Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 7874061, 5 pages
http://dx.doi.org/10.1155/2016/7874061

Research Article

Approximation of Analytic Functions by Solutions of Cauchy-Euler Equation

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Received 2 November 2015; Accepted 15 May 2016

Academic Editor: Gestur Ólafsson

We investigate the approximation properties of a special class of twice continuously differentiable functions by solutions of the Cauchy-Euler equation.

1. Introduction

Throughout this paper, let n be a positive integer, let I be a nondegenerate interval of $\mathbb{R}$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. We will consider the (linear) differential equation of nth order

$$\mathcal{F}\left(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\right) = 0 \tag{1}$$

defined on I, where $y : I \to \mathbb{K}$ is an n times continuously differentiable function.

For arbitrary $\varepsilon > 0$, assume that an n times continuously differentiable function $y : I \to \mathbb{K}$ satisfies the differential inequality

$$\left|\mathcal{F}\left(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\right)\right| \le \varepsilon \tag{2}$$

for all $x \in I$. If for each function $y : I \to \mathbb{K}$ satisfying inequality (2) there exists a solution $y_0 : I \to \mathbb{K}$ of the differential equation (1) such that

$$\left|y(x) - y_0(x)\right| \le K(\varepsilon) \tag{3}$$

for any $x \in I$, where $K(\varepsilon)$ depends on ε only and satisfies $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$, then we say that the differential equation (1) satisfies (or has) the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). If the above statement also holds when we replace ε and $K(\varepsilon)$ with some appropriate $\varphi(x)$ and $\Phi(x)$, respectively, then we say that the differential equation (1) satisfies the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability and recent papers on this subject, refer to [1–4].

Obłoza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations (see [5, 6]). Let $g, r : (a, b) \to \mathbb{R}$ be continuous functions with $\int_a^b |g(x)|dx < \infty$, where a and b are real constants. Assume that $\varepsilon > 0$ is an arbitrary real number. Obłoza proved that if a differentiable function $y : (a, b) \to \mathbb{R}$ satisfies the inequality $|y'(x)+g(x)y(x)-r(x)| \le \varepsilon$ for all $x \in (a, b)$ and if a function $y_0 : (a, b) \to \mathbb{R}$ satisfies $y_0'(x) + g(x)y_0(x) = r(x)$ for all $x \in (a, b)$ and $y(\tau) = y_0(\tau)$ for some $\tau \in (a, b)$, then there exists a constant $\delta > 0$ such that $|y(x) - y_0(x)| \le \delta$ for all $x \in (a, b)$.

Thereafter, Alsina and Ger [7] proved that if a differentiable function $f : (a, b) \to \mathbb{R}$ satisfies the differential inequality $|y'(x) - y(x)| \le \varepsilon$, then there exists a solution $f_0 : (a, b) \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x)-f_0(x)| \le 3\varepsilon$ for any $x \in (a, b)$. This result of Alsina and Ger was generalized by Takahasi et al. They proved in [8] that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [9–13]). For a recent result on the Hyers-Ulam stability for second-order linear differential equations, we refer to [14, 15].

In this paper, we consider the (inhomogeneous) Cauchy-Euler equation

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = r(x), \tag{4}$$

where α and β are real-valued constants and $r : \mathbb{R} \to \mathbb{R}$ is a continuous function, and we investigate the approximation properties of twice continuously differentiable functions by solutions of the Cauchy-Euler equation

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = 0, \tag{5}$$

which is associated with (4).

2. Preliminaries

Recently, Choi and Jung [16, Corollary 4.2] proved the Hyers-Ulam stability of the Cauchy-Euler equation (4) for the case of $(\alpha-1)^2 - 4\beta > 0$.

Theorem 1. *Assume that the real-valued constants α, β are given with $(\alpha-1)^2 - 4\beta > 0$ and ε is an arbitrarily given positive constant. Let c be a positive real-valued constant and let m_1, m_2 be given as*

$$m_1 = \frac{-(\alpha-1) - \sqrt{(\alpha-1)^2 - 4\beta}}{2}, \qquad m_2 = \frac{-(\alpha-1) + \sqrt{(\alpha-1)^2 - 4\beta}}{2}. \tag{6}$$

If $r : (0,\infty) \to \mathbb{R}$ is a differentiable function and $y : (0,\infty) \to \mathbb{R}$ is a twice continuously differentiable function such that the inequality

$$\left| x^2 y''(x) + \alpha x y'(x) + \beta y(x) - r(x) \right| \le \varepsilon \tag{7}$$

holds for any $x \in (0,\infty)$, then there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the inhomogeneous Cauchy-Euler equation (4) such that

$$\left| y(x) - y_c(x) \right| \le \begin{cases} \dfrac{\varepsilon}{m_1 m_2} + \dfrac{\varepsilon}{m_2 - m_1} \left(\dfrac{1}{m_2} \left(\dfrac{x}{c} \right)^{m_2} - \dfrac{1}{m_1} \left(\dfrac{x}{c} \right)^{m_1} \right) & (\text{for } m_1 \neq 0 \neq m_2), \\ \dfrac{\varepsilon}{m_2^2} \left(\left(\dfrac{x}{c} \right)^{m_2} - 1 \right) - \dfrac{\varepsilon}{m_2} \ln \dfrac{x}{c} & (\text{for } m_1 = 0), \\ \dfrac{\varepsilon}{m_1^2} \left(\left(\dfrac{x}{c} \right)^{m_1} - 1 \right) - \dfrac{\varepsilon}{m_1} \ln \dfrac{x}{c} & (\text{for } m_2 = 0) \end{cases} \tag{8}$$

for all $x \in (0,\infty)$.

For the case of $(\alpha-1)^2 - 4\beta = 0$, the Hyers-Ulam stability of the inhomogeneous Cauchy-Euler equation (4) was proved in [16, Corollary 4.4].

Theorem 2. *Assume that the real-valued constants α and β are given with $\alpha \neq 1$, $\beta = (\alpha-1)^2/4$ and ε is an arbitrarily given positive constant. Let c be a positive real-valued constant and let $\lambda = -(\alpha-1)/2$. If $r : (0,\infty) \to \mathbb{R}$ is a differentiable function and $y : (0,\infty) \to \mathbb{R}$ is a twice continuously differentiable function such that the inequality*

$$\left| x^2 y''(x) + \alpha x y'(x) + \frac{(\alpha-1)^2}{4} y(x) - r(x) \right| \le \varepsilon \tag{9}$$

holds for all $x \in (0,\infty)$, then there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the inhomogeneous Cauchy-Euler equation (4) with $\beta = (\alpha-1)^2/4$ such that

$$\left| y(x) - y_c(x) \right| \le \frac{\varepsilon}{\lambda^2} + \frac{\varepsilon}{\lambda} \left(\frac{x}{c} \right)^{\lambda} \left(\ln \frac{x}{c} - \frac{1}{\lambda} \right) \tag{10}$$

for all $x \in (0,\infty)$.

Finally, the Hyers-Ulam stability of the Cauchy-Euler equation (4) was also proven in [16, Theorem 4.5] for the case of $(\alpha-1)^2 - 4\beta < 0$.

Theorem 3. *Assume that the real-valued constants α and β are given with $(\alpha-1)^2 - 4\beta < 0$ and ε is an arbitrarily given positive constant. Let $c > 0$ be a given real-valued constant and let*

$$\lambda = -\frac{\alpha-1}{2}, \qquad \mu = \frac{1}{2}\sqrt{4\beta - (\alpha-1)^2}. \tag{11}$$

If a differentiable function $r : (0,\infty) \to \mathbb{R}$ and a twice continuously differentiable function $y : (0,\infty) \to \mathbb{R}$ satisfy inequality (7) for all $x \in (0,\infty)$, then there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the inhomogeneous Cauchy-Euler equation (4) such that

$$\left| y(x) - y_c(x) \right| \le \frac{\varepsilon}{\mu} \left| \int_c^x \frac{x^{\lambda}}{\zeta^{\lambda+1}} \left| \sin\left(\mu \ln \frac{x}{\zeta} \right) \right| d\zeta \right| \tag{12}$$

for all $x \in (0,\infty)$.

Remark 4. Cîmpean and Popa [14] proved the Hyers-Ulam stability of the linear differential equations of nth order with constant coefficients. Indeed, they proved a general theorem for the Hyers-Ulam stability which includes Theorems 1, 2, and 3 as its corollaries with the inequality

$$|y(x) - y_c(x)| \leq \begin{cases} \dfrac{\varepsilon}{|\beta|} & \left(\text{for either } \alpha^2 - 4\beta > 0,\ \beta \neq 0 \text{ or } \alpha^2 - 4\beta = 0,\ \alpha \neq 0\right), \\ \dfrac{4\varepsilon}{\alpha^2} & \left(\text{for } \alpha^2 - 4\beta < 0,\ \alpha \neq 0\right). \end{cases} \tag{13}$$

However, Theorems 1, 2, and 3 have the advantage of more exact local approximation over the result of Cîmpean and Popa as we see in Theorems 5, 6, and 7.

3. Approximation Properties

We denote by $\mathscr{B}(\alpha;\beta)$ the set of all twice continuously differentiable functions $y : (0,\infty) \to \mathbb{R}$ for which there exists a constant $\varepsilon > 0$ such that

$$\left|x^2 y''(x) + \alpha x y'(x) + \beta y(x)\right| \leq \varepsilon \tag{14}$$

for all $x \in (0,\infty)$, where α and β are real-valued constants.

If we define

$$\begin{aligned} (y_1 + y_2)(x) &:= y_1(x) + y_2(x), \\ (\gamma y_1)(x) &:= \gamma y_1(x) \end{aligned} \tag{15}$$

for all $y_1, y_2 \in \mathscr{B}(\alpha;\beta)$ and $\gamma \in \mathbb{R}$, then $\mathscr{B}(\alpha;\beta)$ is a vector space over $\mathbb{R}$. This fact implies that the set $\mathscr{B}(\alpha;\beta)$ is large enough to be a vector space.

In the following theorems, we investigate approximation properties of functions of $\mathscr{B}(\alpha;\beta)$ by solutions of the Cauchy-Euler equation (5).

Theorem 5. *Let $c > 0$ be a given real number and let $\alpha, \beta \in \mathbb{R}$ be given with $(\alpha-1)^2 - 4\beta > 0$. If $y \in \mathscr{B}(\alpha;\beta)$, then there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the Cauchy-Euler equation (5) such that*

$$|y(x) - y_c(x)| = o(|x - c|) \tag{16}$$

as $x \to c$.

Proof. We define m_1 and m_2 by the formulas given in Theorem 1; that is, m_1 and m_2 are the distinct roots of the indicial equation $m^2 + (\alpha-1)m + \beta = 0$. Since $(\alpha-1)^2 - 4\beta > 0$, we have $m_1 < m_2$. Since $y \in \mathscr{B}(\alpha;\beta)$, there exists a constant $\varepsilon > 0$ such that inequality (14) holds for all $x \in (0,\infty)$.

According to Theorem 1 with $r(x) \equiv 0$, there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the Cauchy-Euler equation (5) such that

$$|y(x) - y_c(x)| \leq \begin{cases} \dfrac{\varepsilon}{m_1 m_2} + \dfrac{\varepsilon}{m_2 - m_1}\left(\dfrac{1}{m_2}\left(\dfrac{x}{c}\right)^{m_2} - \dfrac{1}{m_1}\left(\dfrac{x}{c}\right)^{m_1}\right) & (\text{for } m_1 \neq 0 \neq m_2), \\ \dfrac{\varepsilon}{m_2^2}\left(\left(\dfrac{x}{c}\right)^{m_2} - 1\right) - \dfrac{\varepsilon}{m_2}\ln\dfrac{x}{c} & (\text{for } m_1 = 0), \\ \dfrac{\varepsilon}{m_1^2}\left(\left(\dfrac{x}{c}\right)^{m_1} - 1\right) - \dfrac{\varepsilon}{m_1}\ln\dfrac{x}{c} & (\text{for } m_2 = 0) \end{cases} \tag{17}$$

for any $x \in (0,\infty)$.

We will only estimate the following limit for the case of $m_1 = 0$ by applying L'Hospital's rule:

$$\begin{aligned} &\lim_{x\to c}\left|\frac{y(x) - y_c(x)}{x - c}\right| \\ &\leq \left|\lim_{x\to c}\frac{\left(\varepsilon/m_2^2\right)\left((x/c)^{m_2} - 1\right) - \left(\varepsilon/m_2\right)\ln(x/c)}{x - c}\right| \\ &= \left|\lim_{x\to c}\left(\frac{\varepsilon}{m_2 c}\left(\frac{x}{c}\right)^{m_2-1} - \frac{\varepsilon}{m_2}\frac{1}{x}\right)\right| = 0, \end{aligned} \tag{18}$$

which implies the validity of this theorem. □

We now consider the case of $(\alpha-1)^2 - 4\beta = 0$ and use Theorem 2 to prove the following theorem.

Theorem 6. *Let $c > 0$ and $\alpha \neq 1$ be real numbers and let $\lambda = -(\alpha-1)/2$. If $y \in \mathscr{B}(\alpha;\lambda^2)$, then there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the Cauchy-Euler equation (5) with $\beta = \lambda^2$ such that*

$$|y(x) - y_c(x)| = o(|x - c|) \tag{19}$$

as $x \to c$.

Proof. Since $y \in \mathscr{B}(\alpha;\lambda^2)$, there exists a constant $\varepsilon > 0$ such that inequality (14) holds for all $x \in (0,\infty)$. According to Theorem 2 with $r(x) \equiv 0$, there exists a solution $y_c : (0,\infty) \to \mathbb{R}$ of the Cauchy-Euler equation (5) with $\beta = \lambda^2$ such that

$$|y(x) - y_c(x)| \leq \frac{\varepsilon}{\lambda^2} + \frac{\varepsilon}{\lambda}\left(\frac{x}{c}\right)^{\lambda}\left(\ln\frac{x}{c} - \frac{1}{\lambda}\right) \tag{20}$$

for all $x \in (0,\infty)$.

Therefore, we estimate the limit by applying L'Hospital's rule:

$$\begin{aligned}
&\lim_{x\to c}\left|\frac{y(x)-y_c(x)}{x-c}\right| \\
&\quad\le\left|\lim_{x\to c}\frac{\left(\varepsilon/\lambda^2\right)+(\varepsilon/\lambda)(x/c)^{\lambda}(\ln(x/c)-1/\lambda)}{x-c}\right| \\
&\quad=\left|\lim_{x\to c}\varepsilon\frac{x^{\lambda-1}}{c^{\lambda}}\ln\frac{x}{c}\right|=0,
\end{aligned}\tag{21}$$

which implies the validity of this theorem. □

Finally, we investigate the approximation property of each function of $\mathscr{B}(\alpha;\beta)$ by a solution of the differential equation (5) when $(\alpha-1)^2-4\beta<0$.

Theorem 7. *Let $c>0$ be a given real number and let $\alpha,\beta\in\mathbb{R}$ be given with $(\alpha-1)^2-4\beta<0$. If $y\in\mathscr{B}(\alpha;\beta)$, then there exists a solution $y_c:(0,\infty)\to\mathbb{R}$ of the Cauchy-Euler equation (5) such that*

$$\left|y(x)-y_c(x)\right|=o(|x-c|)\tag{22}$$

as $x\to c$.

Proof. Let us define

$$\begin{aligned}
\lambda&=-\frac{\alpha-1}{2},\\
\mu&=\frac{1}{2}\sqrt{4\beta-(\alpha-1)^2}.
\end{aligned}\tag{23}$$

Since $y\in\mathscr{B}(\alpha;\beta)$, there exists a constant $\varepsilon>0$ such that inequality (14) holds for all $x\in(0,\infty)$. According to Theorem 3 with $r(x)\equiv0$, there exists a solution $y_c:(0,\infty)\to\mathbb{R}$ of the Cauchy-Euler equation (5) such that

$$\left|y(x)-y_c(x)\right|\le\frac{\varepsilon}{\mu}\left|\int_c^x\frac{x^{\lambda}}{\zeta^{\lambda+1}}\left|\sin\left(\mu\ln\frac{x}{\zeta}\right)\right|d\zeta\right|\tag{24}$$

for all $x\in(0,\infty)$.

If we substitute $\eta=cx/\zeta$, then we have

$$\begin{aligned}
&\int_c^x\frac{x^{\lambda}}{\zeta^{\lambda+1}}\left|\sin\left(\mu\ln\frac{x}{\zeta}\right)\right|d\zeta\\
&\quad=\frac{1}{c^{\lambda}}\int_c^x\eta^{\lambda-1}\left|\sin\left(\mu\ln\frac{\eta}{c}\right)\right|d\eta.
\end{aligned}\tag{25}$$

Hence, we further apply L'Hospital's rule to obtain

$$\begin{aligned}
&\lim_{x\to c}\left|\frac{y(x)-y_c(x)}{x-c}\right|\\
&\quad\le\frac{\varepsilon}{\mu}\left|\lim_{x\to c}\frac{1}{x-c}\int_c^x\frac{x^{\lambda}}{\zeta^{\lambda+1}}\left|\sin\left(\mu\ln\frac{x}{\zeta}\right)\right|d\zeta\right|\\
&\quad=\frac{\varepsilon}{\mu c^{\lambda}}\left|\lim_{x\to c}\frac{1}{x-c}\int_c^x\eta^{\lambda-1}\left|\sin\left(\mu\ln\frac{\eta}{c}\right)\right|d\eta\right|\\
&\quad=\frac{\varepsilon}{\mu c^{\lambda}}\left|\lim_{x\to c}x^{\lambda-1}\left|\sin\left(\mu\ln\frac{x}{c}\right)\right|\right|=0,
\end{aligned}\tag{26}$$

which implies the validity of this theorem. □

Competing Interests

The author declares that there are no competing interests.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2013R1A1A2005557). This work was supported by 2014 Hongik University Research Fund.

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