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Research Article

Continuity Results and Error Bounds on Pseudomonotone Vector Variational Inequalities via Scalarization

Xin Zuo, Hong-Zhi Wei, and Chun-Rong Chen

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Correspondence should be addressed to Chun-Rong Chen; chencr1981@163.com

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Continuity (both lower and upper semicontinuities) results of the Pareto/efficient solution mapping for a parametric vector variational inequality with a polyhedral constraint set are established via scalarization approaches, within the framework of strict pseudomonotonicity assumptions. As a direct application, the continuity of the solution mapping to a parametric weak Minty vector variational inequality is also discussed. Furthermore, error bounds for the weak vector variational inequality in terms of two known regularized gap functions are also obtained, under strong pseudomonotonicity assumptions.

1. Introduction

The concept of the vector variational inequality (VVI, for short) was first introduced by Giannessi in his well-known paper [1]. This model has received extensive attentions in the last three decades. Many important results on various kinds of vector variational inequalities (VVIs, for short) have been established; for example, see [2–4] and the references therein.

Nowadays, VVIs as powerful tools appear in many important problems from theory to applications, such as multiobjective/vector optimization, network economics, and financial equilibrium. In such situation it is very important to understand behaviors of solutions of a VVI when the problem's data vary. In other words, we need to know properties of solutions of parametric VVIs when the parameters vary. Therefore, one of the main topics is to investigate stability of the solution mappings for parametric VVIs and vector equilibrium problems (VEPs, for short). Usually, solution stability investigations were devoted to upper and lower semicontinuities, Lipschitz/Hölder continuity, and error bounds; see, for example, [5–22]. Our interest in this paper is to further discuss the continuity (both upper and lower semicontinuities) of solution mappings for parametric VVIs and error bounds for weak VVIs in terms of the known regularized gap functions.

In the available literature on the subject of solution semicontinuity for parametric VVIs and VEPs, there are two phenomena that could be observed. On the one hand, among many approaches dealing with the lower semicontinuity and continuity of solution mappings for parametric VVIs and VEPs, the scalarization method is of considerable interest and effective (see [7, 8, 10–14, 19]). On the other hand, most of the semicontinuity results were devoted to the weak Pareto/efficient solutions of parametric VVIs and VEPs, while there have been only few investigations on the Pareto/efficient solutions of parametric VVIs and VEPs (see [12–14]). Obviously, the latter is more difficult, as the ordering relations involved are neither closed nor open. Based on the above observations, we would study the continuity (both lower and upper semicontinuities) of Pareto/efficient solution mappings for parametric VVIs via scalarization.

It is well known that the monotonicity of mappings plays a vital role in the study of VVIs and VEPs, such as solution existence and stability analysis. In particular, we notice that almost all scalarization methods dealing with the lower semicontinuity of parametric VVIs and VEPs share a common feature: sufficient conditions are guaranteed under strict monotonicity assumptions or some variants (see [7, 8, 11, 12, 14]). Recently, Wang and Huang [10] have discussed the lower semicontinuity of the weak Pareto/efficient solutions to a parametric vector mixed variational inequality under a kind of strict pseudomonotonicity assumptions. To the best of our knowledge, there was nearly no lower semicontinuity result for parametric VVIs and VEPs with strict pseudomonotone mappings via scalarization in the literature. Therefore, we will further study the continuity (both lower and upper semicontinuities) of the Pareto/efficient solution mapping for a parametric VVI with a polyhedral constraint set discussed in our previous work [12], within the framework of strict pseudomonotonicity assumptions. The technique of proofs is adopted by scalarization, based on the useful properties proposed by Lee and Yen [23] and Lee et al. [24]. The results obtained relax strict monotonicity assumptions used in [12] to strict pseudomonotonicity ones. As a direct application, the continuity of the solution mapping to a parametric weak Minty VVI is also discussed.

Additionally, as we know, error bounds for VVIs and VEPs have played important roles in stability analysis. Using error bounds, one can obtain an upper estimate of the distance between an arbitrary feasible point and the solution set of VVIs or VEPs. Gap functions have turned out to be very useful in deriving the error bounds (cf. [18, 20–22]). About error bounds for VVIs and VEPs, there are also two phenomena that should be noticed. On the one hand, most of the error bound results were devoted to scalar variational inequalities, while there still have been only few investigations for VVIs and VEPs. On the other hand, nearly all error bound results for VVIs and VEPs are obtained under strong monotonicity assumptions (see [20–22]). Whence, we would further deduce error bounds for weak VVIs in terms of the known regularized gap functions. Our models are discussed within the framework of strong pseudomonotonicity assumptions, which are properly weaker than strong monotonicity ones used in most papers. Thus, the conclusions obtained improve main results of [20–22].

The rest of the paper is organized as follows. In Section 2, we introduce the weak vector variational inequality (WVVI), the parametric VVI problem (PVVI), and the parametric weak Minty VVI problem (PWMVVI) and recall some necessary concepts and properties. In particular, the concepts of ξ-pseudomonotonicity, strict ξ-pseudomonotonicity, and strong ξ-pseudomonotonicity are presented. In Section 3, we discuss sufficient conditions that guarantee the continuity of solution mappings $S(\cdot)$ for (PVVI) and $S^w_M(\cdot)$ for (PWMVVI) by using scalarization approaches, under strict pseudomonotonicity assumptions. In Section 4, we deduce error bounds for (WVVI) in terms of regularized gap functions ϕ_α and g_α, under strong pseudomonotonicity assumptions. The last section gives some concluding remarks.

2. Preliminaries

Let $K \subset \mathbb{R}^n$ be a nonempty, closed, and convex set. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, \ldots, p)$ be vector-valued functions. For abbreviation we write $F := (F_1, \ldots, F_p)$ and $F(x)(v) := (\langle F_1(x), v\rangle, \ldots, \langle F_p(x), v\rangle)$ for every $x \in K$ and $v \in \mathbb{R}^n$. The scalar product and the Euclidean norm in an Euclidean space are denoted by $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$, respectively. For a set A in an Euclidean space, $\operatorname{int} A$ and $\operatorname{cl} A$ denote the interior and the closure of A, respectively. $C := \mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$. Let Λ and Ω be nonempty subsets of Euclidean spaces, and set $\Delta := \{\xi = (\xi_1, \ldots, \xi_p) \in \mathbb{R}^p_+ : \|\xi\| = 1\}$ and $\Delta^+ := \{\xi \in \operatorname{int} \mathbb{R}^p_+ : \|\xi\| = 1\}$.

Consider the vector variational inequality (VVI) (resp., the weak vector variational inequality (WVVI)), which consists in finding $\overline{x} \in K$ such that

$$F(\overline{x})(x - \overline{x}) \notin -C \setminus \{0_{\mathbb{R}^p}\} \quad (\text{resp., } F(\overline{x})(x - \overline{x}) \notin -\operatorname{int} C), \quad \forall x \in K. \tag{1}$$

The solution sets of (VVI) and (WVVI) are denoted by sol(VVI) and sol(WVVI), respectively. The elements of the first set (resp., the second set) are called the Pareto/efficient solutions (resp., the weak Pareto/efficient solutions) of (VVI).

When the mapping F is perturbed by the parameter $\mu \in \Omega$, we can consider the following parametric vector variational inequality (PVVI) (resp., parametric weak vector variational inequality (PWVVI)) of finding $\overline{x} \in K$ such that

$$F(\overline{x}, \mu)(x - \overline{x}) \notin -C \setminus \{0_{\mathbb{R}^p}\} \quad (\text{resp., } F(\overline{x}, \mu)(x - \overline{x}) \notin -\operatorname{int} C), \quad \forall x \in K, \tag{2}$$

where $F_i : K \times \Omega \to \mathbb{R}^n$ $(i = 1, \ldots, p)$ are vector-valued functions.

For each $\mu \in \Omega$, we denote the solution mappings of (PVVI) and (PWVVI) by $S(\mu)$ and $S^w(\mu)$, respectively; that is,

$$\begin{aligned} S(\mu) &:= \{\overline{x} \in K : F(\overline{x}, \mu)(x - \overline{x}) \notin -C \setminus \{0_{\mathbb{R}^p}\},\ \forall x \in K\}, \\ S^w(\mu) &:= \{\overline{x} \in K : F(\overline{x}, \mu)(x - \overline{x}) \notin -\operatorname{int} C,\ \forall x \in K\}. \end{aligned} \tag{3}$$

For every $\xi \in C \setminus \{0_{\mathbb{R}^p}\}$, we consider the variational inequality $(\mathrm{VI})_\xi$ of finding $\overline{x} \in K$ such that

$$\langle f_\xi(\overline{x}), x - \overline{x}\rangle \geq 0, \quad \forall x \in K, \tag{4}$$

where $f_\xi(x) := \sum_{i=1}^p \xi_i F_i(x)$, with the corresponding parametric variational inequality $(\mathrm{PVI})_\xi$ of finding $\overline{x} \in K$ such that

$$\langle f_\xi(\overline{x}, \mu), x - \overline{x}\rangle \geq 0, \quad \forall x \in K, \tag{5}$$

where $f_\xi(x, \mu) := \sum_{i=1}^p \xi_i F_i(x, \mu)$.

Denote the solution set of $(\mathrm{VI})_\xi$ by $\mathrm{sol}(\mathrm{VI})_\xi$ and the solution mapping of $(\mathrm{PVI})_\xi$ by $S_\xi(\mu)$: that is,

$$S_\xi(\mu) := \{\overline{x} \in K : \langle f_\xi(\overline{x}, \mu), x - \overline{x}\rangle \geq 0,\ \forall x \in K\}. \tag{6}$$

For $i = 1, \ldots, p$, we denote the variational inequality associated with F_i as $(\mathrm{VI})_i$, that is, to find $\overline{x} \in K$ such that

$$\langle F_i(\overline{x}), x - \overline{x}\rangle \geq 0, \quad \forall x \in K. \tag{7}$$

The solution set of $(\mathrm{VI})_i$ $(i = 1, \ldots, p)$ is denoted by $\mathrm{sol}(\mathrm{VI})_i$.

Lemma 1 (see [19, 23, 24]). *It holds that*

$$\bigcup_{\xi\in\Delta^+} \operatorname{sol}(\mathrm{VI})_\xi \subseteq \operatorname{sol}(\mathrm{VVI}) \subseteq \operatorname{sol}(\mathrm{WVVI}) = \bigcup_{\xi\in\Delta} \operatorname{sol}(\mathrm{VI})_\xi . \tag{8}$$

And, sol(WVVI) *is a closed set provided that F is a continuous mapping. If K is a polyhedral convex set, that is, K is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$, then the first inclusion in the above formula holds as equality.*

Remark 2. Let $\Sigma := \{\xi = (\xi_1, \ldots, \xi_p) \in \mathbb{R}^p_+ : \sum_{i=1}^{p} \xi_i = 1\}$ be the unit simplex in C. The relative interior of Σ is described by the formula $\Sigma^+ := \{\xi \in \operatorname{int} \mathbb{R}^p_+ : \sum_{i=1}^{p} \xi_i = 1\}$. If we replace "$\Delta$" by "$\Sigma$" in Lemma 1, then the corresponding result still holds (cf. [19, Theorem 2.1]).

Lemma 3 (see [24, 25]). *Suppose that there exist $\alpha > 0$ such that*

$$\left\langle F_i(x') - F_i(x), x' - x \right\rangle \ge \alpha \left\|x' - x\right\|^2, \quad \forall x, x' \in K,\ \forall i \in \{1, \ldots, p\}, \tag{9}$$

and $l > 0$ such that

$$\left\|F_i(x') - F_i(x)\right\| \le l \left\|x' - x\right\|, \quad \forall x, x' \in K,\ \forall i \in \{1, \ldots, p\}. \tag{10}$$

Then the solution sets sol(VVI) *and* sol(WVVI) *are nonempty bounded and compact, respectively, and*

$$\bigcup_{\xi\in\Delta^+} \operatorname{sol}(\mathrm{VI})_\xi \subseteq \operatorname{sol}(\mathrm{VVI}) \subseteq \operatorname{sol}(\mathrm{WVVI}) = \bigcup_{\xi\in\Delta} \operatorname{sol}(\mathrm{VI})_\xi = \operatorname{cl} \bigcup_{\xi\in\Delta^+} \operatorname{sol}(\mathrm{VI})_\xi . \tag{11}$$

Moreover, for every $\xi \in \Delta$, the variational inequality $(\mathrm{VI})_\xi$ *has a unique solution in K.*

Recalling from [26], we say that the function $H : \mathbb{R}^n \to \mathbb{R}^n$ is pseudomonotone on $K \subset \mathbb{R}^n$ iff

$$\langle H(x), y - x\rangle \ge 0 \Longrightarrow \langle H(y), y - x\rangle \ge 0, \quad \forall x, y \in K. \tag{12}$$

It is called strictly pseudomonotone on K iff

$$\langle H(x), y - x\rangle \ge 0 \Longrightarrow \langle H(y), y - x\rangle > 0, \quad \forall x, y \in K,\ x \ne y. \tag{13}$$

It is called strongly pseudomonotone on K iff there exists a constant $c > 0$ such that

$$\langle H(x), y - x\rangle \ge 0 \Longrightarrow \langle H(y), y - x\rangle \ge c\left\|x - y\right\|^2, \quad \forall x, y \in K. \tag{14}$$

Definition 4. The mapping $F := (F_1, \ldots, F_p)$ is said to be ξ-pseudomonotone (resp., strictly ξ-pseudomonotone and strongly ξ-pseudomonotone) on K iff $\forall \xi = (\xi_1, \ldots, \xi_p) \in \Delta$, f_ξ is pseudomonotone (resp., strictly pseudomonotone and strongly pseudomonotone) on K.

Clearly, the strong ξ-pseudomonotonicity implies the strict ξ-pseudomonotonicity, which, in turn, implies the ξ-pseudomonotonicity. Definition 4 is motivated by [10, Definition 2.3] and [25, Definitions 1 and 2]. Similar to [25], the vector variational inequality (VVI) is said to be pseudomonotone (resp., strictly pseudomonotone and strongly pseudomonotone) iff F is ξ-pseudomonotone (resp., strictly ξ-pseudomonotone, strongly ξ-pseudomonotone) on K. Next, we give an example to illustrate Definition 4.

Example 5. Let $K = \mathbb{R}^2$. Define $F_1, F_2 : K \to \mathbb{R}^2$ as $F_1(x) = (x_1 - 1, x_2)$ and $F_2(x) = ((1/2)x_1, x_2 + 1)$ for every $x = (x_1, x_2) \in \mathbb{R}^2$, respectively. For every $\xi = (\xi_1, \xi_2) \in \Delta$ and $x \in K$,

$$\begin{aligned} f_\xi(x) &= \sum_{i=1}^{2} \xi_i F_i(x) \\ &= \left(\left(\xi_1 + \frac{\xi_2}{2}\right)x_1 - \xi_1, (\xi_1 + \xi_2)x_2 + \xi_2\right). \end{aligned} \tag{15}$$

Thus, we have that $\forall x = (x_1, x_2),\ x' = (x_1', x_2') \in K$,

$$\begin{aligned} &\left\langle f_\xi(x') - f_\xi(x), x' - x\right\rangle \\ &= \left(\xi_1 + \frac{\xi_2}{2}\right)\left(x_1' - x_1\right)^2 + (\xi_1 + \xi_2)\left(x_2' - x_2\right)^2 \\ &\ge \frac{\xi_1 + \xi_2}{2}\left[\left(x_1' - x_1\right)^2 + \left(x_2' - x_2\right)^2\right] \\ &= \frac{\xi_1 + \xi_2}{2}\left\|x' - x\right\|^2 \ge \frac{\xi_1^2 + \xi_2^2}{2}\left\|x' - x\right\|^2 \\ &= \frac{1}{2}\left\|x' - x\right\|^2 . \end{aligned} \tag{16}$$

Hence, $F := (F_1, F_2)$ is strongly ξ-monotone on K with modulus $c = 1/2$. Clearly, a strongly ξ-monotone mapping is strongly ξ-pseudomonotone. So, F is strongly ξ-pseudomonotone on K, of course, it is strictly ξ-pseudomonotone and ξ-pseudomonotone.

Remark 6. If F is ξ-pseudomonotone (resp., strictly ξ-pseudomonotone and strongly ξ-pseudomonotone) on K, then it is obvious that, for every $i = 1, \ldots, p$, F_i is pseudomonotone (resp., strictly pseudomonotone and strongly pseudomonotone) on K. However, the following examples show the converses are not true.

Example 7. Let $K = [1, +\infty[$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = -x$ and $F_2(x) = 1$, respectively. Clearly, F_1, F_2 are pseudomonotone on K. However, we can show that $F := (F_1, F_2)$ is not ξ-pseudomonotone on K. For $\xi = (\xi_1, \xi_2) \in \Delta$,

$f_\xi(x) = \sum_{i=1}^{2} \xi_i F_i(x) = -\xi_1 x + \xi_2$, $\forall x \in K$. Take $\xi = (1/2, \sqrt{3}/2) \in \Delta$ and $x_1 = 1$, $x_2 = 2$ in K. Then $\langle f_\xi(x_1), x_2 - x_1\rangle = (-1/2 + \sqrt{3}/2) \cdot 1 = (\sqrt{3} - 1)/2 > 0$, but $\langle f_\xi(x_2), x_2 - x_1\rangle = (-1 + \sqrt{3}/2) \cdot 1 = (\sqrt{3} - 2)/2 < 0$, which implies that F is not ξ-pseudomonotone on K.

Example 8. Let $K = [-1, 1]$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = |x| + 1$ and $F_2(x) = -1$, respectively. Clearly, F_1, F_2 are strictly pseudomonotone on K. However, we can show that $F := (F_1, F_2)$ is not strictly ξ-pseudomonotone on K. For $\xi = (\xi_1, \xi_2) \in \Delta$, $f_\xi(x) = \xi_1|x| + \xi_1 - \xi_2$, $\forall x \in K$. Take $\xi = (\sqrt{2}/2, \sqrt{2}/2) \in \Delta$ and distinct points $x_1 = -1$, $x_2 = 0$ in K. Then $\langle f_\xi(x_1), x_2 - x_1\rangle = \sqrt{2}/2 > 0$, but $\langle f_\xi(x_2), x_2 - x_1\rangle = 0$, which implies that F is not strictly ξ-pseudomonotone on K.

Example 9. Let $K = [1, 2]$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = x$ and $F_2(x) = -x$, respectively. It is easy to see that F_1, F_2 are strongly pseudomonotone on K with constants 1, respectively. Taking F_1, for example, $\forall x, y \in K$, if $\langle F_1(x), y - x\rangle = x(y - x) \ge 0$, we have $y - x \ge 0$. Noting that $|x - y| \le 1$ for all $x, y \in K$, then, we deduce that $\langle F_1(y), y - x\rangle = y(y - x) \ge y - x = |x - y| \ge |x - y|^2$. For $\xi = (\xi_1, \xi_2) \in \Delta$, $f_\xi(x) = (\xi_1 - \xi_2)x$, $\forall x \in K$. Take $\xi = (\sqrt{2}/2, \sqrt{2}/2) \in \Delta$ and distinct points x_1, x_2 in K. Then $\langle f_\xi(x_1), x_2 - x_1\rangle = 0$, but $\langle f_\xi(x_2), x_2 - x_1\rangle = 0 < c|x_1 - x_2|^2$ for any $c > 0$, which implies that F is not strongly ξ-pseudomonotone on K.

Remark 10. If we use "Σ" to replace "Δ" in Definition 4, then analogous concepts can be introduced, and similar discussions hold as above.

Associated with (WVVI), we consider the following weak Minty vector variational inequality (WMVVI) of finding $\overline{x} \in K$ such that

$$F(x)(\overline{x} - x) \notin \operatorname{int} C, \quad \forall x \in K. \tag{17}$$

When F is perturbed by the parameter $\mu \in \Omega$, we consider the parametric weak Minty vector variational inequality (PWMVVI) of finding $\overline{x} \in K$ such that

$$F(x, \mu)(\overline{x} - x) \notin \operatorname{int} C, \quad \forall x \in K. \tag{18}$$

The solution set of (WMVVI) is denoted as sol(WMVVI), and the solution mapping of (PWMVVI) is denoted by $S_M^w(\mu)$: that is,

$$\begin{aligned} &S_M^w(\mu) \\ &\quad := \{\overline{x} \in K : F(x, \mu)(\overline{x} - x) \notin \operatorname{int} C,\ \forall x \in K\}. \end{aligned} \tag{19}$$

The following result is a direct corollary deduced from [27, Theorem 4.2] and Lemma 1.

Lemma 11. *If each $F_i : K \to \mathbb{R}^n$ $(i = 1, \ldots, p)$ is continuous and pseudomonotone, then*

$$\operatorname{sol}(\text{WMVVI}) = \operatorname{sol}(\text{WVVI}) = \bigcup_{\xi \in \Delta} \operatorname{sol}(\text{VI})_\xi. \tag{20}$$

In what follows, the notation $B(\overline{\lambda}, \delta)$ denotes the open ball with center $\overline{\lambda} \in \Lambda$ and radius $\delta > 0$.

Definition 12 (see [28]). Let $G : \Lambda \rightrightarrows \mathbb{R}^n$ be a set-valued mapping and let $\overline{\lambda} \in \Lambda$ be given.

(i) G is called lower semicontinuous (l.s.c) at $\overline{\lambda}$ iff for any open set V satisfying $V \cap G(\overline{\lambda}) \ne \emptyset$, there exists $\delta > 0$ such that, for every $\lambda \in B(\overline{\lambda}, \delta)$, $V \cap G(\lambda) \ne \emptyset$.

(ii) G is called upper semicontinuous (u.s.c) at $\overline{\lambda}$ iff for any open set V satisfying $G(\overline{\lambda}) \subseteq V$, there exists $\delta > 0$ such that, for every $\lambda \in B(\overline{\lambda}, \delta)$, $G(\lambda) \subseteq V$.

We say G is l.s.c (resp., u.s.c) on Λ, iff it is l.s.c (resp., u.s.c) at each $\lambda \in \Lambda$. G is said to be continuous on Λ iff it is both l.s.c and u.s.c on Λ.

Remark that G is l.s.c at $\overline{\lambda}$ iff for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \overline{\lambda}$ and any $\overline{x} \in G(\overline{\lambda})$, there exists a sequence $x_n \in G(\lambda_n)$ such that $x_n \to \overline{x}$.

If G has compact values (i.e., $G(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then G is u.s.c at $\overline{\lambda}$ iff for any sequences $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \overline{\lambda}$ and $\{x_n\}$ with $x_n \in G(\lambda_n)$, there exist $\overline{x} \in G(\overline{\lambda})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \to \overline{x}$.

The following lemma plays an important role in the proof of the lower semicontinuity of the solution mappings $S(\cdot)$ and $S_M^w(\cdot)$.

Lemma 13 (see [28, page 114]). *The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of l.s.c set-valued mappings Γ_i from a topological space X into a topological space Y is also an l.s.c set-valued mapping from X into Y, where I is an index set.*

3. Continuity Results

Throughout this section, we make the following assumption (A): $\forall \mu \in \Omega$, $S(\mu)$ is nonempty and bounded; $\{\xi \in \Delta^+ : S_\xi(\mu) \ne \emptyset\} = \Delta^+$: that is, $\forall \xi \in \Delta^+$, $S_\xi(\mu) \ne \emptyset$.

For example, based on Lemma 3, assumption (A) is fulfilled if there exist constants $\alpha > 0$ and $l > 0$ such that $\forall \mu, \mu' \in \Omega$,

$$\begin{aligned} &\langle F_i(x', \mu) - F_i(x, \mu), x' - x\rangle \ge \alpha \|x' - x\|^2, \\ &\qquad \forall x, x' \in K,\ \forall i \in \{1, \ldots, p\}, \\ &\|F_i(x', \mu') - F_i(x, \mu)\| \le l(\|x' - x\| + \|\mu' - \mu\|), \\ &\qquad \forall x, x' \in K,\ \forall i \in \{1, \ldots, p\}. \end{aligned} \tag{21}$$

Assumption (A) is also fulfilled if K is a compact convex set and for any $\mu \in \Omega$, $F_i(\cdot, \mu)$, $i = 1, \ldots, p$, are continuous on K (see [24, Theorem 2.2]).

Now we study the lower semicontinuity of $S(\cdot)$ with a strictly ξ-pseudomonotone mapping F, but not strictly monotone functions F_i, $i = 1, \ldots, p$. The latter was considered in our previous work [12].

If $F_i : K \to \mathbb{R}^n$, $i = 1, \ldots, p$, are strictly monotone on K, that is, $\langle F_i(x') - F_i(x), x' - x\rangle > 0$, $\forall x, x' \in K$, $x \ne x'$,

$i = 1, \dots, p$, then it is clear that $\forall \xi = (\xi_1, \dots, \xi_p) \in \Delta$, $F := (F_1, \dots, F_p)$ is strictly ξ-monotone on K; that is,

$$\left\langle f_\xi\left(x'\right) - f_\xi(x), x' - x\right\rangle = \sum_{i=1}^{p} \xi_i \left\langle F_i\left(x'\right) - F_i(x), x' - x\right\rangle > 0, \quad \forall x, x' \in K,\ x \neq x'. \tag{22}$$

Obviously, it implies that F is strictly ξ-pseudomonotone on K. However, the following example shows that the converse is not true. That is, the strict ξ-pseudomonotonicity of F is properly weaker than the strict monotonicity of F_i ($i \in \{1, \dots, p\}$).

Example 14. Let $K = [1, +\infty[$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = x$ and $F_2(x) = 1$, respectively. For $\xi = (\xi_1, \xi_2) \in \Delta$, $f_\xi(x) = \sum_{i=1}^{2} \xi_i F_i(x) = \xi_1 x + \xi_2$, $\forall x \in K$. We show that $F := (F_1, F_2)$ is strictly ξ-pseudomonotone on K. For any $\xi = (\xi_1, \xi_2) \in \Delta$ and $x_1, x_2 \in K$, $x_1 \neq x_2$, suppose that $\langle f_\xi(x_1), x_2 - x_1\rangle = (\xi_1 x_1 + \xi_2)(x_2 - x_1) \geq 0$. As $\xi_1 x_1 + \xi_2 > 0$ and $x_1 \neq x_2$, we have $x_2 - x_1 > 0$. Thus, $\langle f_\xi(x_2), x_2 - x_1\rangle = (\xi_1 x_2 + \xi_2)(x_2 - x_1) > 0$, because $\xi_1 x_2 + \xi_2 > 0$. Hence, F is strictly ξ-pseudomonotone on K. However, F_1, F_2 are not both strictly monotone on K. It is clear that F_2 is not strictly monotone on K, since $\langle F_2(x_2) - F_2(x_1), x_2 - x_1\rangle = 0$, $\forall x_1, x_2 \in K$, $x_1 \neq x_2$. Moreover, F is also not strictly ξ-monotone on K. In fact, taking $\xi = (0, 1) \in \Delta$ and $x_1 \neq x_2$ in K, we get $\langle f_\xi(x_2) - f_\xi(x_1), x_2 - x_1\rangle = \xi_1 (x_2 - x_1)^2 = 0$.

Lemma 15. *Let $\xi \in \Delta^+$. Suppose that assumption (A) holds and the following conditions are satisfied:*

(i) *$F_i(\cdot, \cdot)$, $i = 1, \dots, p$, are continuous on $K \times \Omega$, where $K \subset \mathbb{R}^n$ is a nonempty, closed, and convex set.*

(ii) *For any $\mu \in \Omega$, $F(\cdot, \mu)$ is strictly ξ-pseudomonotone on K: that is,*

$$\left\langle f_\xi(x, \mu), x' - x\right\rangle \geq 0 \Longrightarrow \left\langle f_\xi\left(x', \mu\right), x' - x\right\rangle > 0, \quad \forall x, x' \in K,\ x \neq x'. \tag{23}$$

Then, $S_\xi(\cdot)$ is l.s.c on Ω.

Proof. Suppose to the contrary that there exists $\mu_0 \in \Omega$ such that $S_\xi(\cdot)$ is not l.s.c at μ_0. Then there exist $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in S_\xi(\mu_0)$, such that, for any $x_n \in S_\xi(\mu_n)$, $x_n \nrightarrow x_0$.

Since $x_0 \in K$ and $K \subset \mathbb{R}^n$ is a closed set, there exists $\overline{x}_n \in K$ such that $\overline{x}_n \to x_0$. Fix any $y_n \in S_\xi(\mu_n)$. From Lemma 1, $S_\xi(\mu_n) \subseteq S(\mu_n)$. Hence, the sequence $\{y_n\} \subset K$ is bounded by the boundedness of $S(\mu_n)$. Without loss of generality, we can assume that there is a $y_0 \in \mathbb{R}^n$ such that $y_n \to y_0$. As the set K is closed, $y_0 \in K$. It follows from $x_0 \in S_\xi(\mu_0)$ and $y_0 \in K$ that

$$\left\langle f_\xi(x_0, \mu_0), y_0 - x_0\right\rangle \geq 0. \tag{24}$$

Moreover, since $y_n \in S_\xi(\mu_n)$ and $\overline{x}_n \in K$, we get $\langle f_\xi(y_n, \mu_n), \overline{x}_n - y_n\rangle \geq 0$. By the continuity of F_i ($i = 1, \dots, p$), taking limit on above inequality, we get that

$$\left\langle f_\xi(y_0, \mu_0), x_0 - y_0\right\rangle \geq 0. \tag{25}$$

Assume that $y_0 \neq x_0$. Then by the strict ξ-pseudomonotonicity of F and (24), we have $\langle f_\xi(y_0, \mu_0), x_0 - y_0\rangle < 0$, which contradicts (25). Therefore, $y_0 = x_0$. This is impossible by the wrong assumption. The proof is complete. □

Theorem 16. *Suppose that all conditions of Lemma 15 are satisfied and K is a polyhedral convex set. Then $S(\cdot)$ is continuous on Ω.*

Proof. "l.s.c": Since K is a polyhedral convex set, by virtue of Lemma 1, for each $\mu \in \Omega$,

$$S(\mu) = \bigcup_{\xi \in \Delta^+} S_\xi(\mu). \tag{26}$$

It follows from Lemma 15 that, for each $\xi \in \Delta^+$, $S_\xi(\cdot)$ is l.s.c on Ω. Thus, in view of Lemma 13, we immediately obtain that $S(\cdot)$ is l.s.c on Ω.

"u.s.c": We prove that $S(\cdot)$ is u.s.c on Ω. Suppose that there exists some $\mu_0 \in \Omega$ such that $S(\cdot)$ is not u.s.c at μ_0. Then there exist an open set V satisfying $S(\mu_0) \subseteq V$ and sequences $\mu_n \to \mu_0$ and $x_n \in S(\mu_n)$, such that $x_n \notin V$, $\forall n$.

Notice that because the strict ξ-pseudomonotonicity of F is imposed, it is easy to verify that, for every $\xi \in \Delta^+$ and $\mu \in \Omega$, $S_\xi(\mu)$ is a singleton; namely, $S_\xi(\cdot)$ is single-valued.

By Lemma 1, $x_n \in S(\mu_n) = \bigcup_{\xi \in \Delta^+} S_\xi(\mu_n)$; thus, there exists $\xi' \in \Delta^+$ such that $\{x_n\} = S_{\xi'}(\mu_n)$. Let $\{x_0\} = S_{\xi'}(\mu_0)$. Since $S_{\xi'}(\cdot)$ is single-valued, so it is continuous at μ_0 by Lemma 15; thus, $x_n \to x_0$. Note that $x_0 \in \bigcup_{\xi \in \Delta^+} S_\xi(\mu_0) = S(\mu_0) \subseteq V$. It follows from $x_n \notin V$ and the openness of V that $x_0 \notin V$, which yields a contradiction. Thus, we have proved the upper semicontinuity of $S(\cdot)$. □

Remark 17. Theorem 16 improves [12, Theorem 3.2], by weakening the strict monotonicity of F_i ($i \in \{1, \dots, p\}$) to the strict ξ-pseudomonotonicity of F.

Example 18. Let $K = \Omega = [1, 2]$. Define $F_1, F_2 : K \times \Omega \to \mathbb{R}$ as $F_1(x, \mu) = \mu x$ and $F_2(x, \mu) = \mu$, respectively. Clearly, all conditions of Theorem 16 are satisfied (cf. Example 14), and hence it derives the continuity of the solution mapping S (in fact, $S(\mu) = \{1\}$, $\forall \mu \in \Omega$). However, Theorem 3.2 of [12] is not applicable, because the strict monotonicity of $F_2(\cdot, \mu)$ is violated.

We further give an example to illustrate Theorem 16 when S is set-valued. Based on the union property $S(\mu) = \bigcup_{\xi \in \Delta^+} S_\xi(\mu)$, for any $\mu \in \Omega$, $S(\mu)$ in Theorem 16 need not be a singleton in general, although for each $\xi \in \Delta^+$ the problem $(\mathrm{PVI})_\xi$ has a unique solution by the strict ξ-pseudomonotonicity of F. This is because as we change the parameter ξ the solution of $(\mathrm{PVI})_\xi$ changes as well and all these solutions are in fact solutions of (PVVI).

Example 19. Let $K = [0,1]$ and $\Omega = [1,2]$. Define $F_1, F_2 : K\times\Omega \to \mathbb{R}$ as $F_1(x,\mu) = \mu x$ and $F_2(x,\mu) = -\mu$, respectively. For $\xi = (\xi_1,\xi_2) \in \Delta^+$, $f_\xi(x,\mu) = \sum_{i=1}^2 \xi_i F_i(x,\mu) = (\xi_1 x - \xi_2)\mu$, $\forall x \in K$, $\mu \in \Omega$. It is easy to check that $\forall\xi \in \Delta^+$, $x, x' \in K$, $x \neq x'$, $\langle f_\xi(x',\mu) - f_\xi(x,\mu), x' - x\rangle = \xi_1\mu(x'-x)^2 > 0$, which implies that condition (ii) of Lemma 15 holds. Thus, all conditions of Theorem 16 are satisfied. Direct computations show that $\forall\xi \in \Delta^+$, $\mu \in \Omega$,

$$S_\xi(\mu) = \begin{cases} \left\{\dfrac{\xi_2}{\xi_1}\right\}, & \text{if } \xi_2 \le \xi_1, \\ \{1\}, & \text{if } \xi_2 > \xi_1, \end{cases} \tag{27}$$
$$S(\mu) =]0,1].$$

Clearly, $S(\mu) = \bigcup_{\xi\in\Delta^+} S_\xi(\mu)$ and $S(\cdot)$ is continuous on Ω.

In Lemma 15, the strict ξ-pseudomonotonicity condition is strict that the solution set $S_\xi(\mu)$ is confined to be a singleton. In this paper, like done in our previous work [12], we introduce the following assumption (ii) of Lemma 20 to weaken this condition. In the case, the solution set $S_\xi(\mu)$ may be a general set but not a singleton; that is, the solution mapping $S_\xi(\cdot)$ is set-valued in general.

Lemma 20. *Let $\xi \in \Delta^+$. Suppose that assumption (A) holds and the following conditions are satisfied:*

(i) *$F_i(\cdot,\cdot)$, $i = 1,\dots,p$, are continuous on $K\times\Omega$, where $K \subset \mathbb{R}^n$ is a nonempty, closed, and convex set.*

(ii) *There exists a constant $\alpha > 0$ such that for each $\mu \in \Omega$ and $x \in K \setminus S_\xi(\mu)$, there exists $y \in S_\xi(\mu)$ satisfying $\langle F_i(x,\mu), y - x\rangle \le -\alpha\|x-y\|^2$, $i = 1,\dots,p$.*

Then, $S_\xi(\cdot)$ is l.s.c on Ω.

Proof. Suppose to the contrary that there exists $\mu_0 \in \Omega$ such that $S_\xi(\cdot)$ is not l.s.c at μ_0. Then there exist $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in S_\xi(\mu_0)$, such that, for any $x_n \in S_\xi(\mu_n)$, $x_n \nrightarrow x_0$.

Since $x_0 \in K$ and $K \subset \mathbb{R}^n$ is a closed set, there exists $\overline{x}_n \in K$ such that $\overline{x}_n \to x_0$. Whence, it is clear that $\overline{x}_n \in K\setminus S_\xi(\mu_n)$. Thus, by assumption (ii), there exists $y_n \in S_\xi(\mu_n)$ such that

$$\begin{aligned} \left\langle f_\xi(\overline{x}_n,\mu_n), y_n - \overline{x}_n\right\rangle &= \sum_{i=1}^p \xi_i \left\langle F_i(\overline{x}_n,\mu_n), y_n - \overline{x}_n\right\rangle \\ &\le -\left(\sum_{i=1}^p \xi_i\right)\alpha\left\|y_n - \overline{x}_n\right\|^2 \\ &\le -\left(\sum_{i=1}^p \xi_i^2\right)\alpha\left\|y_n - \overline{x}_n\right\|^2 = -\alpha\left\|y_n - \overline{x}_n\right\|^2. \end{aligned} \tag{28}$$

Similarly as in the proof of Lemma 15, the sequence $\{y_n\}$ is bounded; thus, without loss of generality, we can assume that $y_n \to y_0$. As the set K is closed, $y_0 \in K$. Taking the limit as $n \to \infty$ in above inequality, we have

$$\left\langle f_\xi(x_0,\mu_0), y_0 - x_0\right\rangle \le -\alpha\left\|y_0 - x_0\right\|^2. \tag{29}$$

Now we claim that $y_0 = x_0$. Otherwise, by (29), we obtain that $\langle f_\xi(x_0,\mu_0), y_0 - x_0\rangle < 0$, which contradicts (24), because $x_0 \in S_\xi(\mu_0)$ and $y_0 \in K$. Hence, $y_0 = x_0$. However, it is impossible by the wrong assumption. The proof is complete. □

Remark 21. Condition (ii) of Lemma 20 is a modification of the strong pseudomonotonicity of F_i, which may be called the partially strong pseudomonotonicity of F_i, because our assumption is not imposed on all $x, y \in K$. In addition, if "$\langle F_i(x,\mu), y - x\rangle \le -\alpha\|x-y\|^2$, $i = 1,\dots,p$" is replaced by "$\langle f_\xi(x,\mu), y-x\rangle \le -\alpha\|x-y\|^2$", then the conclusion of Lemma 20 still holds. Whence, condition (ii) of Lemma 20 is also from a modification of the strong ξ-pseudomonotonicity of F. We notice that this kind of monotonicity has been used to deal with solution stability (e.g., Hölder continuity, error bound) of vector variational inequalities and vector equilibrium problems; see, for example, [17, Theorem 3.1] and [18, Theorem 4.2].

We give the following trivial example to illustrate Lemma 20, where S_ξ is set-valued.

Example 22. Let $K = \{0\}\times[0,1]$ and $\Omega = [1,2]$. Define $F_1, F_2 : K\times\Omega \to \mathbb{R}^2$ as $F_1(x,\mu) = (\mu x_2, 0)$ and $F_2(x,\mu) = (\mu x_2^2, 0)$ for every $x = (x_1,x_2) \in \mathbb{R}^2$, respectively. For $\xi = (\xi_1,\xi_2) \in \Delta^+$, $x = (x_1,x_2) \in K$ and $\mu \in \Omega$, $f_\xi(x,\mu) = \sum_{i=1}^2 \xi_i F_i(x,\mu) = (\mu x_2(\xi_1 + \xi_2 x_2), 0)$. By a direct calculation, we obtain that $S_\xi(\mu) = \{0\}\times[0,1] = K$, $\forall\xi \in \Delta^+$, $\mu \in \Omega$. Thus, the partially strong pseudomonotonicity of F_i $(i = 1,2)$ holds trivially, as $K \setminus S_\xi(\mu) = \emptyset$. Whence, all conditions of Lemma 20 are satisfied, and hence it derives the lower semicontinuity of the set-valued solution mapping S_ξ.

Theorem 23. *Suppose that all conditions of Lemma 20 are satisfied and K is a polyhedral convex set. Then $S(\cdot)$ is continuous on Ω.*

Proof. "l.s.c": By virtue of Lemmas 1, 13, and 20, the lower semicontinuity is valid.

"u.s.c": We prove that $S(\cdot)$ is u.s.c on Ω. Suppose that there exists some $\mu_0 \in \Omega$ such that $S(\cdot)$ is not u.s.c at μ_0. Then there exist an open set V satisfying $S(\mu_0) \subseteq V$ and sequences $\mu_n \to \mu_0$ and $x_n \in S(\mu_n)$, such that $x_n \notin V$, $\forall n$.

It is easy to check that, for every $\xi \in \Delta^+$ and $\mu \in \Omega$, because of the closedness of K and the continuity of F_i, $S_\xi(\mu)$ is a closed set in $\mathbb{R}^n$. On the other hand, $S_\xi(\mu) \subseteq S(\mu)$ together with the boundedness of $S(\mu)$ yields that $S_\xi(\mu)$ is also a bounded set in $\mathbb{R}^n$. Thus, for every $\xi \in \Delta^+$ and $\mu \in \Omega$, $S_\xi(\mu)$ is a compact set in $\mathbb{R}^n$.

By Lemma 1, $x_n \in S(\mu_n) = \bigcup_{\xi\in\Delta^+} S_\xi(\mu_n)$; thus, there exists $\xi' \in \Delta^+$ such that $x_n \in S_{\xi'}(\mu_n)$. It follows from [12, Lemma 3.3] that $S_{\xi'}(\cdot)$ is u.s.c at μ_0 with compact values. Whence, for $\{\mu_n\}$ and $\{x_n\}$, there exist $x_0 \in S_{\xi'}(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Note that $x_0 \in \bigcup_{\xi\in\Delta^+} S_\xi(\mu_0) = S(\mu_0) \subseteq V$. It follows from $x_{n_k} \notin V$ and the openness of V that $x_0 \notin V$, which yields a contradiction. Thus, we have proved the upper semicontinuity of $S(\cdot)$. □

Remark 24. Theorem 23 modifies [12, Theorem 3.5], by changing the partially strong monotonicity of F_i ($i \in \{1, \ldots, p\}$) to the partially strong pseudomonotonicity of F_i.

Example 25. Consider Example 18. To illustrate that Theorem 23 is valid, we only need to verify condition (ii) of Lemma 20. It is clear that $S_\xi(\mu) = \{1\}$, $\forall \xi \in \Delta^+$, $\mu \in \Omega$. Choose $\alpha = 1$. For any $\mu \in \Omega = [1, 2]$ and $x \in K \setminus S_\xi(\mu) =]1, 2]$, taking $y = 1 \in S_\xi(\mu)$, we get that

$$\begin{aligned} &\langle F_1(x,\mu), y - x\rangle + \alpha \|x - y\|^2 \\ &\quad = \mu x(1 - x) + |x - 1|^2 = -(x - 1)\left[(\mu - 1)x + 1\right] \\ &\quad < 0, \end{aligned} \tag{30}$$

$$\begin{aligned} &\langle F_2(x,\mu), y - x\rangle + \alpha \|x - y\|^2 = \mu(1 - x) + |x - 1|^2 \\ &\quad = -(x - 1)(\mu + 1 - x) \le 0. \end{aligned}$$

That is, condition (ii) of Lemma 20 holds. Hence, all conditions of Theorem 23 hold and it is valid. However, Theorem 3.5 of [12] is not applicable, because the partially strong monotonicity of $F_2(\cdot, \mu)$ is violated. In fact, for every $\alpha > 0$ and $\mu \in \Omega$, $x \in K \setminus S_\xi(\mu) =]1, 2]$, taking any $y \in S_\xi(\mu) = \{1\}$, we have $\langle F_2(y,\mu) - F_2(x,\mu), y - x\rangle = 0(y - x) = 0 < \alpha|y - x|^2$.

In the sequel, we will show an application to the parametric weak Minty vector variational inequality (PWMVVI). Similarly, we make the following assumption (A′): $\forall \mu \in \Omega$, $S_M^w(\mu)$ is nonempty and bounded; $\{\xi \in \Delta : S_\xi(\mu) \ne \emptyset\} = \Delta$; that is, $\forall \xi \in \Delta$, $S_\xi(\mu) \ne \emptyset$.

Theorem 26. *Suppose that assumption* (A') *holds and the following conditions are satisfied:*

(i) $F_i(\cdot,\cdot)$, $i = 1, \ldots, p$, *are continuous on* $K \times \Omega$, *where* $K \subset \mathbb{R}^n$ *is a nonempty, closed, and convex set.*

(ii) *For any* $\mu \in \Omega$, $F(\cdot, \mu)$ *is strictly* ξ*-pseudomonotone on* K*; that is,* $\forall \xi \in \Delta$,

$$\begin{aligned} \left\langle f_\xi(x,\mu), x' - x\right\rangle \ge 0 \Longrightarrow \left\langle f_\xi(x',\mu), x' - x\right\rangle > 0, \\ \forall x, x' \in K,\ x \ne x'. \end{aligned} \tag{31}$$

Then, $S_M^w(\cdot)$ *is continuous on* Ω.

Proof. "l.s.c": Since for any $\mu \in \Omega$, $F(\cdot,\mu)$ is strictly ξ-pseudomonotone on K, it is clear that $F_i(\cdot,\mu)$, $i = 1, \ldots, p$, are strictly pseudomonotone on K, thus pseudomonotone on K. By virtue of Lemma 11, for each $\mu \in \Omega$,

$$S_M^w(\mu) = \bigcup_{\xi \in \Delta} S_\xi(\mu). \tag{32}$$

Similar to the proof of Lemma 15 we know, for each $\xi \in \Delta$, $S_\xi(\cdot)$ is l.s.c on Ω. Thus, in view of Lemma 13, we immediately obtain that $S_M^w(\cdot)$ is l.s.c on Ω.

"u.s.c": The proof is similar to that of Theorem 16. □

The following result can be deduced by the similar proof of Theorem 23.

Theorem 27. *Suppose that assumption* (A') *holds and the following conditions are satisfied:*

(i) $F_i(\cdot,\cdot)$, $i = 1, \ldots, p$, *are continuous on* $K \times \Omega$, *where* $K \subset \mathbb{R}^n$ *is a nonempty, closed, and convex set, and* $F_i(\cdot,\mu)$, $i = 1, \ldots, p$, *are pseudomonotone on* K *for any* $\mu \in \Omega$.

(ii) *For any* $\xi \in \Delta$, *there exists a constant* $\alpha > 0$ *such that, for each* $\mu \in \Omega$ *and* $x \in K \setminus S_\xi(\mu)$, *there exists* $y \in S_\xi(\mu)$ *satisfying* $\langle F_i(x,\mu), y - x\rangle \le -\alpha\|x - y\|^2$, $i = 1, \ldots, p$.

Then, $S_M^w(\cdot)$ *is continuous on* Ω.

4. Gap Functions and Error Bounds

Throughout this section, assume that $\mathrm{sol(VI)}_\xi \ne \emptyset$ for all $\xi \in \Sigma$. The existence, for instance, can be guaranteed by the compactness and convexity of K, and the continuity of F (e.g., [24, Theorem 2.2]).

We will now introduce a regularized gap function for (WVVI). This gap function was studied by Charitha and Dutta [20].

For $\alpha > 0$, we define the function ϕ_α as

$$\begin{aligned} &\phi_\alpha(x) \\ &\quad := \min_{\xi \in \Sigma} \max_{y \in K} \left\{\left\langle \sum_{i=1}^p \xi_i F_i(x), x - y\right\rangle - \frac{\alpha}{2}\|y - x\|^2\right\}. \end{aligned} \tag{33}$$

For fixed $x \in \mathbb{R}^n$ and $\xi \in C \setminus \{0_{\mathbb{R}^p}\}$ consider the following problem:

$$\begin{aligned} &f_\alpha(x,\xi) \\ &\quad := \max_{y \in K} \left\{\left\langle \sum_{i=1}^p \xi_i F_i(x), x - y\right\rangle - \frac{\alpha}{2}\|y - x\|^2\right\}, \end{aligned} \tag{34}$$

which is equivalently written as

$$\begin{aligned} &f_\alpha(x,\xi) \\ &\quad = -\min_{y \in K} \left\{\left\langle \sum_{i=1}^p \xi_i F_i(x), y - x\right\rangle + \frac{\alpha}{2}\|y - x\|^2\right\}. \end{aligned} \tag{35}$$

Lemma 28 (see [20, Lemma 2.4]). *For any* $x \in \mathbb{R}^n$ *and any* $\xi \in C \setminus \{0_{\mathbb{R}^p}\}$, *let* $\phi_\alpha(x)$ *and* $f_\alpha(x,\xi)$ *be defined by (33) and (34), respectively. Then,* $f_\alpha(x,\xi)$ *is continuous on* $\mathbb{R}^n \times \Sigma$ *and* ϕ_α *is well-defined.*

Note that ϕ_α is finite without the assumption that K is compact (cf. [20, Remark 2.5]).

Lemma 29 (see [20, Theorem 2.7]). *Let* $\phi_\alpha(x)$ *be defined by (33). Then* $\phi_\alpha(x) \ge 0$ *for all* $x \in K$. *Furthermore,* $\phi_\alpha(x^*) = 0$, $x^* \in K$ *if and only if* x^* *solves* (WVVI). *That is,* ϕ_α *is a gap function for* (WVVI).

We will now present an error bound for (WVVI) with a strongly ξ-pseudomonotone mapping F, but not strongly monotone functions F_i $(i = 1, \ldots, p)$ done in [20–22]. In our setting we will devise error bounds in terms of the regularized gap function $\phi_\alpha(x)$. In what follows by the notation $\operatorname{dist}(x, A)$ we mean the distance between the point x and the set A.

If $F_i : K \to \mathbb{R}^n$, $i = 1, \ldots, p$, are strongly monotone with $\mu_i > 0$ on K, that is, $\langle F_i(y) - F_i(x), y - x\rangle \geq \mu_i \|y - x\|^2$, $\forall x, y \in K$, $i = 1, \ldots, p$, and set $\mu := \min_{1\leq i\leq p} \mu_i$, then it is clear that $\forall \xi = (\xi_1, \ldots, \xi_p) \in \Sigma$, F is strongly ξ-monotone on K with $\mu > 0$: that is,

$$\begin{aligned}&\left\langle f_\xi(y) - f_\xi(x), y - x\right\rangle \\ &\quad = \sum_{i=1}^{p} \xi_i \left\langle F_i(y) - F_i(x), y - x\right\rangle \geq \sum_{i=1}^{p} \xi_i \mu_i \|y - x\|^2 \\ &\quad \geq \mu \|y - x\|^2, \quad \forall x, y \in K.\end{aligned} \tag{36}$$

Obviously, it implies that F is strongly ξ-pseudomonotone on K. However, the following example shows the converse is not true. That is, the strong ξ-pseudomonotonicity of F is properly weaker than the strong monotonicity of F_i $(i = 1, \ldots, p)$.

Example 30. Let $K = [1, 2]$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = x$ and $F_2(x) = 1$, respectively. For $\xi = (\xi_1, \xi_2) \in \Sigma$, $f_\xi(x) = \sum_{i=1}^{2} \xi_i F_i(x) = \xi_1 x + \xi_2$, $\forall x \in K$. We show that $F := (F_1, F_2)$ is strongly ξ-pseudomonotone on K. For any $\xi = (\xi_1, \xi_2) \in \Sigma$ and $x_1, x_2 \in K$, suppose that $\langle f_\xi(x_1), x_2 - x_1\rangle = (\xi_1 x_1 + \xi_2)(x_2 - x_1) \geq 0$. As $\xi_1 x_1 + \xi_2 > 0$, we have $x_2 - x_1 \geq 0$. Thus, $\langle f_\xi(x_2), x_2 - x_1\rangle = (\xi_1 x_2 + \xi_2)(x_2 - x_1) \geq x_2 - x_1 \geq (x_2 - x_1)^2$, because $\xi_1 + \xi_2 = 1$; then $\xi_1 x_2 + \xi_2 = 1 + \xi_1(x_2 - 1) \geq 1$, and $x_2 - x_1 \in [0, 1]$. Hence, F is strongly ξ-pseudomonotone on K with the modulus of strong pseudomonotonicity $\mu = 1$. However, F_1, F_2 are not both strongly monotone on K. It is clear that F_2 is not strongly monotone on K, since $\langle F_2(x_2) - F_2(x_1), x_2 - x_1\rangle = 0$, $\forall x_1, x_2 \in K$. Moreover, F is also not strongly ξ-monotone on K. In fact, taking $\xi = (0, 1) \in \Sigma$ and x_1, x_2 in K, we get $\langle f_\xi(x_2) - f_\xi(x_1), x_2 - x_1\rangle = \xi_1 (x_2 - x_1)^2 = 0$.

Theorem 31. *Let $\xi \in \Sigma$. Suppose that F is strongly ξ-pseudomonotone on K with the modulus of strong pseudomonotonicity $\mu > 0$, and let $\alpha > 0$ be chosen so that $\alpha < 2\mu$. Then for any $x \in K$ we have*

$$\operatorname{dist}(x, \operatorname{sol}(\mathrm{WVVI})) \leq \frac{1}{\sqrt{\mu - \alpha/2}} \sqrt{\phi_\alpha(x)}. \tag{37}$$

Proof. We can write the function $\phi_\alpha(x)$ in the way: $\phi_\alpha(x) = \min_{\xi\in\Sigma} f_\alpha(x, \xi)$. From Lemma 28 we know that f_α is continuous on $\mathbb{R}^n \times \Sigma$, so the function $f_\alpha(x, \cdot)$ is continuous on Σ. Noting that Σ is compact, hence there exists $\xi^* \in \Sigma$ (ξ^* will depend on the chosen x) such that $\phi_\alpha(x) = f_\alpha(x, \xi^*)$. Whence, using the definition of $f_\alpha(x, \xi^*)$ we have, for all $y \in K$,

$$\phi_\alpha(x) \geq \left\langle \sum_{i=1}^{p} \xi_i^* F_i(x), x - y\right\rangle - \frac{\alpha}{2} \|y - x\|^2. \tag{38}$$

Since $\operatorname{sol}(\mathrm{VI})_{\xi^*} \neq \emptyset$, letting $x^* \in \operatorname{sol}(\mathrm{VI})_{\xi^*}$, further from Remark 2 and Lemma 1 we know that x^* also solves (WVVI). We set $y = x^*$ in (38):

$$\phi_\alpha(x) \geq \left\langle \sum_{i=1}^{p} \xi_i^* F_i(x), x - x^*\right\rangle - \frac{\alpha}{2} \|x - x^*\|^2. \tag{39}$$

Since x^* solves $(\mathrm{VI})_{\xi^*}$, we have $\langle \sum_{i=1}^{p} \xi_i^* F_i(x^*), x - x^*\rangle \geq 0$. Then by the strong ξ-pseudomonotonicity of F with $\mu > 0$, we get

$$\left\langle \sum_{i=1}^{p} \xi_i^* F_i(x), x - x^*\right\rangle \geq \mu \|x - x^*\|^2. \tag{40}$$

Thus, combining with (39), we obtain $\phi_\alpha(x) \geq (\mu - \alpha/2)\|x - x^*\|^2$. We have noted that $\alpha < 2\mu$; hence $\|x - x^*\| \leq (1/\sqrt{\mu - \alpha/2})\sqrt{\phi_\alpha(x)}$, which implies

$$\operatorname{dist}(x, \operatorname{sol}(\mathrm{WVVI})) \leq \frac{1}{\sqrt{\mu - \alpha/2}} \sqrt{\phi_\alpha(x)}, \quad \forall x \in K. \tag{41}$$

The proof is complete. □

Remark 32. (a) Based on the union property $\operatorname{sol}(\mathrm{WVVI}) = \bigcup_{\xi\in\Sigma} \operatorname{sol}(\mathrm{VI})_\xi$, sol(WVVI) in Theorem 31 need not be a singleton in general, although for each $\xi \in \Sigma$ the scalar variational inequality $(\mathrm{VI})_\xi$ admits a unique solution by the strong ξ-pseudomonotonicity of F. This is because as we change the parameter ξ the solution set $\operatorname{sol}(\mathrm{VI})_\xi$ changes as well and all these solutions are in fact solutions of (WVVI).

(b) Theorem 31 improves [20, Theorem 2.9], since we use the strong ξ-pseudomonotonicity of F but not strong monotonicity of F_i $(i = 1, \ldots, p)$.

We give the following example to illustrate Theorem 31.

Example 33. Let $K = [1, 2]$. Define $F_1, F_2 : K \to \mathbb{R}$ as $F_1(x) = x$ and $F_2(x) = 1$, respectively. For $\xi = (\xi_1, \xi_2) \in \Sigma$, $\xi_1 F_1(x) + \xi_2 F_2(x) = \xi_1 x + \xi_2$, $\forall x \in K$. By virtue of Example 30, F is strongly ξ-pseudomonotone on K with $\mu = 1$, and we let $\alpha = 1$. Moreover, $\operatorname{sol}(\mathrm{WVVI}) = \operatorname{sol}(\mathrm{VI})_\xi = \{1\}$, $\forall \xi \in \Sigma$. Thus, all conditions of Theorem 31 are satisfied.

Direct computations show that the regularized gap function for (WVVI) has the following representation:

$$\begin{aligned}\phi_\alpha(x) &= \min_{\xi\in\Sigma}\max_{y\in K}\left\{(\xi_1 x+\xi_2)(x-y)-\frac{1}{2}|y-x|^2\right\}\\ &= \min_{\xi\in\Sigma}\left\{(\xi_1 x+\xi_2)(x-1)-\frac{1}{2}(x-1)^2\right\}\\ &= \min_{\xi\in\Sigma}\left\{(\xi_1(x-1)+1)(x-1)-\frac{1}{2}(x-1)^2\right\}\\ &= x-1-\frac{1}{2}(x-1)^2=\frac{1}{2}(x-1)(3-x),\\ &\qquad \forall x\in K.\end{aligned} \tag{42}$$

Hence, $(1/\sqrt{\mu-\alpha/2})\sqrt{\phi_\alpha(x)} = \sqrt{(x-1)(3-x)}$. Clearly, $\operatorname{dist}(x,\operatorname{sol}(\mathrm{WVVI})) = |x-1| \le (1/\sqrt{\mu-\alpha/2})\sqrt{\phi_\alpha(x)}, \forall x\in K$. However, Theorem 2.9 of [20] is not applicable, because the strong monotonicity of F_2 is violated.

By Remarks 10 and 6 we know the strong pseudomonotonicity of F_i $(i=1,\ldots,p)$ is properly weaker than the strong ξ-pseudomonotonicity of F. Next, we make some discussions on the error bounds of (WVVI) when F_i $(i=1,\ldots,p)$ are strongly pseudomonotone on K.

It is clear that $\bigcap_{i=1}^p \operatorname{sol}(\mathrm{VI})_i \subseteq \operatorname{sol}(\mathrm{VI})_\xi$, $\forall\xi\in\Sigma$ and we assume that $\operatorname{sol}(\mathrm{VI})_\xi \neq \emptyset$ for all $\xi\in\Sigma$, but it is possible that $\bigcap_{i=1}^p \operatorname{sol}(\mathrm{VI})_i = \emptyset$. Now we make the following assumption $(\widetilde{\mathrm{A}})$: $\bigcap_{i=1}^p \operatorname{sol}(\mathrm{VI})_i \neq \emptyset$.

Theorem 34. *Let $\xi\in\Sigma$. Suppose that assumption $(\widetilde{A})$ holds and F_i $(i=1,\ldots,p)$ are strongly pseudomonotone with the modulus of strong pseudomonotonicity $\mu_i>0$ on K. Moreover, let $\mu=\min_{1\le i\le p}\mu_i$ and let $\alpha>0$ be chosen so that $\alpha<2\mu$. Then for any $x\in K$ we have*

$$\operatorname{dist}(x,\operatorname{sol}(\mathrm{WVVI})) \le \frac{1}{\sqrt{\mu-\alpha/2}}\sqrt{\phi_\alpha(x)}. \tag{43}$$

Proof. Similar to the proof of Theorem 31, there exists $\xi^*\in\Sigma$ such that (38) holds.

Since assumption $(\widetilde{\mathrm{A}})$ holds, we let $x^*\in\bigcap_{i=1}^p \operatorname{sol}(\mathrm{VI})_i$. Obviously $x^*\in\operatorname{sol}(\mathrm{WVVI})$. As x^* solves $(\mathrm{VI})_i$ $(i=1,\ldots,p)$, we have $\langle F_i(x^*),x-x^*\rangle\ge 0$, $i=1,\ldots,p$. Then by the strong pseudomonotonicity of F_i $(i=1,\ldots,p)$, we get $\langle F_i(x),x-x^*\rangle \ge \mu_i\|x-x^*\|^2$. Because $\xi^*\in\Sigma$, we obtain

$$\begin{aligned}\left\langle \sum_{i=1}^p \xi_i^* F_i(x), x-x^*\right\rangle &\ge \sum_{i=1}^p \xi_i^*\mu_i\|x-x^*\|^2\\ &\ge \mu\|x-x^*\|^2.\end{aligned} \tag{44}$$

Thus, combining with (38) by setting $y=x^*$, we obtain

$$\phi_\alpha(x) \ge \left(\mu-\frac{\alpha}{2}\right)\|x-x^*\|^2. \tag{45}$$

Noting that $\alpha<2\mu$, we have $\|x-x^*\|\le(1/\sqrt{\mu-\alpha/2})\sqrt{\phi_\alpha(x)}$, which implies

$$\operatorname{dist}(x,\operatorname{sol}(\mathrm{WVVI})) \le \frac{1}{\sqrt{\mu-\alpha/2}}\sqrt{\phi_\alpha(x)}, \quad \forall x\in K. \tag{46}$$

The proof is complete. □

Remark 35. The assumption $(\widetilde{\mathrm{A}})$ in Theorem 34 has been used to deal with error bounds of (WVVI); for example, see [21, Theorem 4.3] and [22, Theorem 3.3]. In fact, by using the scalar regularized gap function mentioned in [21, 22], we can also obtain a similar error bound for (WVVI) under the assumptions in Theorem 34 (see Theorem 38).

However, under the strong pseudomonotonicity of F_i and assumption $(\widetilde{\mathrm{A}})$, we see that, for every $i=1,\ldots,p$, $\operatorname{sol}(\mathrm{VI})_i$ is a singleton and all $(\mathrm{VI})_i$ $(i=1,\ldots,p)$ must have the same solution, and thus sol(WVVI) should be a singleton.

Example 36. Consider Example 33. Direct computations show that

$$\begin{aligned}\operatorname{sol}(\mathrm{WVVI}) &= \operatorname{sol}(\mathrm{VI})_\xi = \operatorname{sol}(\mathrm{VI})_1 = \operatorname{sol}(\mathrm{VI})_2\\ &= \{1\}, \quad \forall\xi\in\Sigma.\end{aligned} \tag{47}$$

Clearly, all conditions of Theorem 34 are satisfied with $\mu_1=\mu_2=1$ and $\alpha=1$, and hence it derives the error bound of (WVVI) in terms of ϕ_α.

Besides the scalar regularized gap function $\phi_\alpha(x)$ mentioned above, Charitha et al. [21] and Sun and Chai [22] also have constructed another scalar regularized gap function $g_\alpha(x)$ for (WVVI), which is independent with the scalarization parameter ξ.

For $\alpha>0$, we define the function g_α as

$$g_\alpha(x) := \sup_{y\in K}\left\{\min_{1\le i\le p}\langle F_i(x),x-y\rangle-\frac{\alpha}{2}\|y-x\|^2\right\}. \tag{48}$$

Charitha et al. [21] and Sun and Chai [22] have explained that $g_\alpha(x)$ is finite for every x and thus is well-defined.

Lemma 37 (see [21, Theorem 4.2] and [22, Corollary 3.2]). *The function g_α is a gap function for* (WVVI).

Now we use the regularized gap function $g_\alpha(x)$ to develop an error bound for (WVVI) under assumption $(\widetilde{\mathrm{A}})$ and strongly pseudomonotone component functions F_i $(i=1,\ldots,p)$.

Theorem 38. *Suppose that assumption $(\widetilde{A})$ holds and F_i $(i=1,\ldots,p)$ are strongly pseudomonotone with the modulus of strong pseudomonotonicity $\mu_i>0$ on K. Moreover, let $\mu=\min_{1\le i\le p}\mu_i$ and $\alpha>0$ be chosen so that $\alpha<2\mu$. Then for any $x\in K$ we have*

$$\operatorname{dist}(x,\operatorname{sol}(\mathrm{WVVI})) \le \frac{1}{\sqrt{\mu-\alpha/2}}\sqrt{g_\alpha(x)}. \tag{49}$$

Proof. From our notations and the presentation of $g_\alpha(x)$, we have, for all $y \in K$,

$$g_\alpha(x) \geq \min_{1\leq i\leq p} \langle F_i(x), x-y\rangle - \frac{\alpha}{2}\|y-x\|^2. \quad (50)$$

As assumption $(\widetilde{\mathrm{A}})$ holds, we let $x^* \in \bigcap_{i=1}^p \mathrm{sol(VI)}_i$. Obviously $x^* \in \mathrm{sol(WVVI)}$. We set $y = x^*$ in (50):

$$g_\alpha(x) \geq \min_{1\leq i\leq p} \langle F_i(x), x-x^*\rangle - \frac{\alpha}{2}\|x^*-x\|^2. \quad (51)$$

Without loss of generality, we assume that

$$\langle F_1(x), x-x^*\rangle = \min_{1\leq i\leq p} \langle F_i(x), x-x^*\rangle. \quad (52)$$

Since $x^* \in \bigcap_{i=1}^p \mathrm{sol(VI)}_i$, x^* solves $(\mathrm{VI})_1$; that is, $\langle F_1(x^*), x-x^*\rangle \geq 0$. Then, by the strong pseudomonotonicity of F_1, we have $\langle F_1(x), x-x^*\rangle \geq \mu_1\|x-x^*\|^2 \geq \mu\|x-x^*\|^2$. Thus, combining with (51) and (52), we obtain

$$g_\alpha(x) \geq \left(\mu - \frac{\alpha}{2}\right)\|x-x^*\|^2. \quad (53)$$

Noting that $\alpha < 2\mu$, we have $\|x-x^*\| \leq (1/\sqrt{\mu-\alpha/2})\sqrt{g_\alpha(x)}$, which implies

$$\mathrm{dist}(x, \mathrm{sol(WVVI)}) \leq \frac{1}{\sqrt{\mu-\alpha/2}}\sqrt{g_\alpha(x)}, \quad \forall x \in K. \quad (54)$$

The proof is complete. □

Remark 39. Theorem 38 improves [21, Theorem 4.3] and [22, Corollary 3.3], since we use the strong pseudomonotonicity of F_i but not strong monotonicity of F_i $(i = 1, \ldots, p)$.

Example 40. Consider Example 36. All conditions of Theorem 38 are satisfied with $\mu_1 = \mu_2 = 1$ and $\alpha = 1$, and hence it derives the error bound of (WVVI) in terms of g_α. Actually, we can verify that

$$\begin{aligned} g_\alpha(x) &= \sup_{y\in K}\left\{\min\{x(x-y), x-y\} - \frac{1}{2}|y-x|^2\right\} \\ &= \begin{cases} \sup_{y\in K}\left\{x-y-\frac{1}{2}(x-y)^2\right\}, & \text{if } x-y>0 \\ \sup_{y\in K}\left\{x(x-y)-\frac{1}{2}(x-y)^2\right\}, & \text{if } x-y\leq 0 \end{cases} \\ &= \sup_{y\in K}\begin{cases} \frac{1}{2}(x-1)(3-x), & \text{if } 1\leq y<x \\ 0, & \text{if } x\leq y\leq 2 \end{cases} \\ &= \frac{1}{2}(x-1)(3-x), \quad \forall x\in K. \end{aligned} \quad (55)$$

However, Theorem 4.3 of [21] (or Corollary 3.3 of [22]) is not applicable, because the strong monotonicity of F_2 is violated.

5. Conclusions

In the paper, firstly, sufficient conditions for the continuity (both lower and upper semicontinuities) of solution mappings $S(\cdot)$ to (PVVI) and $S_M^w(\cdot)$ to (PWMVVI) have been established, when the mappings on models are strictly ξ-pseudomonotone and partially strong pseudomonotone, respectively. Secondly, error bounds for (WVVI) in terms of regularized gap functions ϕ_α and g_α have been obtained, when the mappings on models are strongly ξ-pseudomonotone and strongly pseudomonotone, respectively. All discussions have been carried out via scalarization approaches. The results obtained improve or modify corresponding ones in [12] and [20–22], respectively. Moreover, numerous examples have been provided to illustrate main conclusions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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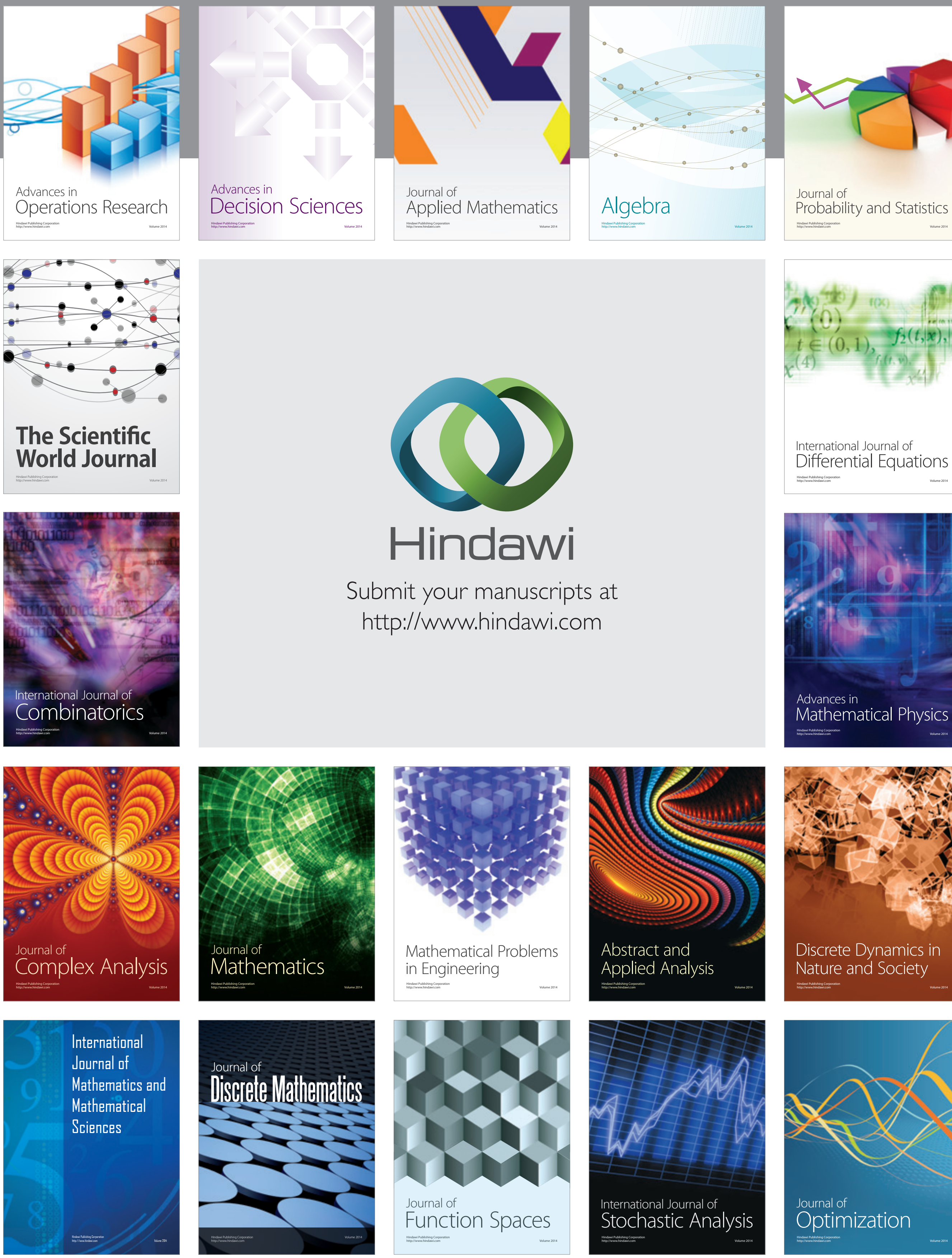

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