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Research Article

Estimates for Parameter Littlewood-Paley g_κ^* Functions on Nonhomogeneous Metric Measure Spaces

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Let $(\mathcal{X}, d, \mu)$ be a metric measure space which satisfies the geometrically doubling measure and the upper doubling measure conditions. In this paper, the authors prove that, under the assumption that the kernel of $\mathfrak{M}_\kappa^*$ satisfies a certain Hörmander-type condition, $\mathfrak{M}_\kappa^{*,\rho}$ is bounded from Lebesgue spaces $L^p(\mu)$ to Lebesgue spaces $L^p(\mu)$ for $p \geq 2$ and is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. As a corollary, $\mathfrak{M}_\kappa^{*,\rho}$ is bounded on $L^p(\mu)$ for $1 < p < 2$. In addition, the authors also obtain that $\mathfrak{M}_\kappa^{*,\rho}$ is bounded from the atomic Hardy space $H^1(\mu)$ into the Lebesgue space $L^1(\mu)$.

1. Introduction

In 1958, Stein in [1] firstly introduced the Littlewood-Paley operators of the higher-dimensional case; meanwhile, the author also obtained the boundedness of the Marcinkiewicz integrals and area integrals. In 1970, Fefferman in [2] proved that the Littlewood-Paley g_κ^* function is weak type (p, p) for $p \in (1, 2)$ and $\kappa = 2/p$. With further research about Littlewood-Paley operators, some authors turn their attentions to study the parameter Littlewood-Paley operators. For example, in 1999, Sakamoto and Yabuta in [3] considered the parameter g_κ^* function. Since then, many papers focus on the behaviours of the operators; among them we refer readers to see [4–6].

In the past ten years or so, most authors mainly study the classical theory of harmonic analysis on $\mathbb{R}^n$ under nondoubling measures which only satisfy the polynomial growth condition; see [7–12]. Exactly, we assume that μ which is a positive Radon measure on $\mathbb{R}^n$ satisfies the following growth conditions; namely, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, there exist constant C and $0 < d \leq n$ such that

$$\mu(B(x, r)) \leq C r^d, \tag{1}$$

where $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The analysis associated with nondoubling measures μ as in (1) has important applications in solving long-standing open Painlevé's problem and Vitushkin's conjecture (see [13, 14]). Besides, Coifman and Weiss have showed that the measure μ is a key assumption in harmonic analysis on homogeneous-type spaces (see [15, 16]).

However, Hytönen in [17] pointed that the measure μ as in (1) may not contain the doubling measure as special cases. To solve the problem, in 2010, Hytönen in [17] introduced a new class of metric measure spaces satisfying the so-called upper doubling conditions and the geometrically doubling (resp., see Definitions 1 and 2 below), which are now claimed nonhomogeneous metric measure spaces. Therefore, if we replace the underlying spaces with nonhomogeneous metric measure spaces, many known-consequences have been proved still true; for example, see [18–22].

In this paper, we always assume that $(\mathcal{X}, d, \mu)$ is a nonhomogeneous metric measure space. In this setting, we will establish the boundedness of the parameter Littlewood-Paley g_κ^* functions on $(\mathcal{X}, d, \mu)$.

In order to state our main results, we firstly recall some necessary notions and notation. Hytönen in [17] gave out the definition of upper doubling metric spaces as follows.

Definition 1 (see [17]). A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling, if μ is Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is nondecreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x,r)) \leq \lambda(x,r) \leq C_\lambda \lambda\left(x, \frac{r}{2}\right). \tag{2}$$

Htyönen et al. in [18] proved that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \leq \lambda$, $C_{\widetilde{\lambda}} \leq C_\lambda$ and

$$\widetilde{\lambda}(x, y) \leq C_{\widetilde{\lambda}} \widetilde{\lambda}(y, r), \tag{3}$$

where $x, y \in \mathcal{X}$ and $d(x, y) \leq r$. Based on this, from now on, let the dominating function in (2) also satisfy (3).

Now we recall the notion of geometrically doubling conditions given in [17].

Definition 2 (see [17]). A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 3 (see [17]). Let $(\mathcal{X}, d)$ be a metric space. Hytönen in [17] showed that the following statements are mutually equivalent:

(1) $(\mathcal{X}, d)$ is geometrically doubling.

(2) For any $\epsilon \in (0, 1)$ and ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$. Here and in what follows, N_0 is as Definition 2 and $n = \log_2 N_0$.

(3) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ can contain at most $N_0 \epsilon^{-n}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$.

(4) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ can contain at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

Hytönen in [17] introduced the following coefficients $K_{B,S}$ analogous to Tolsa's number $K_{Q,R}$ in [7].

Given any two balls $B \subset S$, set

$$K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \mathrm{d}\mu(x), \tag{4}$$

where c_B represents the center of the ball B.

Remark 4. Bui and Duong in [21] firstly introduced the following discrete version $\widetilde{K}_{B,S}$ of $K_{B,S}$ as in (4) on $(\mathcal{X}, d, \mu)$, which is very similar to the number $K_{Q,R}$ introduced in [7] by Tolsa. For any two balls $B \subset S$, $\widetilde{K}_{B,S}$ is defined by

$$\widetilde{K}_{B,S} = 1 + \sum_{i=1}^{N_{B,S}} \frac{\mu(6^i B)}{\lambda(c_B, 6^i r_B)}, \tag{5}$$

where the radii of the balls B and S are denoted by r_B and r_S, respectively, and $N_{B,S}$ is the smallest integer satisfying $6^{N_{B,S}} r_B \geq r_s$. It is easy to obtain $\widetilde{K}_{B,S} \leq C K_{B,S}$. Bui and Duong in [21] also pointed out that it is incorrect that $K_{B,S} \sim \widetilde{K}_{B,S}$.

Now we recall the following notion of (α, β)-doubling property (see [17]).

Definition 5 (see [17]). Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is claimed to be (α, β)-doubling if $\mu(\alpha B) \leq \beta \mu(B)$.

It was stated in [17] that, there exist many balls which have the above (α, β)-doubling property. In the latter part of the paper, if α and β_α are not specified, (α, β_α)-doubling ball always stands for $(6, \beta_6)$-doubling ball with a fixed number $\beta_6 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n := \log_2 N_0$ is considered as a geometric dimension of the space. Moreover, the smallest $(6, \beta_6)$-doubling ball of the form $6^j B$ with $j \in \mathbb{N}$ is denoted by $\widetilde{B}^6$, and sometimes $\widetilde{B}^6$ can be simply denoted by $\widetilde{B}$.

Now we give the definition of the parameter Littlewood-Paley g_κ^* functions on $(\mathcal{X}, d, \mu)$.

Definition 6 (see [22]). Let $K(x, y)$ be a locally integrable function on $(\mathcal{X} \times \mathcal{X}) \setminus \{(x, y) : x = y\}$. Assume that there exists a positive constant C such that, for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$|K(x, y)| \leq C \frac{d(x, y)}{\lambda(x, d(x, y))} \tag{6}$$

and, for all $x, y, y' \in \mathcal{X}$,

$$\int_{d(x,y) \geq 2d(y,y')} \left[\left| K(x, y) - K(x, y') \right| + \left| K(y, x) - K(y', x) \right| \right] \frac{1}{d(x, y)} \mathrm{d}\mu(x) \leq C. \tag{7}$$

The parameter Marcinkiewicz integral $\mathcal{M}^\rho$ associated with the above $K(x, y)$ which satisfies (6) and (7) is defined by

$$\mathcal{M}^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \cdot \int_{d(x,y) \leq t} \frac{K(x, y)}{[d(x, y)]^{1-\rho}} f(y) \mathrm{d}\mu(y) \right|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}, \quad x \in \mathcal{X}, \tag{8}$$

where $\rho \in (0,\infty)$. The parameter g_κ^* function $\mathfrak{M}_\kappa^{*,\rho}$ is defined by

$$\mathfrak{M}_\kappa^{*,\rho}(f)(x) = \left\{ \iint_{\mathscr{X}\times(0,\infty)} \left(\frac{t}{t+d(x,y)}\right)^{\kappa} \left| \frac{1}{t^\rho} \cdot \int_{d(y,z)\le t} \frac{K(y,z)}{[d(y,z)]^{1-\rho}} f(z)\, d\mu(z) \right|^2 \frac{d\mu(y)}{\lambda(y,t)} \frac{dt}{t} \right\}^{1/2}, \quad (9)$$

where $x \in \mathscr{X}$, $\mathscr{X}\times(0,\infty) := \{(y,t) : y \in \mathscr{X},\ t > 0\}$, $\rho > 0$ and $\kappa \in (1,\infty)$.

Remark 7. (1) When $\rho = 1$, the operator $\mathscr{M}^\rho$ as in (8) is just the Marcinkiewicz integral on $(\mathscr{X}, d, \mu)$ (see [22]).

(2) If we take $(\mathscr{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ and $\lambda(y,t) := t^n$, then the parameter g_κ^* function $\mathfrak{M}_\kappa^{*,\rho}$ as in (9) is just a parameter Littlewood-Paley operator with nondoubling measures in [8].

The following definition of the atomic Hardy space was introduced by Htyönen et al. (see [18]).

Definition 8 (see [18]). Let $\zeta \in (1,\infty)$ and $p \in (1,\infty]$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a $(p,1)_\upsilon$-*atomic block* if

(a) there exists a ball B such that $\operatorname{supp} b \subset B$,

(b) $\int_{\mathscr{X}} b(x)\, d\mu(x) = 0$,

(c) for any $i \in \{1,2\}$ there exist a function a_i supported on ball $B_i \subset B$ and a number $\upsilon_i \in \mathbb{C}$ such that

$$b = \upsilon_1 a_1 + \upsilon_2 a_2, \qquad \|a_i\|_{L^p(\mu)} \le [\mu(\zeta B_i)]^{1/p-1} K_{B_i,B}^{-1}. \quad (10)$$

Moreover, let $|b|_{H^{1,p}_{\mathrm{atb}}(\mu)} := |\upsilon_1| + |\upsilon_2|$.

We say a function $f \in L^1(\mu)$ belongs to the *atomic Hardy* space $H^{1,p}_{\mathrm{atb}}(\mu)$ if there are atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ with $\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)} < \infty$. The $H^{1,p}_{\mathrm{atb}}(\mu)$ norm of f is denoted by $\|f\|_{H^{1,p}_{\mathrm{atb}}(\mu)} = \inf\{\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)}\}$, where the infimum is taken over all the possible decompositions of f as above.

It was proved by Htyönen et al. in [18] that the definition of $H^{1,p}_{\mathrm{atb}}(\mu)$ is not related to the choice of ζ and the spaces $H^{1,p}_{\mathrm{atb}}(\mu)$ and $H^{1,\infty}_{\mathrm{atb}}(\mu)$ have the same norms for $p \in (1,\infty]$. Thus, for convenience, we always denote $H^{1,p}_{\mathrm{atb}}(\mu)$ by $H^1(\mu)$.

Now we give the Hörmander-type condition on $(\mathscr{X}, d, \mu)$; that is, there exists a positive C, such that

$$\sup_{\substack{r>0 \\ d(y,y')\le r}} \sum_{i=1}^\infty i \int_{6^i r < d(x,y) \le 6^{i+1} r} \left[\left|K(x,y) - K(x,y')\right| + \left|K(y,x) - K(y',x)\right| \right] \frac{d\mu(x)}{d(x,y)} \le C. \quad (11)$$

Notice this condition is slightly stronger than (7).

Now let us state the main theorems which generalize and improve the corresponding results in [8].

Theorem 9. *Let $K(x,y)$ satisfy (6) and (7), and let $\mathfrak{M}_\kappa^{*,\rho}$ be as in (9) with $\rho \in (0,\infty)$ and $\kappa \in (1,\infty)$. Then $\mathfrak{M}_\kappa^{*,\rho}$ is bounded on $L^p(\mu)$ for any $p \in [2,\infty)$.*

Theorem 10. *Let $K(x,y)$ satisfy (6) and (11), and let $\mathfrak{M}_\kappa^{*,\rho}$ be as in (9) with $\rho \in (1/2,\infty)$ and $\kappa \in (1,\infty)$. Then $\mathfrak{M}_\kappa^{*,\rho}$ is bounded from $L^1(\mu)$ into weak $L^1(\mu)$; namely, there exists a positive constant C such that, for any $\tau > 0$ and $f \in L^1(\mu)$,*

$$\mu\left(\{x \in \mathscr{X} : \mathfrak{M}_\kappa^{*,\rho}(f)(x) > \tau\}\right) \le C \frac{\|f\|_{L^1(\mu)}}{\tau}. \quad (12)$$

Theorem 11. *Let $K(x,y)$ satisfy (6) and (11), and let $\mathfrak{M}_\kappa^{*,\rho}$ be as in (9) with $\rho > 1/2$ and $\kappa > 1$. Suppose that $\mathfrak{M}_\kappa^{*,\rho}$ is bounded on $L^2(\mu)$. Then, $\mathfrak{M}_\kappa^{*,\rho}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$.*

Applying the Marcinkiewicz interpolation theorem and Theorems 9 and 10, it is easy to get the following result.

Corollary 12. *Under the assumption of Theorem 10, $\mathfrak{M}_\kappa^{*,\rho}$ is bounded on $L^p(\mu)$ for $p \in (1,2)$.*

The organization of this paper is as follows. In Section 2, we will give some preliminary lemmas. The proofs of the main theorems will be given in Section 3. Throughout this paper, C stands for a positive constant which is independent of the main parameters, but it may be different from line to line. For any $E \subset \mathscr{X}$, we use χ_E to denote its characteristic function.

2. Preliminary Lemmas

In this section, we make some preliminary lemmas which are used in the proof of the main results. Firstly, we recall some properties of $K_{B,S}$ as in (4) (see [17]).

Lemma 13 (see [17]). *(1) For all balls $B \subset R \subset S$, it holds true that $K_{B,R} \le K_{B,S}$.*

(2) For any $\xi \in [1,\infty)$, there exists a positive constant C_ξ, such that, for all balls $B \subset S$ with $r_S \le \xi r_B$, $K_{B,S} \le C_\xi$.

(3) For any $\varrho \in (1,\infty)$, there exists a positive constant C_ϱ, depending on ϱ, such that, for all balls B, $K_{B,\widetilde{B}^\varrho} \le C_\varrho$.

(4) There exists a positive constant c such that, for all balls $B \subset R \subset S$, $K_{B,S} \le K_{B,R} + cK_{R,S}$. In particular, if B and R are concentric, then $c = 1$.

(5) There exists a positive constant $\widetilde{c}$ such that, for all balls $B \subset R \subset S$, $K_{B,R} \le \widetilde{c} K_{B,S}$; moreover, if B and R are concentric, then $K_{R,S} \le K_{B,S}$.

To state the following lemmas, let us give a known-result (see [19]). For $\eta \in (0,\infty)$, the maximal operator is defined, by setting that, for all $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \mathscr{X}$,

$$M_{(\eta)} f(x) := \sup_{Q \ni x,\, Q \text{ doubling}} \frac{1}{\mu(\eta Q)} \int_Q |f(y)| \, d\mu(y) \tag{13}$$

is bounded on $L^p(\mu)$ provided that $p \in (1,\infty)$ and also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

The following lemma is slightly changed from [8].

Lemma 14. *Let $K(x,y)$ satisfy (6) and (7), and $\eta \in (0,\infty)$. Assume that $\mathscr{M}^\rho$ is as in (8) and $\mathfrak{M}^{*,\rho}_\kappa$ is as in (9) with $\rho \in (0,\infty)$ and $\kappa \in (1,\infty)$. Then for any nonnegative function ϕ, there exists a positive constant C such that, for all $f \in L^p(\mu)$ with $p \in (1,\infty)$,*

$$\begin{aligned} &\int_{\mathscr{X}} \left[\mathfrak{M}^{*,\rho}_\kappa(f)(x)\right]^2 \phi(x) \, d\mu(x) \\ &\quad \le C \int_{\mathscr{X}} \left[\mathscr{M}^\rho(f)(x)\right]^2 M_\eta(\phi)(x) \, d\mu(x). \end{aligned} \tag{14}$$

Proof. By the definition of $\mathfrak{M}^{*,\rho}_\kappa(f)$, we have

$$\begin{aligned} &\int_{\mathscr{X}} \left[\mathfrak{M}^{*,\rho}_\kappa(f)(x)\right]^2 \phi(x) \, d\mu(x) \\ &\quad = \int_{\mathscr{X}} \iint_{\mathscr{X}\times(0,\infty)} \left(\frac{t}{t+d(x,y)}\right)^\beta \left|\frac{1}{t^\rho} \int_{d(y,z)\le t} \frac{K(y,z)}{[d(y,z)]^{1-\rho}} f(y) \, d\mu(z)\right|^2 \frac{d\mu(y)}{\lambda(y,t)} \frac{dt}{t} \phi(x) \, d\mu(x) \\ &\quad \le \int_{\mathscr{X}} \int_0^\infty \left|\frac{1}{t^\rho} \int_{d(y,z)\le t} \frac{K(y,z)}{[d(y,z)]^{1-\rho}} f(y) \, d\mu(z)\right|^2 \frac{dt}{t} \sup_{t>0} \left[\int_{\mathscr{X}} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x)\right] d\mu(y) \\ &\quad = \int_{\mathscr{X}} \left[\mathscr{M}^\rho(f)(y)\right]^2 \sup_{t>0} \left[\int_{\mathscr{X}} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x)\right] d\mu(y). \end{aligned} \tag{15}$$

Thus, to prove Lemma 14, we only need to estimate that

$$\begin{aligned} &\sup_{t>0} \int_{\mathscr{X}} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \\ &\quad \le C M_\eta(\phi)(y). \end{aligned} \tag{16}$$

For any $y \in \mathscr{X}$ and $t > 0$, write

$$\begin{aligned} &\int_{\mathscr{X}} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \\ &\quad = \int_{B(y,t)} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \\ &\qquad + \int_{\mathscr{X}\setminus B(y,t)} \left(\frac{t}{t+d(x,y)}\right)^\beta \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \\ &\quad =: D_1 + D_2. \end{aligned} \tag{17}$$

For D_1, it is not difficult to obtain that

$$\begin{aligned} D_1 &\le \int_{B(y,t)} \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \\ &= \frac{\mu(\eta B(y,t))}{\lambda(y,t)} \frac{1}{\mu(\eta B(y,t))} \int_{B(y,t)} \phi(x) \, d\mu(x) \\ &\le C M_\eta(\phi)(y). \end{aligned} \tag{18}$$

Now we turn to estimate D_2, by (2) and (13); we have

$$\begin{aligned} D_2 &\le \sum_{k=1}^\infty \int_{B(y,6^k t)\setminus B(y,6^{k-1}t)} \left(\frac{t}{t+d(x,y)}\right)^\beta \\ &\quad \cdot \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \le C \sum_{k=1}^\infty 6^{-(k-1)\beta} \\ &\quad \cdot \int_{B(y,6^k t)} \frac{\phi(x)}{\lambda(y,t)} d\mu(x) \le C \sum_{k=1}^\infty 6^{-(k-1)\beta} \\ &\quad \cdot \frac{\mu(B(y,6^k t))}{\lambda(y,t)} \frac{1}{\mu(B(y,6^k t))} \int_{B(y,6^k t)} \phi(x) \, d\mu(x) \\ &\le C \sum_{k=1}^\infty 6^{-(k-1)\beta} \frac{\mu(B(y,6^k t))}{\lambda(y,t)} M_\eta(\phi)(y) \le C \\ &\quad \cdot \frac{\lambda(y,6^k t)}{\lambda(y,t)} M_\eta(\phi)(y) \sum_{k=1}^\infty 6^{-(k-1)\beta} \frac{\lambda(y,6^k t)}{\lambda(y,t)} \le C \\ &\quad \cdot \frac{\lambda(y,6^k t)}{\lambda(y,t)} M_\eta(\phi)(y) \sum_{k=1}^\infty 6^{-(k-1)\beta} \le C M_\eta(\phi)(y). \end{aligned} \tag{19}$$

Combining the estimates for D_1 and D_2, we obtain (16) and hence complete the proof of Lemma 14. □

Finally, we recall the Calderón-Zygmund decomposition theorem (see [21]). Suppose that γ_0 is a fixed positive constant satisfying that $\gamma_0 > \max\{C_\lambda^{3\log_2 6}, 6^{3n}\}$, where C_λ is as in (2) and n as in Remark 3.

Lemma 15 (see [21]). *Let $p \in [1,\infty)$, $f \in L^p(\mu)$, and $t \in (0,\infty)$ ($t > \gamma_0\|f\|_{L^p(\mu)}/\mu(\mathscr{X})$ when $\mu(\mathscr{X}) < \infty$). Then*

(1) *there exists a family of finite overlapping balls $\{6B_i\}_i$ such that $\{B_i\}_i$ is pairwise disjoint:*

$$\frac{1}{\mu(6^2B_i)}\int_{B_i}|f(x)|^p\,d\mu(x) > \frac{t^p}{\gamma_0} \quad \forall i, \tag{20}$$

$$\frac{1}{\mu(6^2\tau B_i)}\int_{\tau B_i}|f(x)|^p\,d\mu(x) \le \frac{t^p}{\gamma_0} \quad \forall i,\ \forall \tau \in (2,\infty), \tag{21}$$

$$|f(x)| \le t \quad \text{for } \mu\text{-almost every } x \in \mathscr{X}\setminus\Big(\bigcup_i 6B_i\Big); \tag{22}$$

(2) *for each i, let S_i be a $(3\times 6^2, C_\lambda^{\log_2(3\times 6^2)+1})$-doubling ball of the family $\{(3\times 6^2)^k B_i\}_{k\in\mathbb{N}}$, and $\omega_i = \chi_{6B_i}/(\sum_k \chi_{6B_k})$. Then there exists a family $\{\varphi_i\}_i$ of functions that, for each i, $\operatorname{supp}(\varphi_i) \subset S_i$, φ_i has a constant sign on S_i and*

$$\int_{\mathscr{X}}\varphi_i(x)\,d\mu(x) = \int_{6B_i} f(x)\,\omega_i(x)\,d\mu(x), \qquad \sum_i|\varphi_i(x)| \le \gamma t \quad \text{for } \mu\text{-almost every } x \in \mathscr{X}, \tag{23}$$

where γ is some positive constant depending only on $(\mathscr{X},\mu)$, and there exists a positive constant C, independent of f, t, and i, such that if $p = 1$, then

$$\|\varphi_i\|_{L^\infty(\mu)}\,\mu(S_i) \le C\int_{\mathscr{X}}|f(x)\,\omega_i(x)|\,d\mu(x), \tag{24}$$

and if $p \in (1,\infty)$,

$$\left(\int_{S_i}|\varphi_i(x)|^p\,d\mu(x)\right)^{1/p}[\mu(S_i)]^{1/p'} \le \frac{C}{t^{p-1}}\int_{\mathscr{X}}|f(x)\,\omega_i(x)|^p\,d\mu(x). \tag{25}$$

3. Proofs of Theorems

Proof of Theorem 9. For the case of $p = 2$, assume $\phi(x) = 1$ in Lemma 14; then it is easy to get that

$$\int_{\mathscr{X}}[\mathfrak{M}_\kappa^{*,\rho}(f)(x)]^2\,\mathrm{d}\mu(x) \le C\int_{\mathscr{X}}[\mathscr{M}^\rho(f)(x)]^2\,\mathrm{d}\mu(x), \tag{26}$$

which, along with $L^2(\mu)$-boundedness of $\mathscr{M}^\rho$, easily yields that Theorem 9 holds.

For the case of $p > 2$, let q be the index conjugate to $p/2$. By applying Hölder inequality and Lemma 14, we can conclude

$$\begin{aligned}\|\mathfrak{M}_\kappa^{*,\rho}(f)\|_{L^p(\mu)}^2 &= \sup_{\substack{\phi\ge 0\\ \|\phi\|_{L^q(\mu)}\le 1}}\int_{\mathscr{X}}[\mathfrak{M}_\kappa^{*,\rho}(f)(x)]^2\,\phi(x)\,\mathrm{d}\mu(x)\\ &\le C\sup_{\substack{\phi\ge 0\\ \|\phi\|_{L^q(\mu)}\le 1}}\int_{\mathscr{X}}[\mathscr{M}^\rho(f)(x)]^2\,M_\eta\phi(x)\,\mathrm{d}\mu(x)\\ &\le C\|\mathscr{M}^\rho(f)\|_{L^p(\mu)}^2\sup_{\substack{\phi\ge 0\\ \|\phi\|_{L^q(\mu)}\le 1}}\|M_\eta(\phi)\|_{L^q(\mu)}\\ &\le C\|f\|_{L^p(\mu)}^2\sup_{\substack{\phi\ge 0\\ \|\phi\|_{L^q(\mu)}\le 1}}\|\phi\|_{L^q(\mu)} \le C\|f\|_{L^p(\mu)}^2,\end{aligned} \tag{27}$$

which is desired. Thus, we complete the proof of Theorem 9. □

Proof of Theorem 10. Without loss of generality, we may assume that $\|f\|_{L^1(\mu)} = 1$. It is easy to see that the conclusion of Theorem 10 naturally holds if $\tau \le \beta_6(\|f\|_{L^1(\mu)}/\mu(\mathscr{X}))$ when $\mu(\mathscr{X}) < \infty$. Thus, we only need to discuss the case that $\tau > \beta_6(\|f\|_{L^1(\mu)}/\mu(\mathscr{X}))$. Applying Lemma 15 to f at the level τ and letting ω_i, φ_i, B_i, and S_i be the same as in Lemma 15, we see that $f(x) = b(x)+h(x)$, where $b(x) := f\chi_{\mathscr{X}\setminus\bigcup_i 6B_i}(x)+\sum_i\varphi_i(x)$ and $h(x) := \sum_i[\omega_i(x)f(x)-\varphi_i(x)] =: \sum_i h_i(x)$. It is easy to obtain that $\|b\|_{L^\infty(\mu)} \le C\tau$ and $\|b\|_{L^1(\mu)} \le C$. By $L^2(\mu)$-boundedness of $\mathfrak{M}_\kappa^{*,\rho}$, we have

$$\mu(\{x\in\mathscr{X} : \mathfrak{M}_\kappa^{*,\rho}(b)(x) > \tau\}) \le \frac{\|\mathfrak{M}_\kappa^{*,\rho}(b)\|_{L^2(\mu)}^2}{\tau^2} \le C\frac{\|b\|_{L^2(\mu)}^2}{\tau^2} \le C\tau^{-1}. \tag{28}$$

On the other hand, by (20) with $p = 1$ and the fact that the sequence of balls, $\{B_i\}_i$, is pairwise disjoint, we see that

$$\mu\Big(\bigcup_i 6^2B_i\Big) \le C\tau^{-1}\int_{\mathscr{X}}|f(x)|\,\mathrm{d}\mu(x) \le C\tau^{-1}, \tag{29}$$

and thus the proof of the Theorem 10 can be reduced to prove that

$$\mu\left(\left\{x\in\mathscr{X}\setminus\bigcup_i 6^2B_i : \mathfrak{M}_\kappa^{*,\rho}(h)(x) > \tau\right\}\right) \le C\tau^{-1}. \tag{30}$$

For each fixed i, denote the center of B_i by x_i, and let N_1 be the positive integer satisfying $S_i = (3\times 6^2)^{N_1}B_i$. We have

$$\mu\left(\left\{x\in\mathscr{X}\setminus\bigcup_i 6^2B_i : \mathfrak{M}_\kappa^{*,\rho}(h)(x) > \tau\right\}\right) \le \tau^{-1}\sum_i\int_{\mathscr{X}\setminus\bigcup_i 6^2B_i}\mathfrak{M}_\kappa^{*,\rho}(h_i)(x)\,\mathrm{d}\mu(x)$$

$$\leq \tau^{-1}\sum_i \int_{\mathscr{X}\setminus 6S_i} \mathfrak{M}_\kappa^{*,\rho}(h_i)(x)\,d\mu(x) + \tau^{-1}\sum_i \int_{6S_i\setminus 6^2B_i} \mathfrak{M}_\kappa^{*,\rho}(h_i)(x)\,d\mu(x) =: \tau^{-1}\sum_i (E_1+E_2). \tag{31}$$

Firstly, let us estimate E_2 and write it as

$$E_2 \leq \int_{6S_i\setminus 6^2B_i} \mathfrak{M}_\kappa^{*,\rho}(f\omega_i)(x)\,d\mu(x) + \int_{6S_i\setminus 6^2B_i} \mathfrak{M}_\kappa^{*,\rho}(\varphi_i)(x)\,d\mu(x) =: E_{21}+E_{22}, \tag{32}$$

where $h_i := \omega_i f - \varphi_i$. By Hölder inequality, (24), and $L^2(\mu)$-boundedness of $\mathfrak{M}_\kappa^{*,\rho}$, we have

$$\begin{aligned} E_{22} &\leq \int_{6S_i} \mathfrak{M}_\kappa^{*,\rho}(\varphi_i)(x)\,d\mu(x) \\ &\leq \left(\int_{6S_i} \left|\mathfrak{M}_\kappa^{*,\rho}(\varphi_i)(x)\right|^2 d\mu(x)\right)^{1/2} \mu(6S_i)^{1/2} \\ &\leq C\left(\int_{6S_i} |\varphi_i(x)|^2 d\mu(x)\right)^{1/2} \mu(6S_i)^{1/2} \\ &\leq C\int_{\mathscr{X}} |f(x)\,\omega_i(x)|\,d\mu(x). \end{aligned} \tag{33}$$

For E_{21}, by Minkowski inequality and (6), write

$$\begin{aligned} E_{21} &= \int_{6S_i\setminus 6^2B_i}\left[\iint_{\mathscr{X}\times(0,\infty)}\left|\left(\frac{t}{t+d(x,y)}\right)^{\kappa/2}\frac{1}{t^\rho}\int_{d(y,z)\leq t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}f(z)\,\omega_i(z)\,d\mu(z)\right|^2\frac{d\mu(y)\,dt}{\lambda(y,t)\,t}\right]^{1/2}d\mu(x) \\ &\leq C\int_{6S_i\setminus 6^2B_i}\int_{\mathscr{X}}|f(z)\,\omega_i(z)|\left[\iint_{d(y,z)\leq t}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(z)\,d\mu(x) \\ &\leq C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\iint_{d(y,z)\leq t}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z) \\ &\leq C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\iint_{\substack{d(y,z)\leq t\\ 2d(y,z)>d(x,z)}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z) \\ &\quad + C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\iint_{\substack{d(y,z)\leq t,\,d(x,y)<t\\ 2d(y,z)\leq d(x,z)}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z) \\ &\quad + C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\iint_{\substack{d(y,z)\leq t,\,d(x,y)\geq t\\ 2d(y,z)\leq d(x,z)}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z) \\ &=: F_1+F_2+F_3. \end{aligned} \tag{34}$$

To this end, let B_i be as in Lemma 15 with c_{B_i} and r_{B_i} being, respectively, its center and radius. For any $x \in 6S_i \setminus 6^2B_i$ and $z \in 6B_i$, by (2) and (3), we have

$$\begin{aligned} F_1 &\leq C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\int_{2d(y,z)>d(x,z)}\int_{d(y,z)}^{\infty}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z) \\ &\leq C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\int_{2d(y,z)>d(x,z)}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{1}{\lambda(y,d(y,z))}\left(\int_{d(y,z)}^{\infty}\frac{dt}{t^{1+2\rho}}\right)d\mu(y)\right]^{1/2}d\mu(x)\,d\mu(z) \\ &\leq C\int_{6B_i}|f(z)|\int_{6S_i\setminus 6^2B_i}\left[\int_{2d(y,z)>d(x,z)}\frac{1}{[\lambda(y,d(y,z))]^3}d\mu(y)\right]^{1/2}d\mu(x)\,d\mu(z) \end{aligned}$$

$$\begin{aligned}
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\int_{2d(y,z)>d(x,z)}\frac{1}{[\lambda(y,d(x,z))]}\frac{d\mu(y)}{[\lambda(y,(1/2)d(x,z))]^2}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\frac{1}{[\lambda(z,(1/2)d(x,z))]^2}\int_{2d(y,z)>d(x,z)}\frac{d\mu(y)}{\lambda(y,d(y,z))}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\sum_{k=1}^{\infty}\int_{B(z,2^{k-1}d(x,z))\backslash B(z,2^{k-2}d(x,z))}\frac{d\mu(y)}{\lambda(y,d(y,z))}\right]^{1/2}\frac{1}{\lambda(z,d(x,z))}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\sum_{k=1}^{\infty}\int_{B(z,2^{k-1}d(x,z))}\frac{d\mu(y)}{\lambda(y,2^{k-2}d(x,z))}\right]^{1/2}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}d\mu(x)\,d\mu(z)\leq C\int_{6B_i}|f(z)|\,d\mu(z),
\end{aligned}\tag{35}$$

where we use the fact that

$$\int_{6S_i\backslash 6^2B_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}d\mu(x)\leq CK_{B_i,S_i}.\tag{36}$$

Next we estimate F_2. For any $x\in 6S_i\setminus 6^2B_i$, $y\in\mathcal{X}$, and $z\in 6B_i$ satisfying $d(y,x)<t$, $2d(y,z)\leq d(x,z)$, and $(1/2)d(x,z)<t$, we have

$$\begin{aligned}
F_2&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\int_{2d(y,z)\leq d(x,z)}\int_{(1/2)d(x,z)}^{\infty}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\int_{2d(y,z)\leq d(x,z)}\frac{1}{[\lambda(y,d(x,z))]^2}\frac{d\mu(y)}{\mu(B(y,d(x,z)))}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}d\mu(x)\,d\mu(z)\leq C\int_{6B_i}|f(z)|\,d\mu(z).
\end{aligned}\tag{37}$$

Finally, for any $x\in 6S_i\setminus 6^2B_i$, $y\in\mathcal{X}$, and $z\in 6B_i$ satisfying $2d(y,z)\leq d(x,z)$, $2d(y,z)\geq d(x,z)$, and $d(x,y)<(3/2)d(x,z)$, by applying (2), we have

$$\begin{aligned}
F_3&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\iint_{\substack{d(y,z)\leq t,\,d(x,y)\geq t\\ 2d(y,z)\leq d(x,z)}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{d\mu(y)\,dt}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\int_{2d(y,z)\leq d(x,z)}\frac{1}{[\lambda(y,d(x,z))]^2}\frac{1}{\lambda(y,d(x,z))}d\mu(y)\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\left[\frac{1}{[\lambda(z,d(x,z))]^2}\frac{\mu(B(z,(1/2)d(x,z)))}{\lambda(z,d(x,z))}\right]^{1/2}d\mu(x)\,d\mu(z)\\
&\leq C\int_{6B_i}|f(z)|\int_{6S_i\backslash 6^2B_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}d\mu(x)\,d\mu(z)\leq C\int_{6B_i}|f(z)|\,d\mu(z).
\end{aligned}\tag{38}$$

Combining the estimates for F_1, F_2, and F_3, we obtain that $E_{21} \le C\int_{6B_i}|f(z)|\mathrm{d}\mu(z)$, where, together with the fact that $E_{22} \le C\int_{6B_i}|f(z)|\mathrm{d}\mu(z)$, we have

$$E_2 \le C\int_{6B_i}|f(x)|\,\mathrm{d}\mu(x). \tag{39}$$

Now we turn to estimate for E_1. Let $Q_i = B(c_{B_i}, r_{S_i})$, and write

$$\begin{aligned} E_1 \le{} & \int_{\mathcal{X}\setminus 6S_i}\left[\iint_{d(x,y)<t}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\le t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x) \\ & + \int_{\mathcal{X}\setminus 6S_i}\left[\iint_{\substack{d(x,y)\ge t\\ y\in Q_i}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\le t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x) \\ & + \int_{\mathcal{X}\setminus 6S_i}\left[\iint_{\substack{d(x,y)\ge t\\ y\in \mathcal{X}\setminus Q_i}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\le t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x) =: E_{11} \\ & + E_{12} + E_{13}. \end{aligned} \tag{40}$$

For each fixed i, decompose E_{11} as

$$\begin{aligned} E_{11} \le{} & \int_{\mathcal{X}\setminus 6S_i}\left[\iint_{\substack{d(x,y)<t\\ y\in 2S_i}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\le t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x) \\ & + \int_{\mathcal{X}\setminus 6S_i}\left[\iint_{\substack{d(x,y)<t\\ y\in \mathcal{X}\setminus 2S_i}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\le t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x) =: I_1 + I_2. \end{aligned} \tag{41}$$

For any $x \in \mathcal{X}\setminus 6S_i$, $y \in 2S_i$ with $d(y,x) < t$, and $z \in S_i$, $d(x, c_{B_i}) - 2r_{S_i} \le d(x,y) < t$ and $d(y,z) < 3r_{S_i}$, together with Minkowski inequality and (6), we can conclude

$$\begin{aligned} I_1 \le{} & C\int_{\mathcal{X}\setminus 6S_i}\int_{6S_i}|h_i(z)|\left[\iint_{\substack{d(x,y)<t,d(y,z)\le t\\ y\in 2S_i}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x) \\ \le{} & C\int_{6S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\left[\int_{d(y,z)\le 3r_{S_i}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\left(\int_{d(x,c_{B_i})-2r_{S_i}}^{\infty}\frac{\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right)\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z) \\ \le{} & C\int_{6S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\left[\int_{d(y,z)\le 3r_{S_i}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{1}{\mu(B(y,d(x,c_{B_i})))}\frac{1}{[d(x,c_{B_i})-2r_{S_i}]^{2\rho}}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z) \\ \le{} & C\int_{6S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\left[\int_{d(y,z)\le 3r_{S_i}}\frac{1}{[\lambda(y,d(y,z))]^2}\frac{1}{\mu(B(y,d(x,c_{B_i})))}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z) \\ \le{} & C\int_{6S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z) \le C\|h_i\|_{L^1(\mu)}. \end{aligned} \tag{42}$$

For I_2, write

$$
\begin{aligned}
I_2 \leq & \int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t\leq d(y,c_{B_i})+r_{S_i}}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\leq t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x)\\
& + \int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t> d(y,c_{B_i})+r_{S_i}}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\left|\frac{1}{t^{\rho}}\int_{d(y,z)\leq t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(x)\\
& =: I_{21}+I_{22}.
\end{aligned} \tag{43}
$$

For I_{21}, by Minkowski inequality and (6), we deduce

$$
\begin{aligned}
I_{21} \leq & C\int_{\mathscr{X}\backslash 6S_i}\int_{S_i}|h_i(z)|\left[\int_{y\in\mathscr{X}\backslash 2S_i}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{1}{\lambda\left(y,d\left(y,c_{B_i}\right)+r_{S_i}\right)}\left(\int_{d(y,z)}^{d(y,c_{B_i})+r_{S_i}}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
\leq & C\int_{\mathscr{X}\backslash 6S_i}\int_{S_i}|h_i(z)|\left[\int_{y\in\mathscr{X}\backslash 2S_i}\frac{1}{[\lambda(y,d(y,z))]^2}\frac{1}{\lambda\left(y,d\left(y,c_{B_i}\right)+r_{S_i}\right)}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
\leq & C\int_{S_i}|h_i(z)|\int_{\mathscr{X}\backslash 6S_i}\frac{1}{\lambda\left(c_{B_i},d\left(x,c_{B_i}\right)\right)}\sum_{k=1}^{\infty}\left[\int_{2^{k+1}6S_i\backslash 2^k 6S_i}\frac{1}{\lambda\left(y,d\left(y,c_{B_i}\right)+r_{S_i}\right)}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z)\leq C\|h_i\|_{L^1(\mu)}.
\end{aligned} \tag{44}
$$

Now we estimate I_{22}. Applying Minkowski inequality and the vanishing moment, we have

$$
\begin{aligned}
I_{22} \leq & C\int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t> d(y,c_{B_i})+r_{S_i}}}\left|\int_{d(y,z)\leq t}\left(\frac{K(y,z)}{[d(y,z)]^{1-\rho}}-\frac{K\left(y,c_{B_i}\right)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}\right)h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(x)\\
\leq & C\int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t> d(y,c_{B_i})+r_{S_i}}}\left|\int_{d(y,z)\leq t}\left(\frac{K(y,z)}{[d(y,z)]^{1-\rho}}-\frac{K(y,z)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}+\frac{K(y,z)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}-\frac{K\left(y,c_{B_i}\right)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}\right)h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(x)\\
\leq & C\int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t> d(y,c_{B_i})+r_{S_i}}}\left|\int_{d(y,z)\leq t}\left(\frac{K(y,z)}{[d(y,z)]^{1-\rho}}-\frac{K(y,z)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}\right)h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(x)\\
& + C\int_{\mathscr{X}\backslash 6S_i}\left[\iint_{\substack{d(x,y)<t,y\in\mathscr{X}\backslash 2S_i\\ t> d(y,c_{B_i})+r_{S_i}}}\left|\int_{d(y,z)\leq t}\left(\frac{K(y,z)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}-\frac{K\left(y,c_{B_i}\right)}{\left[d\left(y,c_{B_i}\right)\right]^{1-\rho}}\right)h_i(z)\,\mathrm{d}\mu(z)\right|^2\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(x)=:J_1+J_2.
\end{aligned} \tag{45}
$$

With a way similar to that used in the proof of I_1, we have $J_1 \leq C\|h_i\|_{L^1(\mu)}$. Thus, we only need to estimate J_2; by Minkowski inequality and (11), it follows that

$$
\begin{aligned}
J_2 \leq & C\int_{\mathscr{X}\backslash 6S_i}\int_{S_i}|h_i(z)|\\
& \cdot\left[\int_{\mathscr{X}\backslash 2S_i}\left|K(y,z)-K\left(y,c_{B_i}\right)\right|^2\frac{1}{\left[d\left(y,c_{B_i}\right)\right]^{2-2\rho}}\frac{1}{\lambda\left(y,d\left(y,c_{B_i}\right)+r_{S_i}\right)}\left(\int_{d(y,c_{B_i})+r_{S_i}}^{\infty}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)
\end{aligned}
$$

$$\begin{aligned}
&\leq C\int_{S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}\left[\int_{\mathcal{X}\setminus 2S_i}\left|K(y,z)-K(y,c_{B_i})\right|^2\frac{1}{[d(y,c_{B_i})]^2}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z)\\
&\leq C\int_{S_i}|h_i(z)|\\
&\quad\cdot\int_{\mathcal{X}\setminus 6S_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}\sum_{k=1}^{\infty}\left[\int_{2^k r_{S_i}<d(y,c_{B_1})\leq 2^{k+1}r_{S_i}}\left|K(y,z)-K(y,c_{B_i})\right|^2\frac{1}{[d(y,c_{B_i})]^2}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z)\\
&\leq C\int_{S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z)\leq C\|h_i\|_{L^1(\mu)}.
\end{aligned}\tag{46}$$

Combining the estimates for J_1, J_2, I_{21}, and I_1, we obtain that

$$E_{11}\leq C\|h_i\|_{L^1(\mu)}.\tag{47}$$

Next we estimate E_{12}. For any $y\in B_i$, $x\in\mathcal{X}\setminus 6S_i$, and $z\in S_i$, we have $d(x,y)\geq(1/2)d(x,c_{B_i})$, $d(y,z)\leq 2r_{S_i}$, and $d(x,y)\sim d(x,c_{B_i})$, and together with this fact, Minkowski inequality, and (6), we get

$$\begin{aligned}
E_{12}&\leq C\int_{\mathcal{X}\setminus 6S_i}\int_{S_i}|h_i(z)|\left[\iint_{\substack{d(x,y)\geq t\\ d(y,z)\leq t\\ y\in Q_i}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t^{1+2\rho}}\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
&\leq C\int_{\mathcal{X}\setminus 6S_i}\int_{S_i}|h_i(z)|\left[\int_{d(y,z)\leq 2r_{S_i}}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{1}{\lambda(y,d(x,c_{B_i}))}\left(\int_{d(y,z)}^{d(x,y)}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
&\leq C\int_{\mathcal{X}\setminus 6S_i}\int_{S_i}|h_i(z)|\left[\int_{d(y,z)\leq 2r_{S_i}}\frac{1}{[\lambda(y,d(y,z))]^2}\frac{1}{\lambda(y,d(x,c_{B_i}))}\mathrm{d}\mu(y)\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
&\leq C\int_{S_i}|h_i(z)|\int_{\mathcal{X}\setminus 6S_i}\frac{1}{\lambda(c_{B_i},d(x,c_{B_i}))}\mathrm{d}\mu(x)\,\mathrm{d}\mu(z)\leq C\|h_i\|_{L^1(\mu)}.
\end{aligned}\tag{48}$$

It remains to estimate E_{13}. Applying Minkowski inequality and (6), we have

$$\begin{aligned}
E_{13}&\leq C\int_{\mathcal{X}\setminus 6S_i}\int_{S_i}|h_i(x)|\left[\iint_{\substack{d(y,z)\leq t\leq d(x,y),y\in\mathcal{X}\setminus Q_i\\ d(x,c_{B_i})\leq 2d(y,c_{B_i})}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
&\quad+C\int_{\mathcal{X}\setminus 6S_i}\int_{S_i}|h_i(x)|\left[\iint_{\substack{d(y,z)\leq t\leq d(x,y),y\in\mathcal{X}\setminus Q_i\\ d(x,c_{B_i})>2d(y,c_{B_i})}}\left(\frac{t}{t+d(x,y)}\right)^{\kappa}\frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2}\frac{\mathrm{d}\mu(y)\,\mathrm{d}t}{\lambda(y,t)\,t}\right]^{1/2}\mathrm{d}\mu(z)\,\mathrm{d}\mu(x)\\
&=:U_1+U_2.
\end{aligned}\tag{49}$$

Now we estimate U_1. For any $y \in \mathscr{X} \setminus Q_i$, $z \in S_i$, and $d(y, z) \le t \le d(x,z)$, it is easy to see $d(y,z) \sim d(y, c_{B_i})$. So we have

$$\begin{aligned} U_1 &\le C \int_{\mathscr{X}\setminus 6S_i} \int_{S_i} |h_i(z)| \left[\int_{\substack{y\in\mathscr{X}\setminus Q_i \\ d(x,c_{B_i})\le 2d(y,c_{B_i})}} \frac{[d(y,z)]^{2\rho}}{[\lambda(y,d(y,z))]^2} \frac{1}{\lambda(y,d(x,y))} \left(\int_{d(y,z)}^{d(x,y)} \frac{dt}{t^{1+2\rho}} \right) d\mu(y) \right]^{1/2} d\mu(z)\, d\mu(x) \\ &\le C \int_{S_i} |h_i(z)| \int_{\mathscr{X}\setminus 6S_i} \left[\int_{\substack{y\in\mathscr{X}\setminus Q_i \\ d(x,c_{B_i})\le 2d(y,c_{B_i})}} \frac{1}{[\lambda(y,d(y,z))]^2} \frac{1}{\lambda(y,d(x,y))} d\mu(y) \right]^{1/2} d\mu(x)\, d\mu(z) \\ &\le C \int_{S_i} |h_i(z)| \int_{\mathscr{X}\setminus 6S_i} \frac{1}{\lambda(c_{B_i}, d(x,c_{B_i}))} \left[\int_{\substack{y\in\mathscr{X}\setminus Q_i \\ d(x,c_{B_i})\le 2d(y,c_{B_i})}} \frac{d\mu(y)}{\lambda(y,d(y,c_{B_i}))} \right]^{1/2} d\mu(x)\, d\mu(z) \\ &\le C \int_{S_i} |h_i(z)| \int_{\mathscr{X}\setminus 6S_i} \frac{1}{\lambda(c_{B_i}, d(x,c_{B_i}))} d\mu(x)\, d\mu(z) \le C \|h_i\|_{L^1(\mu)}. \end{aligned} \tag{50}$$

On the other hand, by a method similar to that used in the proof of U_1, we have

$$U_2 \le C \|h_i\|_{L^1(\mu)}. \tag{51}$$

Combining the estimates U_1, U_2, E_{11}, E_{12}, and the fact that $\|h_i\|_{L^1(\mu)} \le C \int_{6B_i} |f(x)| d\mu(x)$, we conclude that

$$E_1 \le C \int_{6B_i} |f(x)|\, d\mu(x), \tag{52}$$

which, together with E_2, implies (30) and the proof of Theorem 10 is finished. □

Proof of Theorem 11. Without loss of generality, we assume $\zeta = 2$. By a standard argument, it suffices to show that, for any $(\infty, 1)$-atomic block b,

$$\|\mathfrak{M}_\kappa^{*,\rho}(b)\|_{L^1(\mu)} \le C |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}. \tag{53}$$

Assume that $\operatorname{supp} b \subset R$ and $b = \sum_{i=1}^2 v_i a_i$, where, for $i \in \{1,2\}$, a_i is a function supported in $B_i \subset R$ such that $\|a_i\|_{L^\infty(\mu)} \le [\mu(4B_i)]^{-1} K_{B_i,R}^{-1}$ and $|v_1| + |v_2| \sim |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}$. Write

$$\begin{aligned} &\int_{\mathscr{X}} \mathfrak{M}_\kappa^{*,\rho}(b)(x)\, d\mu(x) \\ &\quad = \int_{2R} \mathfrak{M}_\kappa^{*,\rho}(b)(x)\, d\mu(x) \\ &\quad + \int_{\mathscr{X}\setminus 2R} \mathfrak{M}_\kappa^{*,\rho}(b)(x)\, d\mu(x) =: V_1 + V_2. \end{aligned} \tag{54}$$

For V_1, we see that

$$\begin{aligned} V_1 &\le \sum_{i=1}^2 |v_i| \int_{2B_i} \mathfrak{M}_\kappa^{*,\rho}(a_i)(x)\, d\mu(x) \\ &\quad + \sum_{i=1}^2 |v_i| \int_{2R\setminus 2B_i} \mathfrak{M}_\kappa^{*,\rho}(a_i)(x)\, d\mu(x) \\ &=: V_{11} + V_{12}. \end{aligned} \tag{55}$$

Applying the Hölder inequality, $L^2(\mu)$-boundedness of $\mathfrak{M}_\kappa^{*,\rho}$, and the fact that $\|a_i\|_{L^\infty(\mu)} \le C[\mu(4B_i)]^{-1} K_{B_i,R}^{-1}$ for $i \in \{1,2\}$, we have

$$\begin{aligned} &V_{11} \\ &\quad \le \sum_{i=1}^2 |v_i| \left(\int_{2B_i} |\mathfrak{M}_\kappa^{*,\rho}(a_i)(x)|^2\, d\mu(x) \right)^{1/2} \mu(2B_i)^{1/2} \\ &\quad \le C \sum_{i=1}^2 |v_i|\, \|a_i\|_{L^2(\mu)}\, \mu(2B_i)^{1/2} \le C |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}. \end{aligned} \tag{56}$$

Now we estimate V_{12}, with a method similar to that used in the proof of F_1 and $\|a_i\|_{L^\infty(\mu)} \le C[\mu(4B_i)]^{-1} K_{B_i,R}^{-1}$, and we see that

$$\begin{aligned} &V_{12} \\ &\quad \le C \sum_{i=1}^2 |v_i|\, \|a_i\|_{L^1(\mu)} \int_{2R\setminus 2B_i} \frac{1}{\lambda(c_{B_i}, d(x,c_{B_i}))} d\mu(x) \\ &\quad \le C \sum_{i=1}^2 |v_i|\, K_{B_i,R}\, \|a_i\|_{L^\infty(\mu)}\, \mu(B_i) \le C |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}. \end{aligned} \tag{57}$$

Therefore, $V_1 \le C|b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}$.

On the other hand, based on the proof of E_1 and Definition 8, it is easy to obtain that

$$V_2 \leq C \|b\|_{L^1(\mu)} \leq C |b|_{H^{1,\infty}_{\text{atb}}(\mu)}. \quad (58)$$

Combining the estimates for V_1 and V_2, (53) holds. Thus, Theorem 11 is completed. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final paper.

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