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Research Article

Characterization of Reflexivity by Convex Functions

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A new convexity property of convex functions is introduced. This property provides, in particular, a characterization of the class of reflexive Banach spaces.

1. Introduction

Like differentiability, convexity of functions has proven to be a very useful tool to characterize various classes of Banach spaces. We have also known that good progress has been made in this direction. See, for instance, [1–10]. In particular, Zălinescu [10] first proved that a Banach space is reflexive provided that there exists a continuous uniformly convex function on some nonempty open convex subset of the space. Later, Cheng et al. [11] proved that, in fact, such a Banach space is super reflexive, and vice versa (recall that a Banach space is super reflexive if and only if it admits an equivalent uniformly convex norm [12]). Recently, this result has also been obtained independently by Borwein et al. (see [2, Theorem 2.4]). In particular, in [2] they mainly investigated the relationship between the existence of uniformly convex functions $f : X \to \mathbb{R}$ bounded above by some power type of the norm and the existence of an equivalent norm with a certain power type. They showed in particular that there is a uniformly convex function f bounded above by $\|\cdot\|^2$ if and only if there is an equivalent norm on X with power type 2. Recall that a real-valued convex function f defined on a nonempty convex subset D of X is said to be uniformly convex provided that for every $\varepsilon > 0$ there is $\delta > 0$ such that $f(x) + f(y) - 2f((x+y)/2) \geq \delta$ whenever $x, y \in D$ with $\|x - y\| \geq \varepsilon$. An application of uniformly convex functions can be found in [13].

In the present note we are interested in characterizing reflexivity by convex functions. As a result, we prove that a Banach space being reflexive is equivalent to the fact that there exists a continuous function with some kind of convexity on some nonempty open convex subset of the space.

We review recent works in this direction. Odell and Schlumprecht's renorming theorem [14] shows that a separable Banach space X is reflexive if and only if there is an equivalent $2R$ norm $\|\cdot\|$ on X; that is, if a bounded sequence $(x_n) \subset X$ satisfies

$$\lim_m \lim_n \|x_m + x_n\| = 2 \lim_n \|x_n\|, \tag{1}$$

then (x_n) is norm convergent in X. More recently, Hájek and Johanis [15], through introducing a new convexity property of Day's norm on $c_0(\Gamma)$, showed the following renorming characterization of general reflexive spaces. A sufficient and necessary condition for a Banach space X to be reflexive is that it admits an equivalent $w2R$ norm; that is, if a bounded sequence $(x_n) \subset X$ satisfies (1), then (x_n) is weakly convergent in X. The localized versions of the two renorming theorems have been considered in [16].

The purpose of the present note is to provide a new characterization of reflexive Banach spaces by convex functions. More precisely, the following is our main result.

Theorem 1. *(i) A Banach space X is reflexive if (and only if) there exists a continuous $w2R$ convex function on some nonempty open convex set D of X.*

(ii) In particular, if X is separable then X is reflexive if (and only if) there exists a continuous $2R$ convex function on some nonempty open convex set D of X.

Our notation and terminology for Banach spaces are standard, as may be found for example in [17, 18]. All Banach spaces throughout the paper are supposed to be real. The letter X will always denote a Banach space and X^* its dual.

2. Proof of the Main Theorem

In order to complete the proof of our main theorem, we need to introduce some notation and make some preparatory remarks.

The following notion is a natural generalization of $2R$ (resp., $w2R$) norm.

Definition 2. Let K be a nonempty convex set of a Banach space X. A real-valued convex function f defined on K is said to be $2R$ ($w2R$, resp.); if a bounded sequence $(x_n) \subset K$ satisfies

$$\frac{1}{2}\left[f(x_n)+f(x_m)\right]-f\left(\frac{x_n+x_m}{2}\right)\longrightarrow 0 \quad (m,n\longrightarrow\infty) \tag{2}$$

then (x_n) is norm convergent (resp., weakly convergent) in K.

Assume that f ia a real-valued convex function defined on a nonempty convex subset K of X. Recall that the subdifferential of f at $x \in K$ is the set

$$\partial f(x) = \left\{x^* \in X^* : f(y)-f(x) \geq \langle x^*, y-x\rangle \ \ \forall y \in K\right\}. \tag{3}$$

We have already known (see, e.g., [8]) that if the convex function f is continuous at $x \in K$ then $\partial f(x)$ is a nonempty w^*-compact convex subset of X^*.

Lemma 3. *Let K be a bounded closed convex set in $(X, \|\cdot\|)$ with $0 \in K$. Given an $x^* \in X^*$ let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by*

$$g(s) = \sup\left\{\langle x^*, x\rangle : x \in K,\ h(x) \leq s\right\}. \tag{4}$$

Here h is a (real-valued) nonnegative continuous convex function on K with $h(0) = 0$. Then g is continuous on $\mathbb{R}^+ = (0, +\infty)$.

Proof. Indeed, under the assumption of the lemma we claim that g is Lipschitz on $[r_0, +\infty)$ for every $r_0 > 0$. To see this, fix a $r_0 > 0$ and let $s_1, s_2 \in [r_0, +\infty)$ such that $s_1 < s_2$. Put $\varepsilon = s_2 - s_1$. By the definition of g one can find an element $x_\varepsilon \in K$ with $g(x_\varepsilon) \leq s_2$ such that

$$g(s_2) \leq \langle x^*, x_\varepsilon\rangle + \varepsilon. \tag{5}$$

We next distinguish two cases. First, if $h(x_\varepsilon) \leq s_1$, it follows from (3) and (4) that

$$g(s_2) - g(s_1) \leq \langle x^*, x_\varepsilon\rangle - g(s_1) + \varepsilon \leq \varepsilon = s_2 - s_1. \tag{6}$$

Second, if $s_1 < h(x_\varepsilon) \leq s_2$, then we can find a $\lambda_\varepsilon \in (0,1)$ such that $\lambda_\varepsilon x_\varepsilon \in K$ and $h(\lambda_\varepsilon x_\varepsilon) = s_1$. Indeed, let $H(\alpha) = h(\alpha x_\varepsilon)$, for all $\alpha \in [0,1]$. The continuity of h on K ensures that H is also continuous on $[0,1]$. From $H(0) = h(0) = 0$ and $H(1) = h(x_\varepsilon) > s_1 > 0$, it follows that there exists a $\lambda_\varepsilon \in (0,1)$ such that $H(\lambda_\varepsilon) = s_1$. That is, $h(\lambda_\varepsilon x_\varepsilon) = s_1$, and the convexity of K gives $\lambda_\varepsilon x_\varepsilon \in K$. Thus

$$\begin{aligned} g(s_2) - g(s_1) &\leq \langle x^*, x_\varepsilon\rangle + \varepsilon - g(s_1) \\ &\leq \langle x^*, x_\varepsilon\rangle + \varepsilon - \langle x^*, \lambda_\varepsilon x_\varepsilon\rangle \\ &\leq (1-\lambda_\varepsilon) M + \varepsilon, \end{aligned} \tag{7}$$

where $M := \sup\{\langle x^*, x\rangle : x \in K\}$.

In short we have

$$0 \leq g(s_2) - g(s_1) \leq (1-\lambda_\varepsilon) M + \varepsilon. \tag{8}$$

The convexity of h, combined with $h(0) = 0$ together, implies that $\lambda_\varepsilon h(x_\varepsilon) \geq h(\lambda_\varepsilon x_\varepsilon)(= s_1)$. Therefore

$$r_0(1-\lambda_\varepsilon) \leq (1-\lambda_\varepsilon) h(x_\varepsilon) \leq h(x_\varepsilon) - h(\lambda_\varepsilon x_\varepsilon) \leq \varepsilon. \tag{9}$$

Putting this and (8) together, we have

$$\left|g(s_2) - g(s_1)\right| \leq c\left|s_2 - s_1\right|, \tag{10}$$

where $c = 1 + (1/r_0)M$. This completes our claim. Therefore, g is continuous on $\mathbb{R}^+$. □

Lemma 4. *Let K be a bounded closed convex subset of a Banach space X. Assume that there is a continuous $w2R$ convex function on K. Then K is a weakly compact set.*

Proof. Without loss of generality, we can assume that $0 \in K$. The continuity of convex function f on K implies that $\partial f(0)$, the subdifferential of f at 0, is not empty. Take $x^* \in \partial f(0)$ and let $h : K \to \mathbb{R}$ be defined by

$$h(x) = f(x) - \langle x^*, x\rangle - f(0). \tag{11}$$

Then h is a nonnegative continuous convex function on K with $h(0) = 0$. Clearly, h is also $w2R$ convex on K. By the James characterization for weakly compact sets [19, 20], it suffices to prove that every linear functional $x^* \in X^*$ attains its maximum on K. For any given $x^* \in X^*$, we can assume that $\sup_K x^* \equiv \sup\{\langle x^*, x\rangle : x \in K\} > 0$ (otherwise we have $\sup_K x^* = 0 = \langle x^*, 0\rangle$). Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$g(s) = \sup\left\{\langle x^*, x\rangle : x \in K,\ h(x) \leq s\right\}. \tag{12}$$

By Lemma 3, g is continuous on $(0, +\infty)$. Setting

$$s_0 = \inf\left\{s > 0 : g(s) = \sup_K x^*\right\}. \tag{13}$$

In the following, through discussing two cases of s_0, that is, $s_0 = 0$ and $s_0 > 0$, we want to deduce that x^* can always obtain its maximum on K.

Case 1. If $s_0 = 0$, let $0 < s_n \to 0$ such that $g(s_n) = \sup_K x^*$ for all $n \in \mathbb{N}$. Choose a sequence $\{x_n\} \subset K$ such that $h(x_n) \leq s_n$ and such that

$$\langle x^*, x_n\rangle + \frac{1}{n} \geq g(s_n) = \sup_K x^* \quad \forall n \in \mathbb{N}. \tag{14}$$

Hence $\lim_{n\to\infty} h(x_n) = 0$. The convexity of h immediately implies that

$$\lim_{m,n\to\infty} h\left(\frac{x_m + x_n}{2}\right) = \lim_{n\to\infty} h(x_n) = 0. \tag{15}$$

The $w2R$ property of h on K gives that

$$x_n \xrightarrow{w} x_0 \tag{16}$$

for some $x_0 \in K$. This, combined with (14), means that $\langle x^*, x_0\rangle = \sup_K x^*$.

Case 2. If $s_0 > 0$. Choose a sequence $\{s_n\} \subset [s_0, +\infty)$ satisfying $s_n \to s_0$. Then $g(s_n) = \sup_K x^*$ for all $n \in \mathbb{N}$. The continuity of g on $(0, +\infty)$ gives that $g(s_n) \to g(s_0)$, and hence $g(s_0) = \sup_K x^*$. Thus $s_0 = \min\{s > 0 : g(s) = \sup_K x^*\}$. Let $\{x_n\} \subset K$ with $h(x_n) \le s_0$ such that $\lim_{n\to\infty}\langle x^*, x_n\rangle = \sup_K x^*$. Since $\{h(x_n)\}_n \subset \mathbb{R}$ is bounded, there exists a subsequence, still denoted by $\{h(x_n)\}_n$, so that it is convergent. Obviously $h(x_n) \to s_0$, and yet,

$$\lim_{m,n\to\infty} h\left(\frac{x_m + x_n}{2}\right) = s_0 = \lim_{n\to\infty} h(x_n). \tag{17}$$

Therefore $\{x_n\}$ is weakly convergent in K, say, $x_n \xrightarrow{w} x_0$, for some $x_0 \in K$. So we have $\sup_K x^* = \langle x^*, x_0\rangle$. This completes the proof. □

Now, we can prove the main result of this note.

Proof of Theorem 1. The "if" part of (i) is as follows: we assume without loss of generality that $0 \in D$ and that $B(0, r) = \{x \in X : \|x\| \le r\} \subset D$. By Lemma 4, the set $B(0, r)$ is weakly compact, and hence X is reflexive. This also immediately implies that the "if" part of (ii) is valid.

The "only if" parts of (i) and of (ii) are due to the aforementioned renorming theorems of Hájek and Johanis [15] and Odell and Schlumprecht [14], respectively. □

Remark 5. It should be mentioned that, especially recently, several characterizations of reflexivity were also obtained using purely metric properties of Banach spaces (but not the linear structure). See, for example, [21–23]. The interested reader may refer to those papers for further information.

Competing Interests

The authors declare that they have no competing interests.

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