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Research Article

A Characterization of Symmetric Stable Distributions

Wiktor Ejsmont

Department of Mathematics and Cybernetics, Wrocław University of Economics, Ulica Komandorska 118/120, 53-345 Wrocław, Poland

Correspondence should be addressed to Wiktor Ejsmont; wiktor.ejsmont@ue.wroc.pl

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Characterization problems in probability are studied here. Using the characteristic function of an additive convolution we generalize some known characterizations of the normal distribution to stable distributions. More precisely, if a distribution of a linear form depends only on the sum of powers of the certain parameters, then we obtain symmetric stable distributions.

1. Introduction

The original motivation for this paper comes from a desire to understand the results about characterization of normal distribution which were shown in [1]. In this paper, the author provides characterizations of the normal distribution using a certain invariance of the noncentral chi-square distribution. More precisely, let statistic $\sum_{i=1}^{n} a_i X_i + Y + Z$ have a distribution which depends only on $\sum_{i=1}^{n} a_i^2$, with $a_i \in \mathbb{R}$, $1 \leq m < n$, where $(X_1, \ldots, X_m, Y)$ and $(X_{m+1}, \ldots, X_n, Z)$ are independent random vectors with all moments, and X_i are nondegenerate; then X_i are independent and have the same normal distribution with zero means and $\operatorname{cov}(X_i, Y) = \operatorname{cov}(X_i, Z) = 0$ for $i \in \{1, \ldots, n\}$. The proof of the above theorem is divided into two parts: first, it is proved that this result holds for two random variables. Second, it is shown using the properties of multidimensional normal distribution. The additional moment assumption is due to the fact that the author uses a method of cumulants. An alternative method of proof (more direct and straightforward one) allows us to weaken some of the technical assumptions used in the above references and generalize it to a symmetric stable distribution. The paper is organized as follows. In Section 2 we review basic facts about characteristic function. Next in Section 3 we state and prove the main results.

2. A Characteristic Function

In this paper we denote by $\mu_X(dx)$ a probability measure of random variable X. If X is a random variable defined on a probability space (Ω, Σ, P), then the expected value of X, denoted by $E(X)$, is defined as the Lebesgue integral:

$$E(X) = \int_{\Omega} X(\omega)\, \mathbb{P}(d\omega). \tag{1}$$

A characteristic function is simply the Fourier transform, in probabilistic language. The characteristic function of a probability measure μ on $\mathbb{R}$ is the function $\varphi : \mathbb{R} \to \mathbb{C}$:

$$\varphi_{\mu}(t) = \int_{\mathbb{R}} \exp(itx)\, d\mu(dx). \tag{2}$$

When we speak of the characteristic function φ_X of a random variable X, we have the characteristic function φ_{μ_X} of its distribution μ_X in mind. Note, moreover, that

$$\varphi_{\mu}(t) = E[\exp(itX)]. \tag{3}$$

Apparently, it is not accidental that the characteristic function encodes the most important information about the associated random variables. The underlying reason may well reside in the following three important properties:

(i) The Gaussian distribution $N(\mu, \sigma)$ has the characteristic function $\varphi(t) = \exp(i\mu t - \sigma^2 t/2)$.

(ii) The symmetric α-stable distribution has the characteristic function $\varphi(t) = \exp(-c|t|^{\alpha})$, where $\alpha \in (0, 2\rangle$ and $c > 0$. For the special cases of parameter α, we get

(1) the upper bound $\alpha = 2$ corresponding to the normal distribution,

(2) $\alpha = 1$ corresponding to the Cauchy distribution,

(3) for $\alpha = 0.5$ the distribution reduces to a Lévy distribution.

(iii) Random variables $X_1, \ldots, X_n$ are independent if and only if for all $a_1, \ldots, a_n \in \mathbb{R}$ the joint characteristic function (i.e., the linear combination of the X_i's) satisfies

$$\begin{aligned} &E\exp\left(ia_1X_1 + \cdots + ia_nX_n\right) \\ &\quad = E\exp\left(ia_1X_1\right)\cdots E\exp\left(ia_nX_n\right). \end{aligned} \tag{4}$$

3. The Characterization Theorem

The main result of this paper is the following characterization of the symmetric α-stable distribution in terms of independent random vectors.

Theorem 1. *Let $X = (X_1, \ldots, X_m)$ and $(X_{m+1}, \ldots, X_n)$ be independent random vectors, where the distribution of X_i is not Dirac measure and let statistic $\sum_{i=1}^n a_iX_i + Z$ have a distribution which depends only on $\sum_{i=1}^n |a_i|^\alpha$, for all $a_1, \ldots, a_n \in \mathbb{R}$, $1 \le m < n$, Z being an arbitrary random variable, and $\alpha \in (0, 2\rangle$. Then X_i are independent and have the same symmetric α-stable distribution. Additionally, if $Z \in L^2(\Omega, \Sigma, P)$ and $\alpha = 2$ then $\operatorname{cov}(X_i, Z) = 0$ for $i \in \{1, \ldots, n\}$.*

Proof. We write $\Theta = r(\theta_1, \theta_2, \ldots, \theta_n)$, where $|\theta_1|^\alpha + |\theta_2|^\alpha + \cdots + |\theta_n|^\alpha = 1$. We focus on characteristic function $\varphi_{\sum_{i=1}^n r\theta_iX_i+Z}(t)$ of $\sum_{i=1}^n r\theta_iX_i + Z$. Then for $r > 0$ we have

$$\begin{aligned} \varphi_{\sum_{i=1}^n r\theta_iX_i+Z}\left(\frac{t}{r}\right) &= Ee^{i\left(\sum_{i=1}^n r\theta_iX_i+Z\right)t/r} \\ &= Ee^{i(\theta_1S_1+\theta_2S_2+\cdots+\theta_nX_n)t+iZt/r}. \end{aligned} \tag{5}$$

Since the hypothesis implies that left side of (5) does not depend on $\Theta/r = \theta = (\theta_1, \theta_2, \ldots, \theta_n)$, then

$$\lim_{r\to+\infty} \varphi_{\sum_{i=1}^n r\theta_iX_i+Z}\left(\frac{t}{r}\right) = Ee^{i(\theta_1X_1+\theta_2X_2+\cdots+\theta_nX_n)t} \tag{6}$$

does not depend on θ. In particular we have that the distribution of a statistic

$$\sum_{i=1}^n a_iX_i = \left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha}\left(\sum_{i=1}^n \frac{a_i}{\left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha}}X_i\right) \tag{7}$$

depends on $\sum_{i=1}^n |a_i|^\alpha$ only. Let $h(\sum_{i=1}^n |a_i|^q) = Ee^{i\sum_{i=1}^n a_iX_i}$. Because of the independence of $(X_1, \ldots, X_m)$ and $(X_{m+1}, \ldots, X_n)$ we may write

$$h\left(\sum_{i=1}^n |a_i|^\alpha\right) = Ee^{i\sum_{i=1}^m a_iX_i}Ee^{i\sum_{i=m+1}^n a_iX_i}. \tag{8}$$

Evaluating (8) first when $a_{m+1} = \cdots = a_n = 0$ and then when $a_1 = \cdots = a_m = 0$, we get $h(\sum_{i=1}^m |a_i|^\alpha) = Ee^{i\sum_{i=1}^m a_iX_i}$ and $h(\sum_{i=m+1}^n |a_i|^\alpha) = Ee^{i\sum_{i=m+1}^n a_iX_i}$, respectively. Substituting this into (8), we see

$$h\left(\sum_{i=1}^n |a_i|^\alpha\right) = h\left(\sum_{i=1}^m |a_i|^\alpha\right)h\left(\sum_{i=m+1}^n |a_i|^\alpha\right). \tag{9}$$

Note that $h(u)$ is continuous in $u \in [0, \infty)$, which implies $h(u) = e^{cu}$ (see Aczel [2], page 31), and so we have $h(\sum_{i=1}^n |a_i|^\alpha) = e^{c(\sum_{i=1}^n a_i^\alpha)}$. Thus we have actually proved that $X_1, \ldots, X_n$ have the same symmetric α-stable distribution. But since we know that the distributions of X_i are symmetric stable, the independence of random variables $X_1, \ldots, X_n$ follows from the observation that

$$\begin{aligned} h\left(\sum_{i=1}^n |a_i|^\alpha\right) &= Ee^{i\sum_{i=1}^n a_iX_i} = e^{c(\sum_{i=1}^n |a_i|^\alpha)} \\ &= E\exp\left(iX_1a_1\right)\cdots E\exp\left(iX_na_n\right). \end{aligned} \tag{10}$$

If $\alpha = 2$ and $E(Z^2) < \infty$ then $\operatorname{cov}(X_i, Z) < \infty$ and because of the independence of $X_1, \ldots, X_n$ we may write

$$\begin{aligned} V\left(\sum_{i=1}^n a_i^2\right) &= \operatorname{Var}\left(\sum_{i=1}^n a_iX_i + Z\right) \\ &= 2\sum_{i=1}^n a_i\operatorname{cov}\left(X_i, Z\right) + \left(\sum_{i=1}^n a_i^2\right)\operatorname{Var}\left(X_1\right) \\ &\quad + \operatorname{Var}\left(Z\right), \end{aligned} \tag{11}$$

because $\operatorname{cov}(X_i, X_j) = 0$ for $i \ne j$ and $\operatorname{Var}(X_1) = \cdots = \operatorname{Var}(X_n)$. This implies that linear combination $\sum_{i=1}^n a_i\operatorname{cov}(X_i, Z)$ is constant on sphere $\{(a_1, \ldots, a_n) : \sum_{i=1}^n a_i^2 = 1\}$ which gives $\operatorname{cov}(X_i, Z) = 0$. □

The above consideration gives us the following results of Ejsmont [1] and Cook [3], respectively.

Corollary 2 (the main result of Ejsmont [1]). *Let $(X_1, \ldots, X_m, Y)$ and $(X_{m+1}, \ldots, X_n, Z)$ be independent random vectors with all moments, where X_i are nondegenerate, and let statistic $\sum_{i=1}^n a_iX_i + Y + Z$ have a distribution which depends only on $\sum_{i=1}^n a_i^2$ for all $a_i \in \mathbb{R}$ and $1 \le m < n$. Then X_i are independent and have the same normal distribution with zero means and $\operatorname{cov}(X_i, Y) = \operatorname{cov}(X_i, Z) = 0$ for $i \in \{1, \ldots, n\}$.*

Corollary 3 (the main result of Cook [3]). *Let $(X_1, \ldots, X_m)$ and $(X_{m+1}, \ldots, X_n)$ be independent random vectors, where X_i are nondegenerate, and let statistic $\sum_{i=1}^n (X_i + a_i)^2$ have a distribution which depends only on $\sum_{i=1}^n a_i^2$, $a_i \in \mathbb{R}$ and $1 \le m < n$. Then X_i are independent and have the same normal distribution with zero means.*

Proof. If we put $Z = \sum_{i=1}^n X_i^2$ and $\alpha = 2$ in Theorem 1 then we get

$$\sum_{i=1}^m a_iX_i + \sum_{i=m+1}^n a_iX_i + Z = \sum_{i=1}^n \left(X_i + \frac{a_i}{2}\right)^2 - \frac{1}{4}\times\sum_{i=1}^n a_i^2. \tag{12}$$

This means that the distribution of $\sum_{i=1}^{m} a_i X_i + \sum_{i=m+1}^{n} a_i X_i + Z$ depends only on $\sum_{i=1}^{n} a_i^2$, which by Theorem 1 implies the statement. □

Here we state the Herschel-Maxwell theorem in modern notation (see, e.g., [4] or [5]). This theorem can be also obtained from Theorem 1 by considering $m = 1$ and $n = 2$ as well as $Z = 0$ and $\alpha = 2$ (the proof is left to the reader).

Theorem 4. *Let X, Y be independent random variables and a_1, a_2 real numbers such that $a_1^2 + a_2^2 = 1$. Then X, Y are normally distributed with zero means if and only if $a_1 X + a_2 Y$ is distributed identically as X for any $(a_1, a_2) \in \mathbb{R}^2$.*

Open Problem. Kagan and Letac [6] formulate the following theorem: Let n be a fixed integer $n \geq 3$. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables. In the Euclidean space $\mathbb{R}^n$ consider the linear subspace $E = 1^{\perp}$, that is, the set $\{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = 0\}$. Then the following characterizations hold: If for all $a \in E$ the distribution of the random variable

$$\sum_{i=1}^{n} \left(X_i - \overline{X} + a_i\right)^2 \tag{13}$$

depends only on $\|a\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$, then X_i's are normally distributed.

A key role in the proof of these results is played by *Marcinkiewicz' theorem*: if $Q(x)$ is a polynomial and $\exp(Q(x))$ is the characteristic function of some probability distribution, then the degree of Q is less than or equal to two. Finally, we present the conjecture (Theorem 1 cannot be applied here).

Conjecture 5. *Let $(X_1, \ldots, X_m)$ and $(X_{m+1}, \ldots, X_n)$ be independent random vectors, where X_i are nondegenerate, and the distribution of the random variable*

$$\sum_{i=1}^{n} \left(X_i - \overline{X} + a_i\right)^2 \tag{14}$$

depends only on $\|a\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$, where $a = (a_1, \ldots, a_n) \in E$, and then X_i's are normally distributed independent random variables.

Competing Interests

The author declares having no competing interests.

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