_i))\cdot(\phi'_{z_i}(w_i))^{(2+\lambda_i)/2}=(\phi^{\alpha_1}_{z_1}(w_1))\cdot(\phi'_{z_1}(w_1))^{(2+\lambda_1)/2}\otimes\cdots\otimes(\phi^{\alpha_n}_{z_n}(w_n))\cdot(\phi'_{z_n}(w_n))^{(2+\lambda_n)/2}=U_{z_1}w^{\alpha_1}\otimes\cdots\otimes U_{z_n}w^{\alpha_n}=(U_{z_1}\otimes\cdots\otimes U_{z_n})(w^{\alpha_1}\otimes\cdots\otimes w^{\alpha_n})$. If $S\in\mathcal{L}(\mathcal{A}^2_\lambda)$ can be written by $S=S_1\otimes\cdots\otimes S_n$, then $S_z=S_{z_1}\otimes\cdots\otimes S_{z_n}$.

Let $S_1=S_1(\mathcal{A}^2_\lambda)$ denote the class of trace operators on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$. Given $T\in S_1$, we write $\mathrm{tr}[T]$ for its trace and recall that the trace norm of T is given by $\|T\|_{S_1}=\mathrm{tr}[\sqrt{T^*T}]$. Given $f,g\in\mathcal{A}^2_\lambda(\mathbb{D}^n)$, the rank-one-operator $f\otimes g$ acting on $\mathcal{A}^2_\lambda(\mathbb{D}^n)$ by the formula $(f\otimes g)h=\langle h,g\rangle_\lambda f$ obviously belongs to S_1. It is easily proved that $f\otimes g$ is in S_1 and with norm equal to $\|f\otimes g\|_{S_1}=\|f\|_{2,\lambda}\cdot\|g\|_{2,\lambda}$ and $\mathrm{tr}[f\otimes g]=\langle f,g\rangle_\lambda$. Recall as well that if $T\in S_1$ has rank n, then one has the inequality $\|T\|_{S_1}\le\sqrt{n}(\mathrm{tr}[T^*T])^{1/2}$. The pseudo-hyperbolic metric on $\mathbb{D}^n$ is defined as $\rho(z,w)=\max_{1\le i\le n}|\phi_{z_i}(w_i)|$.

Throughout the paper and as a convention we will denote by $C(n,\lambda)$ a positive constant depending only on n and λ and appearing in various estimates and whose value may change at each occurrence.

3. The (m, λ)-Berezin Transform

Recall that m-Berezin transform for unweighted Bergman space over the unit disk and over the unit polydisk was defined in [6] and [2], respectively. We will follow the recipe in [2] and first introduce some notation. Put

$$C_{m,\alpha} = (-1)^{|\alpha|} \binom{m}{\alpha_1} \cdots \binom{m}{\alpha_n}, \tag{6}$$

so that, for $z, w \in \mathbb{D}^n$, we know

$$\sum_{\alpha_1=0}^{m} \cdots \sum_{\alpha_n=0}^{m} C_{m,\alpha} z^\alpha \overline{w}^\alpha = \prod_{i=1}^{n} \left(1 - z_i \overline{w_i}\right)^m. \tag{7}$$

For $z \in \mathbb{D}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, and a positive integer m, let $K_z^{m+\lambda}(u) = \prod_{i=1}^n (1/(1 - \overline{z}_i u_i)^{m+\lambda_i+2})$.

A generalization of the concept of m-Berezin transform to an arbitrary bounded operator on the Bergman space $\mathscr{A}^2(\mathbb{D}^n)$ requires a modification of the definition in [2].

Definition 1. We define the (m, λ)-Berezin transform of $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$ by

$$\begin{aligned}&\left(B_{m,\lambda} S\right)(z) \\ &\quad := \frac{[\lambda + m + 1]}{[\lambda + 1]} \sum_{\alpha_1=0}^{m} \cdots \sum_{\alpha_n=0}^{m} C_{m,\alpha} \left\langle S_z w^\alpha, w^\alpha \right\rangle_\lambda .\end{aligned} \tag{8}$$

It is easy to see that the following pointwise estimate

$$\begin{aligned}\left|\left(B_{m,\lambda} S\right)(z)\right| &\le \|S\| \frac{[\lambda + m + 1]}{[\lambda + 1]} \sum_{\alpha_1=0}^{m} \cdots \sum_{\alpha_n=0}^{m} \left|C_{m,\alpha}\right| \\ &\quad \cdot \left\|w^\alpha\right\|_\lambda := C(\lambda, m, n) \|S\|,\end{aligned} \tag{9}$$

where the constant $C(\lambda, m, n) > 0$ is independent of $z \in \mathbb{D}^n$; that is, $B_{m,\lambda} S$ is a bounded function on $\mathbb{D}^n$ with $\|B_{m,\lambda} S\|_\infty \le C(\lambda, m, n)\|S\|$.

In [5], the (m, λ)-Berezin transform of $S \in \mathscr{L}(\mathscr{A}_\lambda^2(\mathbb{D}))$ is defined by $(B_{m,\lambda} S)(z) = ((\lambda + m + 1)/(\lambda + 1)) \sum_{k=0}^m (-1)^k \binom{m}{k} \langle S_z w^k, w^k \rangle$, for the case of the unit disk $\mathbb{B}^1 = \mathbb{D}$. From the point of view of the tensor product structure, given an elementary tensor $S = S_1 \otimes \cdots \otimes S_n \in \mathscr{L}(\mathscr{A}_{\lambda_1}^2(\mathbb{D}, dA_{\lambda_1}(z_1))) \otimes \cdots \otimes \mathscr{L}(\mathscr{A}_{\lambda_n}^2(\mathbb{D}, dA_{\lambda_n}(z_n)))$, its (m, λ)-Berezin transform for $\lambda = (\lambda_1, \ldots, \lambda_n)$ obviously and naturally has to be defined as

$$\begin{aligned}\left(B_{m,\lambda} S\right)(z) &= \prod_{i=1}^{n} \left(B_{m,\lambda_i} S_i\right)(z_i) = \prod_{i=1}^{n} \frac{\lambda_i + m + 1}{\lambda_i + 1} \\ &\quad \cdot \sum_{\alpha_i=0}^{m} (-1)^{\alpha_i} \binom{m}{\alpha_i} \left\langle \left(S_i\right)_{z_i} w_i^{\alpha_i}, w_i^{\alpha_i} \right\rangle_{\lambda_i} \\ &= \frac{[\lambda + m + 1]}{[\lambda + 1]} \sum_{\alpha_1=0}^{m} \cdots \sum_{\alpha_n=0}^{m} (-1)^{|\alpha|} \binom{m}{\alpha_1} \\ &\quad \cdots \binom{m}{\alpha_n} \left\langle \left(S_1\right)_{z_1} w_1^{\alpha_1}, w_1^{\alpha_1} \right\rangle_{\lambda_1} \\ &\quad \cdots \left\langle \left(S_n\right)_{z_n} w_n^{\alpha_n}, w_n^{\alpha_n} \right\rangle_{\lambda_n} = \frac{[\lambda + m + 1]}{[\lambda + 1]} \sum_{\alpha_1=0}^{m} \\ &\quad \cdots \sum_{\alpha_n=0}^{m} C_{m,\alpha} \left\langle S_z w^\alpha, w^\alpha \right\rangle_\lambda .\end{aligned} \tag{10}$$

Unfortunately, the set of those tensor product operators is not a linear space; that is, for any operator $S \in \mathscr{L}(\mathscr{A}_\lambda^2(\mathbb{D}^n))$, S cannot be written in the form of the tensor product operators. Therefore, we define for any operator $S \in \mathscr{L}(\mathscr{A}_\lambda^2(\mathbb{D}^n))$ with (10), and this coincides with Definition 1. If S can be the tensor product form, this definition is the tensor product of (m, λ_i)-Berezin transform for the case of $n = 1$.

As usual we define the (m, λ)-Berezin transform of a function $a \in L^\infty(\mathbb{D}^n)$ by

$$\begin{aligned}B_{m,\lambda}(a)(z) &:= \left(B_{m,\lambda} T_a\right)(z) \\ &= \frac{[\lambda + m + 1]}{[\lambda + 1]} \sum_{\alpha_1=0}^{m} \cdots \sum_{\alpha_n=0}^{m} C_{m,\alpha} \left\langle \left(T_a\right)_z w^\alpha, w^\alpha \right\rangle_\lambda .\end{aligned} \tag{11}$$

It is easy to see that $B_{m,\lambda}(a)(z) = \int_{\mathbb{D}^n} a \circ \phi_z(w) dA_{\lambda+m}(w)$. Thus, (m, λ)-Berezin transform of a Toeplitz operator T_a acting on $\mathscr{A}_\lambda^2(\mathbb{D}^n)$ coincides with $(0, \lambda+m)$-Berezin transform for T_a now considered on the weighted Bergman space $\mathscr{A}_{\lambda+m}^2(\mathbb{D}^n)$.

From the definition of $\phi_{z_i}(u_i)$, we have the identity $1 - \phi_{z_i}(u_i)\overline{\phi_{z_i}(w_i)} = (1 - |z_i|^2)(1 - \overline{w}_i u_i)/(1 - \overline{z}_i u_i)(1 - \overline{w}_i z_i)$, for $u_i, w_i \in \mathbb{D}$ and $i = 1, \ldots, n$. The following proposition gives an integral representation of the (m, λ)-Berezin transform.

Proposition 2. *Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$, $m \geqslant 0$, and $z \in \mathbb{D}^n$. Then*

$$\begin{aligned}B_{m,\lambda}(S)(z) &= \frac{[\lambda + m + 1]}{[\lambda + 1]} \prod_{i=1}^{n} \left(1 - \left|z_i\right|^2\right)^{\lambda_i + m + 2} \\ &\quad \times \int_{\mathbb{D}^n} \int_{\mathbb{D}^n} \left(1 - \overline{w}_i u_i\right)^m K_z^{m+\lambda}(u) \\ &\quad \cdot \overline{K_z^{m+\lambda}(w) S^* K_w^\lambda(u)} \, dA_\lambda(u) \, dA_\lambda(w) .\end{aligned} \tag{12}$$

Proof. For $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$ and $w \in \mathbb{D}^n$, we have $S(\phi_z^\alpha \cdot \prod_{i=1}^n (\phi_{z_i}')^{(2+\lambda_i)/2})(w) = \langle S(\phi_z^\alpha \cdot \prod_{i=1}^n (\phi_{z_i}')^{(2+\lambda_i)/2}), K_w^\lambda \rangle_\lambda = \langle \phi_z^\alpha \cdot \prod_{i=1}^n (\phi_{z_i}')^{(2+\lambda_i)/2}, S^* K_w^\lambda \rangle_\lambda$ and $U_z w^\alpha = \phi_z^\alpha(w) \cdot \prod_{i=1}^n (\phi_{z_i}')^{(2+\lambda_i)/2}$. Using those by (5) and (7), we have

$$
\begin{aligned}
B_{m,\lambda}(S)(z) &= \frac{[\lambda+m+1]}{[\lambda+1]}\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\left\langle S_z w^{\alpha}, w^{\alpha}\right\rangle_{\lambda} \\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\left\langle S\left(\phi_z^{\alpha}(w)\cdot\prod_{i=1}^{n}\left(\phi_{z_i}'\right)^{(2+\lambda_i)/2}\right), \phi_z^{\alpha}(w)\cdot\prod_{i=1}^{n}\left(\phi_{z_i}'\right)^{(2+\lambda_i)/2}\right\rangle_{\lambda} \\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\times\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\phi_z^{\alpha}(u) \\
&\quad\cdot\prod_{i=1}^{n}\left(\phi_{z_i}'(u_i)\right)^{(2+\lambda_i)/2}\overline{\phi_z^{\alpha}(w)\cdot\prod_{i=1}^{n}\left(\phi_{z_i}'(w_i)\right)^{(2+\lambda_i)/2}} S^{*}K_w^{\lambda}(u)\,dA_{\lambda}(u)\,dA_{\lambda}(w) \\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}\left(1-\phi_{z_i}(u_i)\overline{\phi_{z_i}(w_i)}\right)^{m}\left(\phi_{z_i}'(u_i)\right)^{(2+\lambda_i)/2}\overline{\left(\phi_{z_i}'(w_i)\right)^{(2+\lambda_i)/2}} S^{*}K_w^{\lambda}(u)\,dA_{\lambda}(u)\,dA_{\lambda}(w) \\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}\left(1-|z_i|^2\right)^{m+2+\lambda_i}\left(1-\overline{w}_iu_i\right)^{m} K_z^{m+\lambda}(u)\,\overline{K_z^{m+\lambda}(w)\,S^{*}K_w^{\lambda}(u)}\,dA_{\lambda}(u)\,dA_{\lambda}(w) \\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\prod_{i=1}^{n}\left(1-|z_i|^2\right)^{\lambda_i+m+2}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\left(1-\overline{w}_iu_i\right)^{m} K_z^{m+\lambda}(u)\,\overline{K_z^{m+\lambda}(w)\,S^{*}K_w^{\lambda}(u)}\,dA_{\lambda}(u)\,dA_{\lambda}(w).
\end{aligned} \tag{13}
$$

□

Proposition 3. *Let $S \in \mathscr{L}(\mathscr{A}_{\lambda}^2)$, $m \geqslant 0$, and $z \in \mathbb{D}^n$. Then*

$$
\begin{aligned}
B_{m,\lambda}(S)(z) &:= \frac{[\lambda+m+1]}{[\lambda+1]}\prod_{i=1}^{n}\left(1-|z_i|^2\right)^{\lambda_i+m+2} \\
&\quad\cdot\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\left\langle S\left(w^{\alpha}K_z^{m+\lambda}\right), w^{\alpha}K_z^{m+\lambda}\right\rangle_{\lambda}.
\end{aligned} \tag{14}
$$

Proof. We have

$$
\begin{aligned}
&\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}\left(1-\overline{w}_iu_i\right)^{m} K_z^{m+\lambda}(u) \\
&\quad\cdot\overline{K_z^{m+\lambda}(w)\,S^{*}K_w^{\lambda}(u)}\,dA_{\lambda}(u)\,dA_{\lambda}(w) \\
&\quad= \sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\overline{w}^{\alpha}u^{\alpha}K_z^{m+\lambda}(u) \\
&\quad\cdot\overline{K_z^{m+\lambda}(w)\,S^{*}K_w^{\lambda}(u)}\,dA_{\lambda}(u)\,dA_{\lambda}(w) \\
&\quad= \sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\int_{\mathbb{D}^n} S\left(u^{\alpha}K_z^{m+\lambda}\right)(w) \\
&\quad\cdot\overline{w^{\alpha}K_z^{m+\lambda}(w)}\,dA_{\lambda}(w) = \sum_{\alpha_1=0}^{m} \\
&\quad\cdots\sum_{\alpha_n=0}^{m} C_{m,\alpha}\left\langle S\left(w^{\alpha}K_z^{m+\lambda}\right), w^{\alpha}K_z^{m+\lambda}\right\rangle_{\lambda}.
\end{aligned} \tag{15}
$$

□

For $z, w \in \mathbb{D}^n$, put $t_i = (\phi_{z_i}(w_i)\overline{z_i} - 1)/(1 - z_i\overline{\phi_{z_i}(w_i)})$, for $i = 1, \ldots, n$. In ([2], P98), the map $\phi_{w_i} \circ \phi_{z_i} \circ \phi_{\phi_{w_i}}$ is a unitary map of $\mathbb{D}$ and maps t_i to 1. Let $t = (t_1, t_2, \ldots, t_n)$ and $tu = (t_1u_1, \ldots, t_nu_n)$, for $u \in \mathbb{D}^n$.

Lemma 4. *For $z, w \in \mathbb{D}^n$,*

$$U_wU_z = V_tU_{\phi_z(w)}, \tag{16}$$

where $(V_tf)(u) = t^{(2+\lambda)/2}f(tu)$ and $t^{(2+\lambda)/2} = \prod_{i=1}^{n} t^{(2+\lambda)/2}$, for $f \in \mathscr{A}_{\lambda}^2(\mathbb{D}^n)$.

Proof. Since $V_tw^{\alpha} = t^{(2+\lambda)/2}(tw)^{\alpha} = t_1^{(2+\lambda_1)/2}(t_1w_1)^{\alpha_1} \otimes \cdots \otimes t_n^{(2+\lambda_n)/2}(t_nw_n)^{\alpha_n} = V_{t_1}w_1^{\alpha_1} \otimes \cdots \otimes V_{t_n}w_n^{\alpha_n} = (V_{t_1} \otimes \cdots \otimes V_{t_n})w^{\alpha}$, then $V_t = V_{t_1} \otimes \cdots \otimes V_{t_n}$. It is sufficient to show that $n = 1$.

For $z, w, t \in \mathbb{D}$ and $\lambda > -1$, $\phi_z \circ \phi_{\phi_z(w)}(tu) = \phi_w(u)$. Thus for $k \geq 0$,

$$
\begin{aligned}
U_wU_zu^k &= U_w\left(\phi_z^k(u)\cdot\left(\phi_z'(u)\right)^{(2+\lambda)/2}\right) = (\phi_z \\
&\quad\circ\phi_w(u))^k\cdot\left(\phi_z'\circ\phi_w(u)\right)^{(2+\lambda)/2}\left(\phi_w'(u)\right)^{(2+\lambda)/2} \\
&= \left(\phi_{\phi_z(w)}(tu)\right)^k\cdot\left(\phi_z'\circ\phi_z\circ\phi_{\phi_z(w)}(tu)\cdot\phi_z'\right. \\
&\quad\left.\circ\,\phi_{\phi_z(w)}(tu)\cdot\phi_{\phi_z(w)}'(tu)\cdot t\right)^{(2+\lambda)/2} = \left(\phi_{\phi_z(w)}(tu)\right)^k \\
&\quad\cdot\left(\phi_{\phi_z(w)}'(tu)\cdot t\right)^{(2+\lambda)/2} = V_tU_{\phi_z(w)}u^k.
\end{aligned} \tag{17}
$$

□

Note that $t^{(2+\lambda)/2}$ is a complex number of modulus one.

Theorem 5. *Let $S \in \mathscr{L}(\mathscr{A}_{\lambda}^2)$, $m \geqslant 0$, and $z \in \mathbb{D}^n$; then*

$$B_{m,\lambda}S_z = \left(B_{m,\lambda}S\right)\circ\phi_z. \tag{18}$$

Proof. By definition,

$$\begin{aligned}
&(B_{m,\lambda}S_z)(0)\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m}C_{m,\alpha}\langle S_z U_0 w^{\alpha}, U_0 w^{\alpha}\rangle_\lambda\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\sum_{\alpha_1=0}^{m}\cdots\sum_{\alpha_n=0}^{m}C_{m,\alpha}\langle S_z w^{\alpha}, w^{\alpha}\rangle_\lambda\\
&= (B_{m,\lambda}S)(z) = (B_{m,\lambda}S)\circ\phi_z(0).
\end{aligned} \tag{19}$$

For any $\eta \in \mathbb{D}^n$, by Proposition 2, Lemma 4, and $V_tV_t^* = I$, we have

$$\begin{aligned}
&(B_{m,\lambda}S_z)(\eta) = (B_{m,\lambda}S_z)\circ\phi_\eta(0) = (B_{m,\lambda}(S_z)_\eta)(0)\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}(1-\overline{w_i}u_i)^m\\
&\cdot\overline{((S_z)_\eta)^* K_w^\lambda(u)}\,dA_\lambda(u)\,dA_\lambda(w)\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}(1-\overline{w_i}u_i)^m\\
&\cdot\overline{U_\eta U_z S^* U_z U_\eta K_w^\lambda(u)}\,dA_\lambda(u)\,dA_\lambda(w)\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}\prod_{i=1}^{n}(1-\overline{w_i}u_i)^m\\
&\cdot\overline{V_t U_{\phi_z(\eta)} S^* U_{\phi_z(\eta)} V_t^* K_w^\lambda(u)}\,dA_\lambda(u)\,dA_\lambda(w)\\
&= (B_{m,\lambda}S_{\phi_z(\eta)})(0) = (B_{m,\lambda}S)\circ\phi_{\phi_z(\eta)}(0) = (B_{m,\lambda}S)\\
&\circ\phi_z(\eta).
\end{aligned} \tag{20}$$

Then we have $B_{m,\lambda}S_z = (B_{m,\lambda}S)\circ\phi_z$. ☐

Lemma 6. *Let $S \in \mathcal{L}(\mathcal{A}_\lambda^2)$, and $m,\ j \geqslant 0$. If $|S^*K_z^\lambda(w)| \leqslant C$ for any $w \in \mathbb{D}^n$, then*

$$(B_{m,\lambda}B_{j,\lambda})(S) = (B_{j,\lambda}B_{m,\lambda})(S). \tag{21}$$

Proof. By Theorem 5, we only prove that $(B_{m,\lambda}B_{j,\lambda})(S)(0) = (B_{j,\lambda}B_{m,\lambda})(S)(0)$. Using Proposition 2, Fubini's theorem, and (11), we have

$$\begin{aligned}
&B_{m,\lambda}(B_{j,\lambda}S)(0) = \frac{[\lambda+m+1]}{[\lambda+1]}\int_{\mathbb{D}^n}B_{j,\lambda}S\circ\phi_0(w)\prod_{i=1}^{n}\Big(1\\
&-|w_i|^2\Big)^m dA_\lambda(w) = \frac{[\lambda+m+1]}{[\lambda+1]}\frac{[\lambda+j+1]}{[\lambda+1]}\\
&\cdot\int_{\mathbb{D}^n}\prod_{i=1}^{n}\left(1-|w_i|^2\right)^{\lambda_i+m+j+2}\times\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}(1-\overline{\eta}_i\zeta_i)^j K_w^{j+\lambda}(\zeta)\\
&\cdot\overline{K_w^{j+\lambda}(\eta)}\,S^*K_\eta^\lambda(\zeta)\,dA_\lambda(\zeta)\,dA_\lambda(\eta)\,dA_\lambda(w)\\
&= \frac{[\lambda+m+1]}{[\lambda+1]}\frac{[\lambda+j+1]}{[\lambda+1]}\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}F_{m,j}(\zeta,\eta)\\
&\cdot\overline{S^*K_\eta^\lambda(\zeta)}\,dA_\lambda(\zeta)\,dA_\lambda(\eta),
\end{aligned} \tag{22}$$

where $F_{m,j}(\zeta,\eta) = \overline{\prod_{i=1}^{n}(1-\overline{\eta}_i\zeta_i)^j\int_{\mathbb{D}^n}(1-|w_i|^2)^{\lambda_i+m+j+2}K_w^{j+\lambda}(\zeta)\overline{K_w^{j+\lambda}(\eta)}\,dA_\lambda(w)}$. Then $F_{m,j}(\zeta,\eta) = \sum_{i=1}^{l}H_i(\zeta)\overline{G_i(\eta)}$, where H_i and G_i are holomorphic functions and for some $l \geq 0$. Then, it is sufficient to show that $F_{m,j}(\zeta,\zeta) = F_{j,m}(\zeta,\zeta)$, for $w \in \mathbb{D}^n$.

$$\begin{aligned}
&F_{m,j}(\zeta,\zeta) = [\lambda+1]\prod_{i=1}^{n}\left(1-|\zeta_i|^2\right)^j\\
&\quad\cdot\int_{\mathbb{D}^n}\left(1-|w_i|^2\right)^{2\lambda_i+m+j+2}\left|K_w^{j+\lambda}(\zeta)\right|^2 dA(w)\\
&\quad= [\lambda+1]\prod_{i=1}^{n}\left(1-|\zeta_i|^2\right)^j\\
&\quad\cdot\int_{\mathbb{D}^n}\left(1-\left|\phi_{z_i}(w_i)\right|^2\right)^{2\lambda_i+m+j+2}\left|K_\zeta^{j+\lambda}(\phi_z(w))\right|^2\\
&\quad\cdot|k_z(w)|^2 dA(w) = [\lambda+1]\prod_{i=1}^{n}\left(1-|\zeta_i|^2\right)^m\\
&\quad\cdot\int_{\mathbb{D}^n}\left(1-|w_i|^2\right)^{2\lambda_i+m+j+2}\left|K_w^{m+\lambda}(\zeta)\right|^2 dA(w)\\
&\quad= F_{j,m}(\zeta,\zeta).
\end{aligned} \tag{23}$$

☐

Lemma 7. *For any $S \in \mathcal{L}(\mathcal{A}_\lambda^2)$, there is a sequence $\{S_\beta\}$ satisfying $|S_\beta^*K_z^\lambda(w)| \leq C(\beta)$ for any $w \in \mathbb{D}^n$ and $z \in \mathbb{D}^n$, such that $B_{m,\lambda}S_\beta$ converges to $B_{m,\lambda}S$ pointwise.*

Proof. Let $H^\infty = H^\infty(\mathbb{D}^n)$ denote the algebra of bounded holomorphic functions on $\mathbb{D}^n$. Both the density of H^∞ in $\mathcal{A}_\lambda^2(\mathbb{D}^n)$ and the density of finite rank operator in the ideal $\mathcal{K}$ of compact operators on $\mathcal{L}(\mathcal{A}_\lambda^2)$ imply that the set $\{\sum_{i=1}^{l} f_i\otimes g_i,\ f_i, g_i \in H^\infty\}$ is dense in the ideal $\mathcal{K}$ in the norm topology. Note that $\mathcal{K}$ is dense in the space of bounded operators on $\mathcal{A}_\lambda^2(\mathbb{D}^n)$ with respect to the strong operator topology. Thus, for each $S \in \mathcal{L}(\mathcal{A}_\lambda^2)$, there is a sequence $\{S_\beta\}$ of finite rank operators $S_\beta = \sum_{j=1}^{l(\beta)} f_{\beta,j}\otimes g_{\beta,j}$ converging strongly to S. Then Proposition 2 shows that $B_{m,\lambda}S_\beta$ converges to $B_{m,\lambda}S$ pointwise. To finish the proof we estimate

$$\begin{aligned}
\left|S_\beta^*K_z^\lambda(w)\right| &= \left|\sum_{j=1}^{l(\beta)}\left(g_{\beta,j}\otimes f_{\beta,j}\right)K_z^\lambda(w)\right|\\
&= \left|\sum_{j=1}^{l(\beta)}\left\langle K_z^\lambda(w), f_{\beta,j}(w)\right\rangle_\lambda g_{\beta,j}(w)\right|
\end{aligned}$$

$$\leq \sum_{j=1}^{l(\beta)} \left| f_{\beta,j}(z) \right| \left| g_{\beta,j}(w) \right|$$
$$\leq \sum_{j=1}^{l(\beta)} \left\| f_{\beta,j} \right\|_\infty \left\| g_{\beta,j} \right\|_\infty = C(\beta). \quad (24)$$

□

Proposition 8. *Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$, and $m, j \geqslant 0$; then $(B_{m,\lambda}B_{j,\lambda})(S) = (B_{j,\lambda}B_{m,\lambda})(S)$.*

Proof. Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$; then Lemma 7 implies that there is a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^* K_z^\lambda(w)| \leq C(\alpha)$; hence by Lemma 6

$$B_{m,\lambda}\left(B_{j,\lambda}S_\alpha\right) = B_{j,\lambda}\left(B_{m,\lambda}S_\alpha\right). \quad (25)$$

From (11), we know that $B_{m,\lambda}(B_{j,\lambda}S_\alpha) = \int_{\mathbb{D}^n} (B_{j,\lambda}S_\alpha) \circ \phi_z(w)\, dA_{\lambda+m}(w)$ and $\|(B_{j,\lambda}S_\alpha) \circ \phi_z\|_\infty = \|B_{j,\lambda}S_\alpha\|_\infty \leq \|B_{j,\lambda}\| \cdot \|S_\alpha\| \leq C(j,\lambda)\|S\|$. Furthermore $(B_{j,\lambda}S_\alpha) \circ \phi_z(w)$ converges to $(B_{j,\lambda}S) \circ \phi_z(w)$. As a consequence the functions $B_{m,\lambda}(B_{j,\lambda}S_\alpha)$ and $B_{j,\lambda}(B_{m,\lambda}S_\alpha)$ converge to $(B_{m,\lambda}B_{j,\lambda}S)(z)$ and $(B_{j,\lambda}B_{m,\lambda}S)(z)$, respectively. By the uniqueness of the limit, we have $(B_{m,\lambda}B_{j,\lambda})(S) = (B_{j,\lambda}B_{m,\lambda})(S)$. □

Theorem 9. *Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$; then there is a constant $C(n,\lambda) > 0$, such that*

$$\left|(B_{0,\lambda}S)(z) - (B_{0,\lambda}S)(w)\right| \leq C(n,\lambda)\|S\|\rho(z,w). \quad (26)$$

Proof.

$$\begin{aligned}
&\left|(B_{0,\lambda}S)(z) - (B_{0,\lambda}S)(w)\right| = \left|\langle S_z 1, 1\rangle_\lambda - \langle S_w 1, 1\rangle_\lambda\right| \\
&= \left|\operatorname{tr}\left[S_z(1\otimes 1) - S_w(1\otimes 1)\right]\right| \\
&= \left|\operatorname{tr}\left[S_z(1\otimes 1) - SU_z\left(U_zU_w 1 \otimes U_zU_w 1\right)U_z\right]\right| \\
&= \left|\operatorname{tr}\left[S_z\left(1\otimes 1 - U_{\phi_w(z)}1 \otimes U_{\phi_w(z)}1\right)\right]\right| \\
&\leq \|S_z\| \cdot \left\|1\otimes 1 - U_{\phi_w(z)}1\otimes U_{\phi_w(z)}1\right\|_{S_1} \\
&\leq \sqrt{2}\|S_z\|\left(2 - 2\left|\left\langle 1, \prod_{i=1}^n \left(\phi'_{\phi_{w_i}(z_i)}\right)^{(2+\lambda_i)/2}\right\rangle_\lambda\right|^2\right)^{1/2} \\
&\leq 2\|S\|\left[1 - \prod_{i=1}^n\left(1 - \left|\phi_{w_i}(z_i)\right|^2\right)^{(2+\lambda_i)/2}\right]^{1/2}.
\end{aligned} \quad (27)$$

Let $\alpha_i = \phi_{w_i}(z_i)$; we have

$$\begin{aligned}
&\left[1 - \prod_{i=1}^n\left(1 - |\alpha_i|^2\right)^{(2+\lambda_i)/2}\right] \\
&= 1 - \left(1 - |\alpha_1|^2\right)^{(2+\lambda_1)/2} \\
&+ \left(1 - |\alpha_1|^2\right)^{(2+\lambda_1)/2}\left[1 - \prod_{i=2}^n\left(1 - |\alpha_i|^2\right)^{(2+\lambda_i)/2}\right] \\
&\leq C(\lambda_1)|\alpha_1|^2 + \left[1 - \prod_{i=2}^n\left(1 - |\alpha_i|^2\right)^{(2+\lambda_i)/2}\right]\cdots \\
&\leq C(n,\lambda)\max_{1\leq i\leq n}|\alpha_i|^2,
\end{aligned} \quad (28)$$

where $C(n,\lambda) = n \cdot \max_{1\leq i\leq n} C(\lambda_i)$; we obtain $|(B_{0,\lambda}S)(z) - (B_{0,\lambda}S)(w)| \leq C(n,\lambda)\|S\|\rho(z,w)$. □

Corollary 10. *Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$, and $a := B_{0,\lambda}S \in L^\infty(\mathbb{D}^n)$; then*

$$\lim_{m\to\infty}\left\|B_{m,\lambda}a - a\right\|_\infty = 0. \quad (29)$$

Proof. Let $\varepsilon > 0$; choose $\delta > 0$ with $|a(z) - a(w)| < \varepsilon$ whenever $z, w \in \mathbb{D}^n$ with $\rho(z,w) < \delta$. If $w \in \mathbb{D}^n$, $m \in \mathbb{N}$, by (11), we have

$$\begin{aligned}
&\left|B_{m,\lambda}(a)(w) - a(w)\right| \leq [\lambda + m + 1] \\
&\cdot \int_{\mathbb{D}^n}\left|a\circ\phi_w(z) - a\circ\phi_w(0)\right| \\
&\cdot \prod_{i=1}^n\left(1 - |z_i|^2\right)^{\lambda_i+m} dA(z) \leq [\lambda + m + 1] \\
&\cdot \left\{\int_{\prod_{i=1}^n 0\leq|z_i|\leq\delta} + \int_{\prod_{i=1}^n \delta\leq|z_i|\leq 1}\right\}\left|a\circ\phi_w(z) - a\right. \\
&\left.\circ\,\phi_w(0)\right|\prod_{i=1}^n\left(1 - |z_i|^2\right)^{\lambda_i+m} dA(z).
\end{aligned} \quad (30)$$

Denote by I the first integral, and

$$\begin{aligned}
I &= [\lambda + m + 1]\int_{\prod_{i=1}^n 0\leq|z_i|\leq\delta}\left|B_{0,\lambda}S\left(\phi_w(z) - \phi_w(0)\right)\right| \\
&\cdot \prod_{i=1}^n\left(1 - |z_i|^2\right)^{\lambda_i+m} dA(z) \leq C(n,\lambda,\delta)\|S\|\rho(z,0) \\
&< \varepsilon.
\end{aligned} \quad (31)$$

In the first inequality we use that $\rho(\cdot,\cdot)$ is invariant under the automorphisms ϕ_w and by the Lipschitz continuity of $B_{0,\lambda}S$.

Now we estimate the second integral above.

$$\begin{aligned}
&[\lambda + m + 1]\int_{\prod_{i=1}^n \delta\leq|z_i|\leq 1}\left|a\circ\phi_w(z) - a\circ\phi_w(0)\right| \\
&\cdot \prod_{i=1}^n\left(1 - |z_i|^2\right)^{\lambda_i+m} dA(z) \leq 2[\lambda + m + 1]\cdot\|a\|_\infty \\
&\cdot \int_{\prod_{i=1}^n \delta\leq|z_i|\leq 1}\prod_{i=1}^n\left(1 - |z_i|^2\right)^{\lambda_i+m} dA(z) \\
&= 2[\lambda + m + 1]\|a\|_\infty\left(1 - \delta^2\right)^{|\lambda|+m}\operatorname{vol}\left(\mathbb{D}^n\right).
\end{aligned} \quad (32)$$

It is clear that the right-hand side converges to zero as $m \to \infty$. □

4. Toeplitz Operators to Approximate the Bounded Operators

In this section we will give a criterion for an operator approximated by Toeplitz operators with symbol equal to their (m,λ)-Berezin transforms. From Proposition 1.4.10 in [7] there exists Lemma 3.1 in [2].

Lemma 11 (see [2]). *Suppose $a<1$ and $a+b<2$. Then*

$$\sup_{z\in\mathbb{D}^n}\int_{\mathbb{D}^n}\frac{dA(w)}{\prod_{i=1}^n\left(1-|w_i|^2\right)^a\left|1-\overline{z}_iw_i\right|^b}<\infty. \tag{33}$$

Let $1<q<\infty$ and let p be the conjugate exponent of q. Note that the inequality

$$\begin{aligned}q=1+\frac{1}{p-1}&<\max_{1\le i\le n}\left\{\frac{1+2(\lambda_i+1)}{2+\lambda_i}\right\}\\&=\max_{1\le i\le n}\left\{1+\frac{1+\lambda_i}{2+\lambda_i}\right\}\end{aligned} \tag{34}$$

is equivalent to $p>\max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\}$.

Lemma 11 gives the following lemma.

Lemma 12. *Let $S\in\mathscr{L}(\mathscr{A}_\lambda^2)$ and $p>\max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\}$ and put $h(z)=\prod_{i=1}^n(1-|z_i|^2)^{-a_i}$ with $a_i=(1+\lambda_i)(2+\lambda_i)/(1+2(\lambda_i+1))$, for $i=1,\ldots,n$. Then there exists $C(n,p,\lambda)$ such that*

$$\begin{aligned}&\int_{\mathbb{D}^n}\left|\left(SK_z^\lambda\right)(w)\right|h(w)\,dA_\lambda(w)\\&\quad\le C(n,p,\lambda)\left\|S_z1\right\|_{p,\lambda}h(z)\end{aligned} \tag{35}$$

for all $z\in\mathbb{D}^n$, and

$$\begin{aligned}&\int_{\mathbb{D}^n}\left|\left(SK_z^\lambda\right)(w)\right|h(z)\,dA_\lambda(z)\\&\quad\le C(n,p,\lambda)\left\|S_w^*1\right\|_{p,\lambda}h(w)\end{aligned} \tag{36}$$

for all $w\in\mathbb{D}^n$.

Proof. Given $z\in\mathbb{D}^n$, the equality $U_z1=\prod_{i=1}^n(\phi'_{z_i})^{(2+\lambda_i)/2}=\prod_{i=1}^n(-1)^{(2+\lambda_i)/2}(1-|z_i|^2)^{(2+\lambda_i)/2}K_z^\lambda=(-1)^{|2+\lambda|/2}\prod_{i=1}^n(1-|z_i|^2)^{(2+\lambda_i)/2}K_z^\lambda$ implies that

$$\begin{aligned}SK_z^\lambda&=\frac{(-1)^{|2+\lambda|/2}}{\prod_{i=1}^n\left(1-|z_i|^2\right)^{(2+\lambda_i)/2}}SU_z1\\&=\frac{(-1)^{|2+\lambda|/2}}{\prod_{i=1}^n\left(1-|z_i|^2\right)^{(2+\lambda_i)/2}}\left(S_z1\circ\phi_z\right)\prod_{i=1}^n\left(\phi'_{z_i}\right)^{(2+\lambda_i)/2}\\&=S_z1\circ\phi_zK_z^\lambda.\end{aligned} \tag{37}$$

Thus, let $u=\phi_z(w)$, and apply the Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{D}^n}\frac{\left|\left(SK_z^\lambda\right)(w)\right|}{\prod_{i=1}^n\left(1-|w_i|^2\right)^{a_i}}dA_\lambda(w)=[\lambda+1]\\&\quad\cdot\int_{\mathbb{D}^n}\frac{\left|S_z1\circ\phi_z(w)K_z^\lambda(w)\right|}{\prod_{i=1}^n\left(1-|w_i|^2\right)^{a_i}}\prod_{i=1}^n\left(1-|w_i|^2\right)^{\lambda_i}dA(w)\\&=[\lambda+1]\\&\quad\cdot\int_{\mathbb{D}^n}\frac{\left|S_z1(u)\right|}{\prod_{i=1}^n\left(1-\left|\phi_{z_i}(u_i)\right|^2\right)^{a_i-\lambda_i}}\prod_{i=1}^n\frac{\left|1-\overline{z}_iu_i\right|^{2+\lambda_i}}{\left(1-|z_i|^2\right)^{2+\lambda_i}}\\&\quad\cdot\frac{\left(1-|z_i|^2\right)^2}{\left|1-\overline{z}_iu_i\right|^4}dA(u)=\frac{[\lambda+1]}{\prod_{i=1}^n\left(1-|z_i|^2\right)^{a_i}}\\&\quad\cdot\int_{\mathbb{D}^n}\frac{\left|S_z1(u)\right|}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{a_i-\lambda_i}\left|1-\overline{z}_iu_i\right|^{2+\lambda_i-2a_i}}dA(u)\\&=\frac{1}{\prod_{i=1}^n\left(1-|z_i|^2\right)^{a_i}}\\&\quad\cdot\int_{\mathbb{D}^n}\frac{\left|S_z1(u)\right|}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{a_i}\left|1-\overline{z}_iu_i\right|^{2+\lambda_i-2a_i}}dA_\lambda(u)\\&\le\frac{\left\|S_z1\right\|_p}{\prod_{i=1}^n\left(1-|z_i|^2\right)^{a_i}}\Bigg([\lambda+1]\\&\quad\cdot\int_{\mathbb{D}^n}\frac{dA(u)}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{a_iq-\lambda_i}\left|1-\overline{z}_iu_i\right|^{(2+\lambda_i-2a_i)q}}\Bigg)^{1/q}.\end{aligned} \tag{38}$$

According to (34) we have $a_iq-\lambda_i<1$ and $a_iq-\lambda_i+(2+\lambda_i-2a_i)q<2$, for any $i=1,\ldots,n$. Hence inequality (35) follows from Lemma 11.

The second inequality (36) follows from (35) after replacing S by S^*, interchanging w and z, and making use of

$$\begin{aligned}\left(S^*K_w^\lambda\right)(z)&=\left\langle S^*K_w^\lambda,K_z^\lambda\right\rangle_\lambda=\left\langle K_w^\lambda,SK_z^\lambda\right\rangle_\lambda\\&=\overline{SK_z^\lambda(w)},\end{aligned} \tag{39}$$

which holds for all $z,w\in\mathbb{D}^n$. □

Lemma 13. *Let $S\in\mathscr{L}(\mathscr{A}_\lambda^2)$ and $p>\max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\}$; then*

$$\begin{aligned}&\|S\|\\&\quad\le C(n,p,\lambda)\left(\sup_{z\in\mathbb{D}^n}\left\|S_z1\right\|_{p,\lambda}\right)^{1/2}\left(\sup_{z\in\mathbb{D}^n}\left\|S_z^*1\right\|_{p,\lambda}\right)^{1/2},\end{aligned} \tag{40}$$

where $C(n,p,\lambda)$ is the constant of Lemma 12.

Proof. For $f \in \mathscr{A}^2_\lambda(\mathbb{D}^n)$, $(Sf)(w) = \langle Sf, K^\lambda_w\rangle_\lambda = \int_{\mathbb{D}^n} f(z) S^* K^\lambda_z(w)\, dA_\lambda(z)$. Thus S is the integral operator with kernel function $S^* K^\lambda_z(w)$. By classical Schur's Theorem [8], it is sufficient to prove that there exist positive constants $C_1 = C(n,p,\lambda)\sup_{z\in\mathbb{D}^n}\|S_z 1\|_{p,\lambda}$ and $C_2 = C(n,p,\lambda)\sup_{z\in\mathbb{D}^n}\|S_z^* 1\|_{p,\lambda}$ and a positive measurable function g on $\mathbb{D}^n$ such that $\int_{\mathbb{D}^n}|S^*K^\lambda_z(w)| g^2(w)\, dA_\lambda(w) \le C_1 g^2(z)$ for almost every $z \in \mathbb{D}^n$ and $\int_{\mathbb{D}^n}|S^*K^\lambda_z(w)| g^2(z)\, dA_\lambda(z) \le C_1 g^2(w)$ for almost every $w\in\mathbb{D}^n$. By Lemma 12, let $g = h^{1/2}$; we get the conclusion. □

Lemma 14. *Let $\{S_m\}$ be a bounded sequence in $\mathscr{L}(\mathscr{A}^2_\lambda)$ such that $\|B_{0,\lambda}S_m\|_\infty \to 0$ as $m \to \infty$. Then $\sup_{z\in\mathbb{D}^n}|\langle (S_m)_z 1, f\rangle| \to 0$ as $m\to\infty$ for any $f \in \mathscr{A}^2_\lambda(\mathbb{D}^n)$, and $\sup_{z\in\mathbb{D}^n}|(S_m)_z 1(\cdot)| \to 0$ uniformly on compact subsets of $\mathbb{D}^n$ as $m\to\infty$.*

Proof. To prove the first statement it is sufficient to check that for each multi-index k

$$\sup_{z\in\mathbb{D}^n}\left|\left\langle (S_m)_z 1, w^k\right\rangle\right| \longrightarrow 0 \tag{41}$$

as $m \to \infty$. Since $K^\lambda_u(w) = \prod_{i=1}^n (1/(1-\overline{u_i}w_i)^{\lambda_i+2}) = \prod_{i=1}^n \sum_{\alpha_i=0}^\infty (\Gamma(\alpha_i+\lambda_i+2)/(\alpha_i)!\Gamma(\lambda_i+2))\overline{u_i}^{\alpha_i} w_i^{\alpha_i} = \sum_{|\alpha|=0}^\infty (1/\alpha!)[\prod_{i=1}^n(\Gamma(\alpha_i+\lambda_i+2)/\Gamma(\lambda_i+2))\overline{u}^\alpha w^\alpha]$, by Proposition 3 and Theorem 5 we have

$$\begin{aligned}B_{0,\lambda}S_m(\phi_z(u)) &= B_{0,\lambda}(S_m)_z(u) = \prod_{i=1}^n\left(1-|u_i|^2\right)^{2+\lambda_i} \\ &\cdot \left\langle (S_m)_z K^\lambda_u, K^\lambda_u\right\rangle_\lambda = \prod_{i=1}^n \left(1-|u_i|^2\right)^{2+\lambda_i} \\ &\cdot \sum_{|\alpha|=0}^\infty\sum_{|\beta|=0}^\infty \frac{1}{\alpha!}\frac{1}{\beta!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\prod_{j=1}^n\frac{\Gamma(\beta_j+\lambda_j+2)}{\Gamma(\lambda_j+2)} \\ &\cdot \left\langle (S_m)_z w^\alpha, w^\beta\right\rangle_\lambda \overline{u}^\alpha u^\beta.\end{aligned} \tag{42}$$

Given a multi-index k and $r\in(0,1)$, we have

$$\begin{aligned}&\int_{r\mathbb{D}^n}\frac{B_{0,\lambda}S_m(\phi_z(u))\overline{u}^k}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{2+\lambda_i}}dA_\lambda(u) = \sum_{|\alpha|=0}^\infty\sum_{|\beta|=0}^\infty\frac{1}{\alpha!}\frac{1}{\beta!} \\ &\cdot\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\prod_{j=1}^n\frac{\Gamma(\beta_j+\lambda_j+2)}{\Gamma(\lambda_j+2)}\Big\langle (S_m)_z \\ &\cdot w^\alpha, w^\beta\Big\rangle_\lambda\int_{r\mathbb{D}^n}\overline{u}^{\alpha+k}u^\beta dA_\lambda(u) = \sum_{|\alpha|=0}^\infty\frac{1}{\alpha!(\alpha+k)!} \\ &\cdot\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\Big\langle (S_m)_z \\ &\cdot w^\alpha, w^{\alpha+k}\Big\rangle_\lambda\times\int_{r\mathbb{D}^n}\left|u^{\alpha+k}\right|^2 dA_\lambda(u) \\ &= \sum_{|\alpha|=0}^\infty\frac{[\lambda+1]}{\alpha!(\alpha+k)!} \\ &\cdot\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\Big\langle (S_m)_z \\ &\cdot w^\alpha, w^{\alpha+k}\Big\rangle_\lambda\times\int_{r\mathbb{D}^n}\prod_{i=1}^n\left|u_i^{\alpha_i+\beta_i}\right|^2\left(1-|u_i|^2\right)^{\lambda_i}dA(u).\end{aligned} \tag{43}$$

Passing to the polar coordinates, the integral part is $\prod_{i=1}^n\int_0^{r^2}\rho^{\alpha_i+k_i+1-1}(1-\rho)^{\lambda_i+1-1}d\rho$. Define $I_x(a,b)$ in the standard way ([9], Formula 8.392), $I_x(a,b) = (\Gamma(a+b)/\Gamma(a)\Gamma(b))\int_0^x t^{a-1}(1-t)^{b-1}dt$; then we have

$$\begin{aligned}&\prod_{i=1}^n\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\alpha_i+k_i+1)\Gamma(\lambda_i+1)} \\ &\cdot\int_0^{r^2}\rho^{\alpha_i+k_i+1-1}(1-\rho)^{\lambda_i+1-1}d\rho \\ &= \prod_{i=1}^n I_{r^2}(\alpha_i+k_i+1,\lambda_i+1).\end{aligned} \tag{44}$$

In addition, (43) equals

$$\begin{aligned}&\sum_{|\alpha|=0}^\infty\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \\ &\cdot I_{r^2}(\alpha_i+k_i+1,\lambda_i+1) = \left\langle (S_m)_z 1, w^k\right\rangle_\lambda \\ &\cdot\prod_{i=1}^n I_{r^2}(k_i+1,\lambda_i+1) + \sum_{|\alpha|=1}^\infty\frac{1}{\alpha!} \\ &\cdot\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \\ &\cdot I_{r^2}(\alpha_i+k_i+1,\lambda_i+1).\end{aligned} \tag{45}$$

Thus we have

$$\begin{aligned}&\left|\left\langle (S_m)_z 1, w^k\right\rangle_\lambda\right| \\ &\le\frac{1}{\prod_{i=1}^n I_{r^2}(k_i+1,\lambda_i+1)}\left|\int_{r\mathbb{D}^n}\frac{B_{0,\lambda}S_m(\phi_z(u))\overline{u}^k}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{2+\lambda_i}}dA_\lambda(u)\right| \\ &+\left|\sum_{|\alpha|=1}^\infty\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda\right.\end{aligned}$$

$$\cdot \left.\frac{I_{r^2}(\alpha_i+k_i+1,\lambda_i+1)}{I_{r^2}(k_i+1,\lambda_i+1)}\right|$$

$$\le \frac{[\lambda+1]\,\|B_{0,\lambda}S_m\|_\infty}{\prod_{i=1}^n I_{r^2}(k_i+1,\lambda_i+1)}\left|\int_{r\mathbb{D}^n}\frac{|u^k|}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{2+\lambda_i}}dA(u)\right|$$
$$+\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\|(S_m)_z\|\cdot\|w^\alpha\|_{2,\lambda}\cdot\|w^{\alpha+k}\|_{2,\lambda}$$
$$\cdot\frac{I_{r^2}(\alpha_i+k_i+1,\lambda_i+1)}{I_{r^2}(k_i+1,\lambda_i+1)}$$
$$\le \frac{[\lambda+1]\,\|B_{0,\lambda}S_m\|_\infty}{\prod_{i=1}^n I_{r^2}(k_i+1,\lambda_i+1)}\left|\int_{r\mathbb{D}^n}\frac{|u^k|}{\prod_{i=1}^n\left(1-|u_i|^2\right)^{2+\lambda_i}}dA(u)\right|$$
$$+C\sum_{|\alpha|=1}^{\infty}\prod_{i=1}^n\frac{I_{r^2}(\alpha_i+k_i+1,\lambda_i+1)}{I_{r^2}(k_i+1,\lambda_i+1)}=I+C\Sigma, \tag{46}$$

where $C>0$ is a constant independent of m and z. In the last estimate we used the boundedness of the sequence S_m and the inequality

$$\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\|w^\alpha\|_{2,\lambda}\cdot\|w^{\alpha+k}\|_{2,\lambda}$$
$$=\prod_{i=1}^n\sqrt{\frac{\Gamma(\alpha_i+\lambda_i+2)\,\Gamma(\alpha_i+k_i+1)}{(\alpha_i)!\,\Gamma(\alpha_i+k_i+\lambda_i+2)}}\le 1 \tag{47}$$

which easily follows from (3). The first summand I above tends to zero as $m\to\infty$. It is sufficient to estimate the second summand Σ.

$$\Sigma=\sum_{|\alpha|=1}^{\infty}\prod_{i=1}^n\frac{I_{r^2}(\alpha_i+k_i+1,\lambda_i+1)}{I_{r^2}(k_i+1,\lambda_i+1)}$$
$$=\left(\prod_{i=1}^n\frac{\Gamma(k_i+\lambda_i+2)}{\Gamma(k_i+1)\,\Gamma(\lambda_i+1)}\right.$$
$$\left.\cdot\int_0^{r^2}\rho^{k_i}(1-\rho)^{\lambda_i}\,d\rho\right)^{-1} \tag{48}$$
$$\times\sum_{|\alpha|=1}^{\infty}\prod_{i=1}^n\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\alpha_i+k_i+1)\,\Gamma(\lambda_i+1)}$$
$$\cdot\int_0^{r^2}\rho^{\alpha_i+k_i}(1-\rho)^{\lambda_i}\,d\rho.$$

Estimating the multiple $(1-\rho)^{\lambda_i}$ for any $i=1,\ldots,n$ in both integrals $(1-r^2)^{\lambda_i}\le(1-\rho)^{\lambda_i}\le 1$, for $\lambda_i\ge 0$, and $1\le(1-\rho)^{\lambda_i}\le(1-r^2)^{\lambda_i}$, for $\lambda_i\in(-1,0)$. By ([9], Formula 8.328.2),

$$\lim_{\alpha_i\to\infty}\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\alpha_i+k_i+1)(\alpha_i+k_i+1)^{\lambda_i+1}}=1, \tag{49}$$

and thus there exists $C_i>0$ such that, for all $\alpha\in\mathbb{Z}_+^n$, $\Gamma(\alpha_i+k_i+\lambda_i+2)/\Gamma(\alpha_i+k_i+1)(\alpha_i+k_i+1)^{\lambda_i+1}<C_i$. Then

$$\Sigma\le\prod_{i=1}^n\frac{\Gamma(k_i+1)\,\Gamma(\lambda_i+1)\,(k_i+1)}{\Gamma(k_i+\lambda_i+2)}\frac{\left(1-r^2\right)^{-|\lambda_i|}}{r^{2k_i+2}}$$
$$\cdot\sum_{|\alpha|=1}^{\infty}\prod_{i=1}^n\frac{\Gamma(\alpha_i+k_i+\lambda_i+2)}{\Gamma(\alpha_i+k_i+1)\,\Gamma(\lambda_i+1)}\frac{r^{2(\alpha_i+k_i)+2}}{(\alpha_i+k_i+1)}$$
$$\le\prod_{i=1}^n C_i\frac{\Gamma(k_i+2)}{\Gamma(k_i+\lambda_i+2)}\left(1-r^2\right)^{-|\lambda_i|} \tag{50}$$
$$\cdot\sum_{|\alpha|=1}^{\infty}\prod_{i=1}^n(\alpha_i+k_i+1)^{\lambda_i}r^{2\alpha_i}.$$

The power series in r in the last line has radius of convergence equal to 1 and vanishes at 0. Thus the value of Σ becomes small if one takes r sufficiently closed to 0.

In order to prove the second statement we use the series representations (3) and (4) again,

$$\left|(S_m)_z 1(u)\right|=\left|\left\langle (S_m)_z 1,K_u^\lambda\right\rangle_\lambda\right|$$
$$\le\left|\left\langle (S_m)_z 1,\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\left[\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\overline{u}^\alpha w^\alpha\right]\right\rangle_\lambda\right|$$
$$\le\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left|\left\langle (S_m)_z 1,w^\alpha\right\rangle_\lambda\right|\cdot|u^\alpha|$$
$$\le\sum_{|\alpha|=0}^{l-1}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left|\left\langle (S_m)_z 1,w^\alpha\right\rangle_\lambda\right|$$
$$+\sum_{|\alpha|=l}^{\infty}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left|\left\langle (S_m)_z 1,w^\alpha\right\rangle_\lambda\right|\cdot|u^\alpha| \tag{51}$$
$$\le\sum_{|\alpha|=0}^{l-1}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left|\left\langle (S_m)_z 1,w^\alpha\right\rangle_\lambda\right|$$
$$+\sum_{|\alpha|=l}^{\infty}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\|S_m\|\cdot\|w^\alpha\|_\lambda\cdot|u^\alpha|$$
$$\le\sum_{|\alpha|=0}^{l-1}\frac{1}{\alpha!}\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{\Gamma(\lambda_i+2)}\left|\left\langle (S_m)_z 1,w^\alpha\right\rangle_\lambda\right|$$
$$+C\sum_{|\alpha|=l}^{\infty}\left(\prod_{i=1}^n\frac{\Gamma(\alpha_i+\lambda_i+2)}{(\alpha_i)!\,\Gamma(\lambda_i+2)}\right)^{1/2}|u^\alpha|=\Sigma_1+\Sigma_2.$$

By the first statement of the lemma the expression Σ_1 uniformly tends to zero as $m\to\infty$ with l being already fixed. To estimate Σ_2 we use the Cauchy-Schwarz inequality,

$$\begin{aligned}\Sigma_2 &= C\sum_{j=l}^{\infty}\sum_{|\alpha|=j}\left(\prod_{i=1}^{n}\frac{\Gamma(\alpha_i+\lambda_i+2)}{(\alpha_i)!\Gamma(\lambda_i+2)}\right)^{1/2}|u^\alpha| \\ &\le C\sum_{j=l}^{\infty}\left(\frac{1}{j!}\right)^{1/2}\prod_{i=1}^{n}\left(\frac{\Gamma(j+\lambda_i+2)}{\Gamma(\lambda_i+2)}\right)^{1/2} \\ &\cdot\sum_{|\alpha|=j}\left[\frac{j!}{\alpha!}\right]^{1/2}|u^\alpha| \le C\sum_{j=l}^{\infty}\left(\frac{1}{j!}\right)^{1/2} \\ &\cdot\prod_{i=1}^{n}\left(\frac{\Gamma(j+\lambda_i+2)}{\Gamma(\lambda_i+2)}\right)^{1/2}\left(\sum_{|\alpha|=j}\frac{j!}{\alpha!}|u^\alpha|^2\right)^{1/2} \\ &\cdot\left(\sum_{|\alpha|=j}1\right)^{1/2} = C\sum_{j=l}^{\infty}\left(\frac{1}{j!}\right)^{1/2} \\ &\cdot\prod_{i=1}^{n}\left(\frac{\Gamma(j+\lambda_i+2)}{\Gamma(\lambda_i+2)}\right)^{1/2}\left(\sum_{|\alpha|=j}\frac{j!}{\alpha!}|u^\alpha|^2\right)^{1/2} \\ &\cdot\left(\frac{(n+j-1)!}{j!(n-1)!}\right)^{1/2}.\end{aligned} \tag{52}$$

In [5], we get $\sum_{|\alpha|=j}(j!/\alpha!)|u^\alpha|^2 = |u|^{2j}$; we finally have $\Sigma_2 \le C\sum_{j=l}^{\infty}(1/j!)^{1/2}((n+j-1)!/j!(n-1)!)^{1/2}\prod_{i=1}^{n}(\Gamma(j+\lambda_i+2)/\Gamma(\lambda_i+2))^{1/2}r^j$. By choosing l sufficiently large we can make Σ_2 as small as needed. This ends the proof. □

Lemma 15. *Let $\{S_m\}$ be a bounded sequence in $\mathcal{L}(\mathcal{A}_\lambda^2)$ such that $\|B_{0,\lambda}S_m\|_\infty \to 0$ as $m\to\infty$. Assume that, for some $p > \max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\}$, the following inequalities hold:*

$$\begin{aligned}\sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{p,\lambda} &\le C, \\ \sup_{z\in\mathbb{D}^n}\left\|(S_m^*)_z 1\right\|_{p,\lambda} &\le C,\end{aligned} \tag{53}$$

where $C>0$ is independent of m. Then $S_m\to 0$ as $m\to\infty$ in the $\mathcal{L}(\mathcal{A}_\lambda^2)$-norm.

Proof. By Lemma 13,

$$\begin{aligned}\|S_m\| &\le C(n,p,\lambda)\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{p,\lambda}\right)^{1/2} \\ &\cdot\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m^*)_z 1\right\|_{p,\lambda}\right)^{1/2} \le C(n,p,\lambda).\end{aligned} \tag{54}$$

Then, for $\max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\} < s < p$, Hölder's inequality gives

$$\begin{aligned}\sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{s,\lambda}^s &\le \sup_{z\in\mathbb{D}^n}\int_{\mathbb{D}^n\setminus r\overline{\mathbb{D}^n}}\left|(S_m)_z 1(w)\right|^s dA_\lambda(w) \\ &+ \sup_{z\in\mathbb{D}^n}\int_{r\overline{\mathbb{D}^n}}\left|(S_m)_z 1(w)\right|^s dA_\lambda(w) \\ &\le \left(\sup_{z\in\mathbb{D}^n}\int_{\mathbb{D}^n\setminus r\overline{\mathbb{D}^n}}\left(\left|(S_m)_z 1(w)\right|^s\right)^{p/s} dA_\lambda(w)\right)^{s/p} \\ &\cdot\left(\int_{\mathbb{D}^n\setminus r\overline{\mathbb{D}^n}} dA_\lambda(w)\right)^{(1-s/p)} \\ &+ \sup_{z\in\mathbb{D}^n}\int_{r\overline{\mathbb{D}^n}}\left|(S_m)_z 1(w)\right|^s dA_\lambda(w) \\ &\le \sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{p,\lambda}^s\left(\int_{\mathbb{D}^n\setminus r\overline{\mathbb{D}^n}} dA_\lambda(w)\right)^{(1-s/p)} \\ &+ \sup_{z\in\mathbb{D}^n}\int_{r\overline{\mathbb{D}^n}}\left|(S_m)_z 1(w)\right|^s dA_\lambda(w).\end{aligned} \tag{55}$$

Then the first term is less than or equal to $C^s(n(1-r^2)^{\lambda+1}/(\lambda+1))^{(1-s/p)}$ which converges to 0 as r goes to 1 and the second term tends to 0 as $m\to\infty$ by Lemma 14. Finally, Lemma 13 gives

$$\begin{aligned}\|S_m\| &\le C(n,s,\lambda)\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{s,\lambda}\right)^{1/2} \\ &\cdot\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m^*)_z 1\right\|_{s,\lambda}\right)^{1/2} \le C(n,s,\lambda) \\ &\cdot\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m)_z 1\right\|_{s,\lambda}\right)^{1/2}\left(\sup_{z\in\mathbb{D}^n}\left\|(S_m^*)_z 1\right\|_{p,\lambda}\right)^{1/2} \\ &\longrightarrow 0,\end{aligned} \tag{56}$$

as $m\to 0$, proving the statement of the lemma. □

Corollary 16. *Let $S\in\mathcal{L}(\mathcal{A}_\lambda^2)$ such that, for some $p > \max_{1\le i\le n}\{(1+2(\lambda_i+1))/(1+\lambda_i)\}$, the following inequalities hold*

$$\begin{aligned}\sup_{z\in\mathbb{D}^n}\left\|S_z 1-\left(T_{B_{m,\lambda}S}\right)_z 1\right\|_{p,\lambda} &\le C, \\ \sup_{z\in\mathbb{D}^n}\left\|S_z^* 1-\left(T_{B_{m,\lambda}S^*}\right)_z 1\right\|_{p,\lambda} &\le C,\end{aligned} \tag{57}$$

where $C>0$ is independent of m. Then $T_{B_{m,\lambda}S}\to S$ as $m\to\infty$ in the $\mathcal{L}(\mathcal{A}_\lambda^2)$-norm.

Proof. Let $S_m = S - T_{B_{m,\lambda}S}$ and by Proposition 8, we have

$$\begin{aligned}B_{0,\lambda}S_m &= B_{0,\lambda}S - B_{0,\lambda}T_{B_{m,\lambda}S} = B_{0,\lambda}S - B_{0,\lambda}(B_{m,\lambda}S) \\ &= B_{0,\lambda}S - B_{m,\lambda}(B_{0,\lambda}S).\end{aligned} \tag{58}$$

By Corollary 10, the right of equation uniformly tends to 0 as $m \to \infty$; that is, $\|B_{0,\lambda} S_m\|_\infty \to 0$. An application of Lemma 15 finishes the proof. □

Theorem 17. *Let $S \in \mathscr{L}(\mathscr{A}_\lambda^2)$. If there is $p > \max_{1 \le i \le n}\{(1 + 2(\lambda_i + 1))/(1 + \lambda_i)\}$, such that*

$$\sup_{z \in \mathbb{D}^n} \left\| T_{(B_{m,\lambda} S) \circ \phi_z} 1 \right\|_{p,\lambda} \le C,$$
$$\sup_{z \in \mathbb{D}^n} \left\| T^*_{(B_{m,\lambda} S) \circ \phi_z} 1 \right\|_{p,\lambda} \le C, \tag{59}$$

where $C > 0$ is independent of m, then $T_{B_{m,\lambda} S} \to S$ as $m \to \infty$ in the $\mathscr{L}(\mathscr{A}_\lambda^2)$-norm.

Proof. Since $T_{(B_{m,\lambda} S) \circ \phi_z} = (T_{B_{m,\lambda} S})_z$ and $T^*_{(B_{m,\lambda} S) \circ \phi_z} = T^*_{B_{m,\lambda} S_z} = T_{\overline{B_{m,\lambda} S_z}} = T_{B_{m,\lambda} S_z^*} = T_{(B_{m,\lambda} S^*) \circ \phi_z}$, by Corollary 16, we only need to prove that (59) implies (53). Hence, it is sufficient to prove that

$$\sup_{z \in \mathbb{D}^n} \left\| S_z 1 \right\|_{p,\lambda} < \infty. \tag{60}$$

By Lemma 13, we have

$$\left\| T_{B_{m,\lambda} S} \right\| \le C(n, p, \lambda) \left(\sup_{z \in \mathbb{D}^n} \left\| T_{(B_{m,\lambda} S) \circ \phi_z} 1 \right\|_{p,\lambda} \right)^{1/2} \cdot \left(\sup_{z \in \mathbb{D}^n} \left\| T^*_{(B_{m,\lambda} S) \circ \phi_z} 1 \right\|_{p,\lambda} \right)^{1/2} < C, \tag{61}$$

where C is independent of m. Let $S_m = S - T_{B_{m,\lambda} S}$; then $\|S_m\| \le C$, where C is independent of m. According to the proof of Corollary 16, we get $\lim_{m \to \infty} \|B_{0,\lambda} S_m\|_\infty = 0$. Let f be an analytic polynomial with $\|f\|_{q,\lambda} = 1$; Lemma 14 implies $\sup_{z \in \mathbb{D}^n} |\langle (S_m)_z 1, f \rangle| \to 0$ as $m \to \infty$. Then, for any $\varepsilon > 0$ and any $z_0 \in \mathbb{D}^n$, there is a sufficiently large m such that

$$\begin{aligned} \left| \left\langle S_{z_0} 1, f \right\rangle_\lambda \right| &\le \left| \left\langle (S_m)_{z_0} 1, f \right\rangle_\lambda \right| + \left| \left\langle \left(T_{B_{m,\lambda} S} \right)_{z_0} 1, f \right\rangle_\lambda \right| \\ &\le \sup_{z \in \mathbb{D}^n} \left| \left\langle (S_m)_z 1, f \right\rangle_\lambda \right| \\ &\quad + \left| \left\langle \left(T_{B_{m,\lambda} S} \right)_{z_0} 1, f \right\rangle_\lambda \right| \le \varepsilon + C, \end{aligned} \tag{62}$$

where C is independent of m and z_0. Since ε is arbitrary, we have inequality (60). □

Competing Interests

The authors declare that they have no competing interests.

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