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Research Article

Filling Disks of Hayman Type of Meromorphic Functions

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We obtain the existence of the filling disks with respect to Hayman directions. We prove that, under the condition $\limsup_{r\to\infty}(T(r,f)/(\log r)^3) = \infty$, there exists a sequence of filling disks of Hayman type, and these filling disks can determine a Hayman direction. Every meromorphic function of positive and finite order ρ has a sequence of filling disks of Hayman type, which can also determine a Hayman direction of order ρ.

1. Introduction and Results

For a meromorphic function f, disks of the form

$$\Delta\left(z_n, \varepsilon_n |z_n|\right) = \left\{z : |z - z_n| < \varepsilon_n |z_n|\right\}, \tag{1}$$

where $z_n \to \infty$ and $\varepsilon_n \to 0$, are called filling disks of f if f takes every extended complex value with at most two exceptions infinitely often in any infinite subcollection of them, which are also called the filling disks of Julia type. We can prove that if a meromorphic function f satisfies

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = \infty, \tag{2}$$

then f must possess a sequence of filling disks of Julia type, and these filling disks can determine a Julia direction. A direction $\theta \in [0, 2\pi]$ is said to be a Julia direction for a meromorphic function f, if, given $\varepsilon > 0$, f takes all complex values infinitely often in the region $D = \{z : |\arg z - \theta| < \varepsilon\}$ except possibly two exceptions.

For a meromorphic function f of order $0 < \rho < \infty$, disks of the form (1), where $z_n \to \infty$ and $\varepsilon_n \to 0$, are called filling disks of Borel type of f if f takes every extended complex value at least $|z_n|^{\rho-\delta_n}$ times, except some complex values which can be contained in a disk whose spherical radius is $e^{-|z_n|^{\rho-\delta_n}}$, where $\delta_n \to \infty$. A meromorphic function with positive and finite order must possess a sequence of Borel type filling disks, which determine a ρ-order Borel direction, and a ρ-order Borel direction also determines a sequence of Borel type filling disks. A direction $\theta \in [0, 2\pi]$ is said to be a Borel direction for a meromorphic function f of order ρ, if, given $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{\log n\left(r, Z_\varepsilon(\theta), f = a\right)}{\log r} = \rho \tag{3}$$

for all complex values a, at most with two possible exceptions. Here and throughout the paper, $n(r, Z_\varepsilon(\theta), f = a)$ denotes the numbers of the roots of $f = a$ in the region $Z_\varepsilon(\theta) = \{z : |\arg z - \theta| < \varepsilon\}$.

For the case of Hayman direction, we pose a question whether there exist filling disks of Hayman type. In this paper, we mainly obtain the following two theorems.

Theorem 1. *Let $f(z)$ be a meromorphic function in the plane satisfying*

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^3} = \infty. \tag{4}$$

Then, there exists a sequence of filling disks with the form (1), where $z_n \to \infty$ and $\varepsilon_n \to 0$; in each disk, f takes all complex values at least k_n times or else there exists an integer number k

such that $f^{(k)}$ takes all complex values, except possibly zero at least k_n times, where $\limsup_{n\to\infty}(k_n/\log|z_n|)=\infty$. *Moreover, these filling disks can determine a Hayman direction* $\arg z=\theta$, *such that for arbitrary small $\varepsilon>0$, positive integer k, and complex numbers a and $b\neq 0$ we have*

$$\limsup_{r\to\infty}\left(n\left(r,Z_\varepsilon(\theta),f=a\right)+n\left(r,Z_\varepsilon(\theta),f^{(k)}=b\right)\right)=\infty. \tag{5}$$

Theorem 2. *Let $f(z)$ be a meromorphic function in the plane of order $0<\rho<\infty$. Then, there exists a sequence of filling disks with the form (1), where $z_n\to\infty$ and $\varepsilon_n\to 0$; in each disk, f takes all complex values at least $|z_n|^{\rho-\delta_n}$ times or else there exists an integer number k such that $f^{(k)}$ takes all complex values, except possibly zero at least $|z_n|^{\rho-\delta_n}$ times, where $\delta_n\to 0$. Moreover, these filling disks can determine a Hayman direction* $\arg z=\theta$, *such that, for arbitrary small $\varepsilon>0$, positive integer k, and complex numbers a and $b\neq 0$, we have*

$$\limsup_{r\to\infty}\frac{\log\left(n\left(r,Z_\varepsilon(\theta),f=a\right)+n\left(r,Z_\varepsilon(\theta),f^{(k)}=b\right)\right)}{\log r}=\rho. \tag{6}$$

Remark. The filling disks in Theorem 1 are called the filling disks of Hayman-Julia type. Moreover, if we add the growth condition $0<\rho<\infty$ to f, we can obtain the filling disks of Hayman-Borel type in Theorem 2. Hayman-Borel type filling disks are more precise than Hayman-Julia type filling disks.

2. Proof of Theorem 1

First of all, let us recall the definition of Ahlfors-Shimizu characteristic in an angle (see [1]). Let $f(z)$ be a meromorphic function defined in an angle $\Omega=\{z:\alpha\le\arg z\le\beta\}$. Set $\Omega(r)=\Omega\cap\{z:1<|z|<r\}$. Define

$$\mathcal{S}(r,\Omega,f)=\frac{1}{\pi}\int\int_{\Omega(r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma,$$
$$\mathcal{T}(r,\Omega,f)=\int_1^r\frac{\mathcal{S}(t,\Omega,f)}{t}dt. \tag{7}$$

To prove our theorems, we need the following lemma which was first established by Chen and Guo [2] and essentially comes from the Hayman inequality and the estimation of primary values appeared in the inequality and was used to confirm the existence of Hayman T directions of meromorphic functions by Zheng and the first author [3].

Lemma 3. *Let f be meromorphic in $|z|<R$ and let*

$$N=n(R,f=a)+n\left(R,f^{(k)}=b\right) \tag{8}$$

for two complex numbers a and b with $b\neq 0$. Then, we have

$$\mathcal{S}\left(\frac{R}{256},f\right)<C_k\left\{N+\log R+\frac{3}{2}\right\}, \tag{9}$$

where C_k is a positive constant depending only on k.

Lemma 3 can be obtained by using Lemma 8 of Chen and Guo [2] to the function $(bR^k)^{-1}(f(z)-a)$ in the unit disk $\{z:|z|<1\}$ and noticing the following:

$$\mathcal{S}(r,\Omega,f)=\frac{1}{\pi}\iint_{|z|\le\infty}n(r,\Omega,f=z)\,d\omega(z),$$
$$\frac{1}{\pi}\iint_{|z|\le\infty}d\omega(z)=1, \tag{10}$$
$$\frac{1}{\pi}\iint_{|z|\le\infty}\log\frac{1}{|x,z|}d\omega(z)=\frac{1}{2},$$

where $d\omega(z)$ is the area element on the Riemann sphere and $|x,z|$ is the chordal distance between x and z.

The following lemma comes from [4], which is very useful for our study.

Lemma 4. *Let f be meromorphic in the plane, and let η, R, and M be arbitrary positive numbers with $\eta<1$ and $R>2$. If f satisfies*

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^3}=\infty, \tag{11}$$

then there exists a positive number $r>R$ such that

$$\mathcal{S}\left(\Gamma\left((1-\eta)r,(1+\eta)r\right),f\right)>M\log r, \tag{12}$$

where $\Gamma((1-\eta)r,(1+\eta)r)=\{z:(1-\eta)r<|z|<(1+\eta)r\}$.

In view of Lemmas 3 and 4, we can prove Theorem 1.

Proof of Theorem 1. For each $n\in N$, let $M_n>0$, $0<\eta_n<1$, and $R_n>2$. Then, there exists $r_n>R_n$, such that

$$\mathcal{S}\left(\Gamma\left((1-\eta_n)r_n,(1+\eta_n)r_n\right),f\right)>M_n\log r_n. \tag{13}$$

Divide $\Gamma((1-\eta_n)r_n,(1+\eta_n)r_n)$ into K_n domains D_{nj} ($j=1,2,\ldots,K_n$), where

$$D_{nj}=\left\{z:(1-\eta_n)r_n<|z|<(1+\eta_n)r_n,\ \frac{2j\pi}{K_n}\le\arg z\le\frac{2(j+1)\pi}{K_n}\right\}. \tag{14}$$

Then, there exists at least one $j_0=j_0(n)$ ($1\le j_0\le K_n$), such that

$$\mathcal{S}\left(D_{nj_0},f\right)\ge\frac{M_n}{K_n}\log r_n,$$
$$D_{nj_0}\subset B_n'=\left\{z:|z-z_n|<\frac{\pi}{K_n}|z_n|\right\}\subset B_n \tag{15}$$
$$=\left\{z:|z-z_n|<\frac{256\pi}{K_n}|z_n|\right\},$$

where $z_n=r_n\exp(i((2j_0+1)/K_n)\pi)$. In view of

$$\mathcal{S}\left(B_n',f\right)<C_k\left\{n\left(B_n,f=a\right)+n\left(B_n,f^{(k)}=b\right)+\log\frac{256\pi r_n}{K_n}+\frac{3}{2}\right\}, \tag{16}$$

for large enough n, we have

$$\frac{M_n}{K_n} < C_k \frac{n\left(B_n, f = a\right) + n\left(B_n, f^{(k)} = b\right)}{\log r_n}. \tag{17}$$

Here we choose the proper K_n, making $M_n/K_n \to \infty$. The proof is complete. □

Here we point out that, under the current technology, we cannot replace the growth condition (4) with (2). Rossi [4] also obtained that if the growth condition (4) was replaced by (2), then the inequality in Lemma 4 should be replaced by $\mathcal{S}(\Gamma((1-\eta)r, (1+\eta)r), f) > M$. Unfortunately, the inequality of Lemma 3 has the term $\log R$; owing to this term, we cannot replace the growth condition, because we notice that if the growth condition is (2) then the term $\log R$ should be bounded. If we choose $K_n = r_n$, then the term $\log R$ can be bounded but we cannot assure that $M_n/K_n \to \infty$ as $n \to \infty$.

3. Proof of Theorem 2

The following lemma can be proved by the same method of Lemma 4, and we prove it for the completeness.

Lemma 5. *Let f be meromorphic in the plane with order $0 < \rho < \infty$, and let η, R, and M be arbitrary positive numbers with $\eta < 1$ and $R > 2$. Then, for any $\varepsilon > 0$, there exists a positive number $r > R$ such that*

$$\frac{\mathcal{S}\left(\Gamma\left(\left(1-\eta\right)r, \left(1+\eta\right)r\right), f\right)}{r^{\rho-\varepsilon}} > M. \tag{18}$$

Proof. Suppose that the lemma is not true. Then, for every $t > R$,

$$\frac{\mathcal{S}\left(\Gamma\left(\left(1-\eta\right)t, \left(1+\eta\right)t\right), f\right)}{t^{\rho-\varepsilon}} \le M. \tag{19}$$

Choose r larger than $2R$ and set $\beta = (1-\eta)/(1+\eta)$. Then, there exists $\alpha > 0$ such that

$$R < \left(1+\eta\right)\beta^{\alpha} r < 2R. \tag{20}$$

Let $r_0 = r$ and $r_n = \beta r_{n-1}$. Combining (19) with (20), we have

$$\begin{aligned} &\mathcal{S}\left(\Gamma\left(2R, r\right), f\right) \\ &\quad \le \sum_{j=0}^{[\alpha]+1} \mathcal{S}\left(\Gamma\left(\left(1-\eta\right)r_j, \left(1+\eta\right)r_j\right), f\right) \\ &\quad \le M\left([\alpha]+2\right) r^{\rho-\varepsilon}. \end{aligned} \tag{21}$$

It follows from (20) that

$$\alpha \le \frac{\log r + \log\left(1+\eta\right)}{-\log\beta}. \tag{22}$$

Substituting (22) into (21) leads to the order of f being $\rho - \varepsilon$ by the fact that R and η are fixed. □

Proof of Theorem 2. For each $n \in N$, let $M_n > 0$, $0 < \eta_n < 1$, $\varepsilon_n > 0$, and $R_n > 2$. Then, there exists $r_n > R_n$, such that

$$\frac{\mathcal{S}\left(\Gamma\left(\left(1-\eta_n\right)r_n, \left(1+\eta_n\right)r_n\right), f\right)}{r_n^{\rho-\varepsilon_n}} > M_n. \tag{23}$$

Divide $\Gamma((1-\eta_n)r_n, (1+\eta_n)r_n)$ into K_n domains D_{nj} ($j = 1, 2, \ldots, K_n$), where

$$\begin{aligned} D_{nj} = \Bigg\{ z : &\left(1-\eta_n\right)r_n < |z| < \left(1+\eta_n\right)r_n, \frac{2j\pi}{K_n} \le \arg z \\ &\le \frac{2\left(j+1\right)\pi}{K_n} \Bigg\}. \end{aligned} \tag{24}$$

Then, there exists at least one $j_0 = j_0(n)$ ($1 \le j_0 \le K_n$), such that

$$\begin{aligned} &S\left(D_{nj_0}, f\right) \ge \frac{M_n}{K_n} r_n^{\rho-\varepsilon_n}, \\ &D_{nj_0} \subset B_n' = \left\{ z : \left|z - z_n\right| < \frac{\pi}{K_n}\left|z_n\right| \right\} \subset B_n \\ &\quad = \left\{ z : \left|z - z_n\right| < \frac{256\pi}{K_n}\left|z_n\right| \right\}, \end{aligned} \tag{25}$$

where $z_n = r_n \exp(i((2j_0+1)/K_n)\pi)$. In view of

$$\begin{aligned} \mathcal{S}\left(B_n', f\right) < C_k \Bigg\{ &n\left(B_n, f = a\right) + n\left(B_n, f^{(k)} = b\right) \\ &+ \log\frac{256\pi r_n}{K_n} + \frac{3}{2} \Bigg\}, \end{aligned} \tag{26}$$

when n is large enough, we have

$$\frac{1}{C_k}\frac{M_n}{K_n} r_n^{\rho-\varepsilon_n} < n\left(B_n, f = a\right) + n\left(B_n, f^{(k)} = b\right). \tag{27}$$

Choosing $K_n = M_n$, we can obtain the result. Hence, we complete the proof of Theorem 2. It is not difficult to see that these filling disks can determine a Borel direction of order ρ. □

Zhang and Yang [5] also obtained a sequence of filling disks, whose result is as follows: let f be a meromorphic function of positive and finite order ρ. Then, there exists a sequence

$$\begin{aligned} &\Gamma_j : \left|z - z_j\right| < \varepsilon_j \left|z_j\right|, \\ &\lim_{j\to\infty} \varepsilon_j = 0, \\ &\left|z_{j+1}\right| > 2\left|z_j\right|, \\ &\arg z_j = \theta, \\ &\qquad j = 1, 2, 3, \ldots \end{aligned} \tag{28}$$

such that at least one of the following holds:

(1) $f(z)$ in Γ_j takes all complex values at least $|z_j|^{\rho-\varepsilon_j}$ times, with some exceptional values which can be contained in a spherical disk with center ∞ and radius $e^{-|z_j|^{\rho-\varepsilon_j}}$.

(2) For any fixed positive integer k, $f^{(k)}(z)$ in Γ_j takes all complex values at least $|z_j|^{\rho-\varepsilon_j}$ times, with some exceptional values which can be contained in two spherical disks with center ∞ and 0 and radius $e^{-|z_j|^{\rho-\varepsilon_j}}$, where $\lim_{j\to\infty}\varepsilon_j = 0$.

We can see that our result is different from theirs.

4. Filling Disks on an Angular Region

In Yang's book [6], he said that a Borel direction of $0 < \rho < \infty$ can determine a sequence of Borel type filling disks. In this paper, we can see that a ρ order Borel direction can determine not only a sequence of Borel type filling disks but also a sequence of Hayman type filling disks. Actually, if a meromorphic function f on any angular region $\Omega(\alpha,\beta)$ satisfies some growth conditions, it possesses a sequence of filling disks. Zhang [7] obtained the following lemma, which established the growth condition on any angular domain $\Omega(\alpha,\beta)$ containing the Borel direction $\arg z = \theta$.

Lemma 6. *Let f be a meromorphic function in the plane of order $0 < \rho < \infty$. Then, a half line $L : \arg z = \theta$ is a ρ-order Borel direction if and only if it satisfies*

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log \mathcal{T}\left(r, Z_\varepsilon(\theta), f\right)}{\log r} \\ = \limsup_{r\to\infty}\frac{\log \mathcal{S}\left(r, Z_\varepsilon(\theta), f\right)}{\log r} = \rho.\end{aligned} \tag{29}$$

In view of Lemma 6, we have the following result.

Theorem 7. *Let $f(z)$ be a meromorphic function in the plane and let $\arg z = \theta$ be a Borel direction of order ρ. Then, there exists a sequence of disks of the form*

$$C_n = \left\{z : |z - z_n| < \frac{2\delta_n}{K_n}|z_n|\right\}, \tag{30}$$

where $z_n = r_n \exp[i(\theta-\delta_n+((2j_0+1)/K_n)\delta_n)] \to \infty$, $K_n \to \infty$. In C_n, f takes all but possibly two extended complex values with exponent $(M_n/K_n)|z_n|^{\rho-\varepsilon_n}$ and f takes a and $f^{(k)}$ takes b with exponent $(M_n/K_n)|z_n|^{\rho-\varepsilon_n}$, where $M_n \to \infty$ and $\delta_n \to 0$.

Proof. For each $n \in N$, let $M_n > 0$, $0 < \eta_n < 1$, $\varepsilon_n > 0$, and $R_n > 2$. It follows from (29) and Lemma 5 that there exists $r_n > R_n$, such that

$$\frac{\mathcal{S}\left(\Gamma\left((1-\eta_n)r_n, (1+\eta_n)r_n\right), Z_{\delta_n}(\theta), f\right)}{r_n^{\rho-\varepsilon_n}} > M_n. \tag{31}$$

Divide $\Gamma((1-\eta_n)r_n, (1+\eta_n)r_n, Z_{\delta_n}(\theta))$ into K_n domains D_{nj} $(j = 0, 2, \ldots, K_n - 1)$, where

$$\begin{aligned}D_{nj} = \Big\{z : (1-\eta_n)r_n < |z| < (1+\eta_n)r_n, \theta - \delta_n \\ + \frac{2j\delta_n}{K_n} \le \arg z \le \theta - \delta_n + \frac{2(j+1)\delta_n}{K_n}\Big\}.\end{aligned} \tag{32}$$

Then, there exists at least one $j_0 = j_0(n)$ $(1 \le j_0 \le K_n)$, such that

$$\begin{aligned}S\left(D_{nj_0}, f\right) \ge \frac{M_n}{K_n} r_n^{\rho-\varepsilon_n}, \\ D_{nj_0} \subset B_n' = \left\{z : |z - z_n| < \frac{\delta_n}{K_n}|z_n|\right\} \subset B_n \\ = \left\{z : |z - z_n| < \frac{256\delta_n}{K_n}|z_n|\right\},\end{aligned} \tag{33}$$

where $z_n = r_n \exp[i(\theta - \delta_n + ((2j_0+1)/K_n)\delta_n)]$. In view of

$$\begin{aligned}\mathcal{S}\left(B_n', f\right) < C_k \Big\{n\left(B_n, f = a\right) + n\left(B_n, f^{(k)} = b\right) \\ + \log\frac{256\pi r_n}{K_n} + \frac{3}{2}\Big\},\end{aligned} \tag{34}$$

when n is large enough, we have

$$\frac{1}{C_k}\frac{M_n}{K_n} r_n^{\rho-\varepsilon_n} < n\left(B_n, f = a\right) + n\left(B_n, f^{(k)} = b\right). \tag{35}$$

Choosing $K_n = M_n$, we can obtain the result. □

At last, we pose a question:

Does a Hayman direction of order $0 < \rho < \infty$ have a sequence of filling disks of Hayman type?

Competing Interests

The authors declare that they have no competing interests.

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