Hindawi
Journal of Function Spaces
Volume 2017, Article ID 1409642, 14 pages
https://doi.org/10.1155/2017/1409642

*Research Article*

# Essential Norms of Volterra Type Operators between *Zygmund* Type Spaces

**Shanli Ye[1] and Caishu Lin[2]**

[1] *School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China*
[2] *Department of Mathematics, Fujian Normal University, Fuzhou 350007, China*

Correspondence should be addressed to Shanli Ye; ye_shanli@aliyun.com

Received 5 January 2017; Revised 4 March 2017; Accepted 7 March 2017; Published 11 April 2017

Academic Editor: Ruhan Zhao



We investigate the boundedness of some Volterra type operators between *Zygmund* type spaces. Then, we give the essential norms of such operators in terms of $g, \varphi$, their derivatives, and the $n$th power $\varphi^n$ of $\varphi$.

## 1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$ and let $\partial D = \{z : |z| = 1\}$ be its boundary, and $H(D)$ denote the set of all analytic functions on $D$.

For every $0 < \alpha < \infty$, we denote by $\mathscr{B}^\alpha$ the Bloch type space of all functions $f \in H(D)$ satisfying

$$b^\alpha(f) = \sup_{z\in D}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty \tag{1}$$

endowed with the norm $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + b^\alpha(f)$. The little Bloch type space $\mathscr{B}_0^\alpha$ consists of all $f \in \mathscr{B}^\alpha$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| = 0$, and $\mathscr{B}_0^\alpha$ is obviously the closed subspace of $\mathscr{B}^\alpha$. When $\alpha = 1$, we get the classical Bloch space $\mathscr{B}^1 = \mathscr{B}$ and little Bloch space $\mathscr{B}_0^1 = \mathscr{B}_0$. It is well known that, for $0 < \alpha < 1$, $\mathscr{B}^\alpha$ is a subspace of $H^\infty$, the Banach space of bounded analytic functions on $D$. Some sources for results and references about the Bloch type functions are the papers of Yoneda [1], Stevic [2, 3], and the first author [4–7].

For $0 < \alpha < \infty$ we denote by $\mathscr{Z}^\alpha$ the *Zygmund type space* of those functions $f \in H(D)$ satisfying

$$\sup_{z\in D}\left(1-|z|^2\right)^\alpha \left|f''(z)\right| < \infty, \tag{2}$$

and the little Zygmund type space $\mathscr{Z}_0^\alpha$ consists of all $f \in \mathscr{Z}^\alpha$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f''(z)| = 0$, and $\mathscr{Z}_0^\alpha$ is obviously the closed subspace of $\mathscr{Z}^\alpha$. It can easily be proved that $\mathscr{Z}^\alpha$ is a Banach space under the norm

$$\|f\|_{\mathscr{Z}^\alpha} = |f(0)| + \left|f'(0)\right| + \sup_{z\in D}\left(1-|z|^2\right)^\alpha \left|f''(z)\right| \tag{3}$$

and that $\mathscr{Z}_0^\alpha$ is a closed subspace of $\mathscr{Z}^\alpha$. When $\alpha = 1$, we get the classical Zygmund space $\mathscr{Z}^1 = \mathscr{Z}$ and the little Zygmund space $\mathscr{Z}_0^1 = \mathscr{Z}_0$. It is clear that $f \in \mathscr{Z}$ if and only if $f' \in \mathscr{B}^1$.

We consider the weighted Banach spaces of analytic functions

$$H_v^\infty = \left\{f \in H(D) : \|F\|_v = \sup_{z\in D} v(z)\left|f(z)\right| < \infty\right\} \tag{4}$$

endowed with norm $\|\cdot\|_v$, where the weight $v : D \to R_+$ is a continuous, strictly positive, and bounded function. The weight $v$ is called radial, if $v(z) = v(|z|)$ for all $z \in D$. For a weight $v$ the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) = \left(\sup\left\{\left|f(z)\right| : f \in H_v^\infty, \|f\|_v \le 1\right\}\right)^{-1}, \quad z \in D. \tag{5}$$

We notice the standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, where $0 < \alpha < \infty$, and it is well known that $\widetilde{v}_\alpha = v_\alpha$. We also consider the logarithmic weight

$$v_{\log} = \left(\log \frac{2}{1-|z|^2}\right)^{-1}, \quad z \in D. \tag{6}$$

It is straightforward to show that $\widetilde{v}_{\log} = v_{\log}$.

For an analytic self-map $\varphi$ of $D$ and a function $u \in H(D)$, we define the weighted composition operator as $uC_\varphi f = u \cdot (f \circ \varphi)$ for $f \in H(D)$. Weighted composition operators have been extensively studied recently. It is interesting to provide a function theoretic characterization when $\varphi$ and $u$ induce a bounded or compact composition operator on various function spaces. Some results on the boundedness and compactness of concrete operators between some spaces of analytic functions one of which is of Zygmund type can be found, for example, in [8–19].

Suppose that $g : D \to \mathbb{C}$ is an analytic map. Let $T_g$ and $I_g$ denote the Volterra type operators with the analytic symbol $g$ on $D$, respectively:

$$\begin{aligned} T_g f(z) &= \int_0^z f(\xi)\, g'(\xi)\, d\xi, \quad z \in D, \\ I_g f(z) &= \int_0^z f'(\xi)\, g(\xi)\, d\xi, \quad z \in D. \end{aligned} \tag{7}$$

If $g(z) = z$, then $T_g$ is an integral operator. While $g(z) = \ln(1/(1-z))$, then $T_g$ is Cesàro operator. Pommerenke introduced the *Volterra* type operator $T_g$ and characterized the boundedness of $T_g$ between $H^2$ spaces in [20]. More recently, boundedness and compactness of *Volterra* type operators between several spaces of analytic functions have been studied by many authors; one may see [21, 22].

In this paper, we consider the following integral type operators, which were introduced by Li and Stevic (see, e.g., [10, 23]); they can be defined by

$$\begin{aligned} \left(C_\varphi T_g f\right)(z) &= \int_0^{\varphi(z)} f(\xi)\, g'(\xi)\, d\xi, \\ \left(C_\varphi I_g f\right)(z) &= \int_0^{\varphi(z)} f'(\xi)\, g(\xi)\, d\xi, \\ \left(T_g C_\varphi f\right)(z) &= \int_0^{z} f(\varphi(\xi))\, g'(\xi)\, d\xi, \\ \left(I_g C_\varphi f\right)(z) &= \int_0^{z} f'(\varphi(\xi))\, g(\xi)\, d\xi. \end{aligned} \tag{8}$$

We will characterize the boundedness of those integral type operators between Zygmund type spaces and also estimate their essential norms. The boundedness and compactness of these operators on the logarithmic Bloch space have been characterized in [22].

Recall that essential norm $\|T\|_{e,X\to Y}$ of a bounded linear operator $T : X \to Y$ is defined as the distance from $T$ to $\mathscr{K}(X,Y)$, the space of compact operators from $X$ to $Y$, namely,

$$\begin{aligned} &\|T\|_{e,X\to Y} \\ &\quad = \inf\left\{\|T+K\|_{X\to Y} : K : X \to Y \text{ is compact}\right\}. \end{aligned} \tag{9}$$

It provides a measure of noncompactness of $T$. Clearly, $T$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there are positive constants $C_1, C_2$ such that $C_1 a \le b \le C_2 a$.

## 2. Boundedness

In order to prove the main results of this paper. We need some auxiliary results.

**Lemma 1** (see [8, 13]). *For $0 < \alpha < 2$ and let $\{f_n\}$ be a bounded sequence in $\mathscr{Z}^\alpha$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n\to\infty}\sup_{z\in D}|f_n(z)| = 0$.*

**Lemma 2** (see [8, 13]). *For every $f \in \mathscr{Z}^\alpha$, where $\alpha > 0$, one has*

(i) $|f'(z)| \le (2/(1-\alpha))\|f\|_{\mathscr{Z}^\alpha}$ *and* $|f(z)| \le (2/(1-\alpha))\|f\|_{\mathscr{Z}^\alpha}$ *for every* $0 < \alpha < 1$;

(ii) $|f'(z)| \le 2\log(2/(1-|z|))\|f\|_{\mathscr{Z}^\alpha}$ *and* $|f(z)| \le \|f\|_{\mathscr{Z}^\alpha}$ *for* $\alpha = 1$;

(iii) $|f'(z)| \le (2/(1-\alpha))(\|f\|_{\mathscr{Z}^\alpha}/(1-|z|)^{\alpha-1})$, *for every* $\alpha > 1$;

(iv) $|f(z)| \le (2/(\alpha-1)(2-\alpha))\|f\|_{\mathscr{Z}^\alpha}$, *for every* $1 < \alpha < 2$;

(v) $|f(z)| \le 2\log(2/(1-|z|))\|f\|_{\mathscr{Z}^\alpha}$, *for every* $\alpha = 2$;

(vi) $|f(z)| \le (2/(\alpha-1)(\alpha-2))(\|f\|_{\mathscr{Z}^\alpha}/(1-|z|)^{\alpha-2})$, *for every* $\alpha > 2$.

**Lemma 3** (see [8]). *Let $0 < \alpha < \infty$ and $v$ a radial, nonincreasing weight tending to 0 at boundary of $D$, and let the weighted composition operator $uC_\varphi : \mathscr{Z}_\alpha \to H_v^\infty$ be bounded.*

(i) *If $0 < \alpha < 2$, then $uC_\varphi$ is a compact operator.*

(ii)

$$\begin{aligned} \left\|uC_\varphi\right\|_{e,\mathscr{Z}_2\to H_v^\infty} &\asymp \limsup_{n\to\infty} (\log n)\, \|u\varphi^n\|_v \\ &\asymp \limsup_{|\varphi(z)|\to 1} v(z)\,|u(z)| \log \frac{2}{1-|\varphi(z)|^2}. \end{aligned} \tag{10}$$

(iii) *If $\alpha > 2$, then*

$$\begin{aligned} \left\|uC_\varphi\right\|_{e,\mathscr{Z}_\alpha\to H_v^\infty} &\asymp \limsup_{n\to\infty} (n+1)^{\alpha-2}\, \|u\varphi^n\|_v \\ &\asymp \limsup_{|\varphi(z)|\to 1} \frac{v(z)\,|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-2}}. \end{aligned} \tag{11}$$

The following lemma is due to [24, 25].

**Lemma 4.** *Let $v$ and $w$ be radial, nonincreasing weights tending to zero at the boundary of $D$. Then*

(i) *The weighted composition operator $uC_\varphi$ maps $H_v^\infty$ into $H_w^\infty$ if and only if*

$$\sup_{n\geq 0}\frac{\|u\varphi^n\|_w}{\|z^n\|_v}\asymp\sup_{z\in D}\frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|<\infty, \tag{12}$$

*with norm comparable to the above supermum.*

(ii)

$$\begin{aligned}\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty}&=\limsup_{n\to\infty}\frac{\|u\varphi^n\|_w}{\|z^n\|_v}\\&=\limsup_{|\varphi(z)|\to 1}\frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.\end{aligned} \tag{13}$$

**Lemma 5** (see [26]). *For every $0<\alpha<\infty$, one has*

$$\lim_{n\to\infty}(n+1)^\alpha\|z^n\|_{v_\alpha}=\left(\frac{2\alpha}{e}\right)^\alpha, \tag{14}$$

$$\lim_{n\to\infty}(\log n)\|z^n\|_{v_{\log}}=1. \tag{15}$$

**Theorem 6.** *Let $\varphi$ be an analytic self-map of $D$ and $g\in H(D)$.*

(i) *If $0<\alpha<1$, then $I_gC_\varphi:\mathcal{Z}^\alpha\to\mathcal{Z}^\beta$ is a bounded operator if and only if $g'\in H_{v_\beta}^\infty$ and*

$$\begin{aligned}&\sup_{n\geq 0}(n+1)^\alpha\|g(\varphi')\varphi^n\|_{v_\beta}\\&\quad\asymp\sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|g(z)\varphi'(z)|<\infty.\end{aligned} \tag{16}$$

(ii) *If $\alpha=1$, then $I_gC_\varphi:\mathcal{Z}^\alpha\to\mathcal{Z}^\beta$ is a bounded operator if and only if*

$$\begin{aligned}&\sup_{n\geq 0}(n+1)\|g(\varphi')\varphi^n\|_{v_\beta}\\&\quad\asymp\sup_{z\in D}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|g(z)\varphi'(z)|<\infty,\end{aligned} \tag{17}$$

$$\begin{aligned}&\sup_{n\geq 0}(\log n)\|g'\varphi^n\|_{v_\beta}\\&\quad\asymp\sup_{z\in D}(1-|z|^2)^\beta\log\left(\frac{2}{1-|\varphi(z)|^2}\right)|g'(z)|<\infty.\end{aligned} \tag{18}$$

(iii) *If $\alpha>1$, then $I_gC_\varphi:\mathcal{Z}^\alpha\to\mathcal{Z}^\beta$ is a bounded operator if and only if*

$$\begin{aligned}&\sup_{n\geq 0}(n+1)^\alpha\|g(\varphi')\varphi^n\|_{v_\beta}\\&\quad\asymp\sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|g(z)\varphi'(z)|<\infty,\end{aligned} \tag{19}$$

$$\begin{aligned}&\sup_{n\geq 0}(n+1)^{\alpha-1}\|g'\varphi^n\|_{v_\beta}\\&\quad\asymp\sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|g'(z)|<\infty.\end{aligned} \tag{20}$$

*Proof.* Suppose that $I_gC_\varphi$ is bounded from $\mathcal{Z}^\alpha$ to $\mathcal{Z}^\beta$. Using the test functions $f(z)=z$ and $f(z)=z^2$, we have

$$(1-|z|^2)^\beta|(I_gC_\varphi z)''|=(1-|z|^2)^\beta|g'(z)|<\infty, \tag{21}$$

$$\begin{aligned}&(1-|z|^2)^\beta|(I_gC_\varphi z^2)''|\\&\quad=(1-|z|^2)^\beta|2\varphi'(z)g(z)+2\varphi(z)g'(z)|<\infty.\end{aligned} \tag{22}$$

Since $\varphi$ is a self-map, we get that $g'\in H_{v_\beta}^\infty$, $\varphi'g\in H_{v_\beta}^\infty$.

For every $0<\alpha<\infty$ and given nonzero $a\in D$, we take the test functions

$$f_a(z)=\frac{1}{\overline{a}^2}\left[\frac{(1-|a|^2)^2}{(1-\overline{a}z)^\alpha}-\frac{1-|a|^2}{(1-\overline{a}z)^{\alpha-1}}\right], \tag{23}$$

$$h_a(z)=\frac{1}{\overline{a}}\int_0^z\frac{1-|a|^2}{(1-\overline{a}w)^\alpha}dw, \tag{24}$$

$$g_a(z)=f_a(z)-h_a(z), \tag{25}$$

for every $z\in D$. One can show that $f_a$, $h_a$, and $g_a$ are in $\mathcal{Z}_0^\alpha$, $\sup_{1/2<a<1}\|f_a\|_{\mathcal{Z}^\alpha}<\infty$, and $\sup_{1/2<a<1}\|h_a\|_{\mathcal{Z}^\alpha}<\infty$. Since $g_a'(a)=0$, $g_a''(a)=\alpha/(1-|a|^2)^\alpha$, it follows that for all $z\in D$ with $|\varphi(z)|>1/2$, we have

$$\begin{aligned}+\infty>C\|g_a\|_{\mathcal{Z}^\alpha}&\geq\|I_gC_\varphi(g_{\varphi(z)})\|_{\mathcal{Z}^\beta}\\&\geq(1-|z|^2)^\beta|\varphi'(z)g(z)||g''_{\varphi(z)}(\varphi(z))|\\&\quad-(1-|z|^2)^\beta|g'(z)||g'_{\varphi(z)}(\varphi(z))|\\&=(1-|z|^2)^\beta|\varphi'(z)g(z)|\left|\frac{\alpha}{(1-|\varphi(z)|^2)^\alpha}\right|.\end{aligned} \tag{26}$$

Then

$$\begin{aligned}
&\sup_{z\in D}\left|\varphi'(z)\,g(z)\right|\frac{\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\\
&\quad\le \sup_{|\varphi(z)|\le 1/2}\frac{\left|\varphi'(z)\,g(z)\right|\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\\
&\qquad+\sup_{|\varphi(z)|>1/2}\frac{\left|\varphi'(z)\,g(z)\right|\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\\
&\quad\le\left(\frac{4}{3}\right)^{\alpha}\left\|\varphi' g\right\|_{v_\beta}+C\left\|g_a\right\|_{\mathcal{Z}^\alpha}<\infty.
\end{aligned}\tag{27}$$

Now we use (14) and Lemma 4 to conclude that

$$\begin{aligned}
&\sup_{n\ge 0}(n+1)^{\alpha}\left\|g\left(\varphi'\right)\varphi^{n}\right\|_{v_\beta}\\
&\quad\asymp\sup_{z\in D}\frac{\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\left|g(z)\,\varphi'(z)\right|<\infty,
\end{aligned}\tag{28}$$

which shows that (16) is necessary for all case.

Conversely, suppose that $g'\in H^{\infty}_{v_\beta}$ and (16) holds. Assume that $f\in\mathcal{Z}^{\alpha}$. From Lemma 2, it follows that

$$\begin{aligned}
&\left(1-|z|^2\right)^{\beta}\left|\left(I_gC_\varphi f\right)''(z)\right|=\left(1-|z|^2\right)^{\beta}\\
&\quad\cdot\left|\varphi'(z)\,g(z)\,f''(\varphi(z))+g'(z)\,f'(\varphi(z))\right|\\
&\quad\le\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\,f''(\varphi(z))\right|+\left(1-|z|^2\right)^{\beta}\\
&\quad\cdot\left|g'(z)\,f'(\varphi(z))\right|\le\frac{\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\left|\varphi'(z)\,g(z)\right|\\
&\quad\cdot\|f\|_{\mathcal{Z}^\alpha}+C\left(1-|z|^2\right)^{\beta}\left|g'(z)\right|\|f\|_{\mathcal{Z}^\alpha}\le C\|f\|_{\mathcal{Z}^\alpha},\\
&\left|I_gC_\varphi(f)(0)\right|=0,\\
&\left|\left(I_gC_\varphi f\right)'(0)\right|=\left|f'(\varphi(0))\,g(0)\right|\le\|f\|_{\mathcal{Z}^\alpha}\\
&\quad\cdot\left|g(\varphi(0))\right|,
\end{aligned}\tag{29}$$

which implies that $I_gC_\varphi$ is bounded. This completes the proof of (i).

Next we will prove (ii). The necessity in condition (17) has been proved above. Fixing $a\in D$ with $|a|>1/2$, we take the function

$$k_a(z)=\frac{p(\bar{a}z)}{\bar{a}}\left(\log\frac{1}{1-|a|}\right)^{-1},\tag{30}$$

for $z\in D$, where

$$p(z)=(z-1)\left(\left(1+\log\frac{1}{1-z}\right)^{2}+1\right).\tag{31}$$

Then we have $\sup_{1/2<|a|<1}\|k_a\|_{\mathcal{Z}^\alpha}\le C$ by [11]. Let $a=\varphi(z)$. It follows that

$$\begin{aligned}
&\left\|I_gC_\varphi\left(k_{\varphi(z)}\right)\right\|_{\mathcal{Z}^\beta}\\
&\quad\ge\left(1-|z|^2\right)^{\beta}\left|g'(z)\right|\left|k'_{\varphi(z)}(\varphi(z))\right|\\
&\qquad-\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|k''_{\varphi(z)}(\varphi(z))\right|\\
&\quad=\left(1-|z|^2\right)^{\beta}\left|g'(z)\right|\log\frac{1}{1-|\varphi(z)|^2}\\
&\qquad-\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|\frac{\alpha}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\right|.
\end{aligned}\tag{32}$$

Since (17) holds and $I_gC_\varphi$ is bounded, we obtain that

$$\begin{aligned}
&\sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta}\log\frac{1}{1-|\varphi(z)|^2}\left|g'(z)\right|\\
&\quad\le\sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|\frac{\alpha}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\right|\\
&\qquad+\sup_{|\varphi(z)|>1/2}\left\|I_gC_\varphi\left(k_{\varphi(z)}\right)\right\|_{\mathcal{Z}^\beta}<\infty.
\end{aligned}\tag{33}$$

Noting $g'\in H^{\infty}_{v_\beta}$ and together with (15) and Lemma 4, we conclude that (18) holds.

The converse implication can be shown as in the proof of (i).

Finally we will prove (iii). We have proved that (19) holds above. To prove (20), we take function $f_{\varphi(z)}$ defined in (23) for every $z\in D$ with $|\varphi(z)|>1/2$ and obtain that

$$\begin{aligned}
&\left\|I_gC_\varphi\left(f_{\varphi(z)}\right)\right\|_{\mathcal{Z}^\beta}\\
&\quad\ge\left(1-|z|^2\right)^{\beta}\left|g'(z)\right|\left|f'_{\varphi(z)}(\varphi(z))\right|\\
&\qquad-\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|f''_{\varphi(z)}(\varphi(z))\right|\\
&\quad=\left(1-|z|^2\right)^{\beta}\left|g'(z)\right|\frac{1}{\left|\overline{\varphi(z)}\right|\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\\
&\qquad-\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|\frac{2\varphi(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\right|.
\end{aligned}\tag{34}$$

Since $I_gC_\varphi$ is bounded and (19) holds, we obtain that

$$\begin{aligned}
&\sup_{|\varphi(z)|>1/2}\frac{\left(1-|z|^2\right)^{\beta}}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\left|g'(z)\right|\\
&\quad\le 2\sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)\,g(z)\right|\left|\frac{2}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\right|\\
&\qquad+2\sup_{|\varphi(z)|>1/2}\left\|I_gC_\varphi\left(f_{\varphi(z)}\right)\right\|_{\mathcal{Z}^\beta}<\infty;
\end{aligned}\tag{35}$$

therefore, we deduce that (20) holds by (14) and Lemma 4.

The converse implication can be shown as in the proof of (i). □

**Theorem 7.** *Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$.*

(i) *If $0 < \alpha < 1$, then $C_\varphi I_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $((g\circ\varphi)(\varphi'')+(g'\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$ and*

$$\sup_{n\geq 0}(n+1)^\alpha \left\|(g\circ\varphi)(\varphi')^2\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\left|g(\varphi(z))(\varphi'(z))^2\right| < \infty. \tag{36}$$

(ii) *If $\alpha = 1$, then $C_\varphi I_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if*

$$\begin{aligned}\sup_{n\geq 0}(n+1)\left\|(g\circ\varphi)(\varphi')^2\varphi^n\right\|_{v_\beta} &\asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\\ &\cdot\left|g(\varphi(z))(\varphi'(z))^2\right| < \infty,\\ \sup_{n\geq 0}(\log n)\left\|\left((g'\circ\varphi)(\varphi')^2+(g\circ\varphi)(\varphi'')\right)\varphi^n\right\|_{v_\beta} &\\ \asymp \sup_{z\in D}(1-|z|^2)^\beta\log\frac{2}{1-|\varphi(z)|^2}&\\ \cdot\left|g'(\varphi(z))(\varphi'(z))^2+g(\varphi(z))\varphi''(z)\right| &< \infty.\end{aligned} \tag{37}$$

(iii) *If $\alpha > 1$, then $C_\varphi I_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if (36) holds and*

$$\begin{aligned}\sup_{n\geq 0}(n+1)^{\alpha-1}\left\|\left((g'\circ\varphi)(\varphi')^2+(g\circ\varphi)(\varphi'')\right)\varphi^n\right\|_{v_\beta} &\\ \asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}&\\ \cdot\left|g'(\varphi(z))(\varphi'(z))^2+g(\varphi(z))\varphi''(z)\right| &< \infty.\end{aligned} \tag{38}$$

The proof is similar to that of Theorem 6, and the details are omitted.

**Theorem 8.** *Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$.*

(i) *If $0 < \alpha < 1$, then $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$ and $(g'\circ\varphi)(\varphi')^2 \in H^\infty_{v_\beta}$.*

(ii) *If $\alpha = 1$, then $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$ and*

$$\begin{aligned}\sup_{n\geq 0}(\log n)\left\|(g'\circ\varphi)(\varphi')^2\varphi^n\right\|_{v_\beta} &\asymp \sup_{z\in D}(1-|z|^2)^\beta\\ \cdot\log\left(\frac{2}{1-|\varphi(z)|^2}\right)&\left|g'(\varphi(z))(\varphi'(z))^2\right| < \infty;\end{aligned} \tag{39}$$

(iii) *If $1 < \alpha < 2$, then $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$ and*

$$\sup_{n\geq 0}(n+1)^{\alpha-1}\left\|\left((g'\circ\varphi)(\varphi')^2\right)\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}\left|g'(\varphi(z))(\varphi'(z))^2\right| < \infty. \tag{40}$$

(iv) *If $\alpha = 2$, then $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if*

$$\begin{aligned}\sup_{n\geq 0}(n+1)\left\|\left((g'\circ\varphi)(\varphi')^2\right)\varphi^n\right\|_{v_\beta} &\\ \asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)}&\left|g'(\varphi(z))(\varphi'(z))^2\right| < \infty,\end{aligned} \tag{41}$$

$$\begin{aligned}\sup_{n\geq 0}(\log n)\left\|\left((g''\circ\varphi)(\varphi')^2+(g'\circ\varphi)\varphi''\right)\varphi^n\right\|_{v_\beta} &\\ \asymp \sup_{z\in D}(1-|z|^2)^\beta\log\left(\frac{2}{(1-|\varphi(z)|^2)}\right)&\\ \cdot\left|g''(\varphi(z))(\varphi'(z))^2+g'(\varphi(z))\varphi''(z)\right| &< \infty.\end{aligned} \tag{42}$$

(v) *If $\alpha > 2$, then $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if (40) holds and*

$$\begin{aligned}\sup_{n\geq 0}(n+1)^{\alpha-2}\left\|\left((g''\circ\varphi)(\varphi')^2+(g'\circ\varphi)\varphi''\right)\varphi^n\right\|_{v_\beta} &\\ \asymp \sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-2}}&\left|g''(\varphi(z))(\varphi'(z))^2\right.\\ \left.+g'(\varphi(z))\varphi''(z)\right| &< \infty.\end{aligned} \tag{43}$$

*Proof.* Suppose that $C_\varphi T_g$ is bounded from $\mathcal{Z}^\alpha$ to $\mathcal{Z}^\beta$ space.

*(i) Case* $0 < \alpha < 1$. Using functions $f = 1 \in \mathcal{Z}^\alpha$ and $f = z \in \mathcal{Z}^\alpha$, we obtain

$$\begin{aligned} &\sup_{z\in D}\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|g''(\varphi(z))\left(\varphi'(z)\right)^2+g'(\varphi(z))\varphi''(z)\right|<\infty, \\ &\sup_{z\in D}\left(1-|z|^2\right)^\beta\left|g'(\varphi(z))\left(\varphi'(z)\right)^2\right|<\infty. \end{aligned} \tag{44}$$

Then we obtain that $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2)\in H^\infty_{v_\beta}$ and $(g'\circ\varphi)(\varphi')^2\in H^\infty_{v_\beta}$ are necessary for all case.

For the converse implication, suppose that $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2)\in H^\infty_{v_\beta}$ and $(g'\circ\varphi)(\varphi')^2\in H^\infty_{v_\beta}$. For $f\in\mathcal{Z}^\alpha$, it follows from Lemma 2 that

$$\begin{aligned} &\left(1-|z|^2\right)^\beta\left|\left(C_\varphi T_g f\right)''(z)\right| = \left(1-|z|^2\right)^\beta\left|\left(\varphi'(z)\right)^2\right. \\ &\quad\cdot g'(\varphi(z))f'(\varphi(z)) \\ &\quad+\left(g''(\varphi(z))\left(\varphi'(z)\right)^2+g'(\varphi(z))\varphi''(z)\right) \\ &\quad\cdot f(\varphi(z))\Big| \le \left(1-|z|^2\right)^\beta\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right. \\ &\quad\cdot f'(\varphi(z))\Big| + \left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|\left(g''(\varphi(z))\left(\varphi'(z)\right)^2+g'(\varphi(z))\varphi''(z)\right)\right. \\ &\quad\cdot f(\varphi(z))\Big| \le \left(1-|z|^2\right)^\beta\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right| \\ &\quad\cdot\|f\|_{\mathcal{Z}^\alpha}+\left(1-|z|^2\right)^\beta\left|g''(\varphi(z))\left(\varphi'(z)\right)^2\right. \\ &\quad+g'(\varphi(z))\varphi''(z)\Big|\,\|f\|_{\mathcal{Z}^\alpha}\le C\|f\|_{\mathcal{Z}^\alpha}, \\ &\left|C_\varphi T_g(f)(0)\right| = \left|\int_0^{\varphi(0)} f(\zeta)g'(\zeta)\,d\zeta\right| \\ &\quad\le \max_{|\zeta|\le|\varphi(0)|}|f(\zeta)|\max_{|\zeta|\le|\varphi(0)|}\left|g'(\zeta)\right|\le\frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha} \\ &\quad\cdot\max_{|\zeta|\le|\varphi(0)|}\left|g'(\zeta)\right|, \\ &\left|\left(C_\varphi T_g f\right)'(0)\right| = \left|f(\varphi(0))\varphi'(0)g'(\varphi(0))\right|\le\|f\|_{\mathcal{Z}^\alpha} \\ &\quad\cdot\left|\varphi'(0)\right|\left|g'(\varphi(0))\right|. \end{aligned} \tag{45}$$

Then $C_\varphi T_g$ is bounded. This completes the proof of (i).

*(ii) Case* $\alpha = 1$. We consider the test function $k_{\varphi(z)}(z)$ defined in (30) for every $z\in D$ with $|\varphi(z)|>1/2$. It follows that

$$\begin{aligned} &\left\|C_\varphi T_g k_{\varphi(z)}\right\|_{\mathcal{Z}^\beta}\ge\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right|-\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|\varphi''(z)g'(\varphi(z))+g''(\varphi(z))\left(\varphi(z)\right)^2\right| \\ &\quad\cdot\left|k_{\varphi(z)}(\varphi(z))\right|. \end{aligned} \tag{46}$$

Since $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2)\in H^\infty_{v_\beta}$ and $\sup_{|\varphi(z)|>1/2}\|C_\varphi T_g k_{\varphi(z)}\|_{\mathcal{Z}^\beta}\le C$, we get

$$\begin{aligned} &\sup_{|\varphi(z)|>1/2}\log\frac{1}{1-|\varphi(z)|^2}\left(1-|z|^2\right)^\beta\left|\left(\varphi'(z)\right)^2\right. \\ &\quad\cdot g'(\varphi(z))\Big|\le\sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|\varphi''(z)g'(\varphi(z))+g''(\varphi(z))\left(\varphi(z)\right)^2\right| \\ &\quad\cdot\left|k_{\varphi(z)}(\varphi(z))\right|+\left\|C_\varphi T_g k_{\varphi(z)}\right\|_{\mathcal{Z}^\beta}<\infty. \end{aligned} \tag{47}$$

Then we have

$$\begin{aligned} &\sup_{z\in D}\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right| \\ &\quad\le\sup_{|\varphi(z)|\le1/2}\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\left(1-|z|^2\right)^\beta\left|\left(\varphi'(z)\right)^2\right. \\ &\quad\cdot g'(\varphi(z))\Big|+\sup_{|\varphi(z)|>1/2}\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\Big(1 \\ &\quad-|z|^2\Big)^\beta\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right|\le C+\log\left(\frac{4}{3}\right)\Big\|\Big(g' \\ &\quad\circ\varphi\Big)\left(\varphi'\right)^2\Big\|_{v_\beta}<\infty. \end{aligned} \tag{48}$$

On the other hand, from (15) and Lemma 4, we have

$$\begin{aligned} &\sup_{n\ge0}(\log n)\left\|\left(g'\circ\varphi\right)\left(\varphi'\right)^2\varphi^n\right\|_{v_\beta}\asymp\sup_{z\in D}\left(1-|z|^2\right)^\beta \\ &\quad\cdot\log\left(\frac{2}{1-|\varphi(z)|^2}\right)\left|g'(\varphi(z))\left(\varphi'(z)\right)^2\right|. \end{aligned} \tag{49}$$

Hence (39) holds.

The converse implication can be shown as in the proof of (i).

*(iii) Case* $1<\alpha<2$. $((g'\circ\varphi)(\varphi'')+(g''\circ\varphi)(\varphi')^2)\in H^\infty_{v_\beta}$ has been proved above. We take the test function $f_{\varphi(z)}$ in (23) for

every $z \in D$ with $|\varphi(z)| > 1/2$; by the same way as (ii), we can obtain that (40) holds.

The converse implication can be shown as in the proof of (i).

*(iv) Case* $\alpha = 2$. We have proved that (41) holds above. To prove (42), we consider another test function $t_a(z) = \log(2/(1-\overline{a}z))$. Clearly $t_a \in \mathcal{Z}^2$ and $\sup_{1/2<|a|<1}\|t_a\|_{\mathcal{Z}^2} < \infty$. For every $z \in D$ with $|\varphi(z)| > 1/2$, it follows that

$$\begin{aligned}
&\sup_{z\in D}\left(1-|z|^2\right)^{\beta}\left|\left(C_\varphi T_g t_a\right)''(z)\right| \\
&\quad\geq \sup_{|\varphi(z)|>1/2}\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\left(1-|z|^2\right)^{\beta} \\
&\quad\cdot\left|\left(\varphi'(z)\right)^2 g''(\varphi(z)) + g'(\varphi(z))\varphi''(z)\right| \\
&\quad- \sup_{|\varphi(z)|>1/2}\frac{\left(1-|z|^2\right)^{\beta}}{1-|\varphi(z)|^2}\left|g'(\varphi(z))(\varphi(z))^2\right|.
\end{aligned} \tag{50}$$

Applying (41) we get

$$\begin{aligned}
&\sup_{|\varphi(z)|>1/2}\log\left(\frac{1}{1-|\varphi(z)|^2}\right)\left(1-|z|^2\right)^{\beta} \\
&\quad\cdot\left|\left(\varphi'(z)\right)^2 g''(\varphi(z)) + g'(\varphi(z))\varphi''(z)\right| \\
&\quad\leq \sup_{|\varphi(z)|>1/2}\frac{\left(1-|z|^2\right)^{\beta}}{1-|\varphi(z)|^2}\left|g'(\varphi(z))(\varphi(z))^2\right| \\
&\quad+\left\|C_\varphi T_g t_a\right\|_{\mathcal{Z}^\beta} < \infty.
\end{aligned} \tag{51}$$

Noting $((g'\circ\varphi)(\varphi'') + (g''\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$ and using Lemma 4 and (15), we conclude that (42) holds.

*(v) Case* $\alpha > 2$. We have proved that (40) holds above. Applying test function $f_{\varphi(z)}$ in (23) for every $z \in D$ with $|\varphi(z)| > 1/2$, we have

$$\begin{aligned}
S_1 &= \sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta}\left|\frac{\left(\varphi'(z)\right)^2 g'(\varphi(z))}{\overline{\varphi(z)}\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\right| \\
&= \sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta}\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right| \\
&\quad\cdot\left|f'_{\varphi(z)}(\varphi(z))\right| \leq \sup_{|\varphi(z)|>1/2}\left\|C_\varphi T_g\left(f_{\varphi(z)}\right)\right\|_{\mathcal{Z}^\beta} < \infty.
\end{aligned} \tag{52}$$

With the same calculation for test function $t_{\varphi(z)}(\varphi(z)) = (1-|\varphi(z)|^2)^2/(1-\overline{\varphi(z)}z)^\alpha$ with $|\varphi(z)| > 1/2$, then $\sup_{|\varphi(z)|>(1/2)}\|t_{\varphi(z)}\|_{\mathcal{Z}^\alpha} \leq C$, and we have that

$$\begin{aligned}
S_2 &= \sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta} \\
&\quad\cdot\left|\frac{\left(\varphi'(z)\right)^2 g''(\varphi(z)) + g'(\varphi(z))\varphi''(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha-2}}\right. \\
&\quad\left.+\alpha\overline{\varphi(z)}\frac{\left(\varphi'(z)\right)^2 g'(\varphi(z))}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\right| = \sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta} \\
&\quad\cdot\left|\left(C_\varphi T_g\left(t_{\varphi(z)}\right)\right)''(\varphi(z))\right| \leq \left\|C_\varphi T_g\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \\
&\quad\cdot\sup_{|\varphi(z)|>1/2}\left\|t_{\varphi(z)}\right\|_{\mathcal{Z}^\alpha} < \infty.
\end{aligned} \tag{53}$$

Therefore,

$$\begin{aligned}
&\sup_{|\varphi(z)|>1/2}\left(1-|z|^2\right)^{\beta} \\
&\quad\cdot\left|\frac{\left(\varphi'(z)\right)^2 g''(\varphi(z)) + g'(\varphi(z))\varphi''(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha-2}}\right| \leq S_2 \\
&\quad+\alpha\sup_{|\varphi(z)|>1/2}\frac{\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}} \leq S_2+\alpha \\
&\quad\cdot\sup_{|\varphi(z)|>1/2}\frac{\left|\left(\varphi'(z)\right)^2 g'(\varphi(z))\right|}{\left|\overline{\varphi(z)}\right|\left(1-|\varphi(z)|^2\right)^{\alpha-1}} \leq S_2+\alpha S_1 \\
&\quad< \infty.
\end{aligned} \tag{54}$$

Since $((g'\circ\varphi)(\varphi'') + (g''\circ\varphi)(\varphi')^2) \in H^\infty_{v_\beta}$, we conclude that (43) holds. □

**Theorem 9.** *Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$.*

(i) *If $0 < \alpha < 1$, then $T_g C_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $g'\varphi' \in H^\infty_{v_\beta}$ and $g'' \in H^\infty_{v_\beta}$.*

(ii) *If $\alpha = 1$, then $T_g C_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $g'' \in H^\infty_{v_\beta}$ and*

$$\begin{aligned}
&\sup_{n\geq 0}(\log n)\left\|\left(g'\varphi'\right)\varphi^n\right\|_{v_\beta} \\
&\quad\asymp \sup_{z\in D}\left(1-|z|^2\right)^{\beta}\log\frac{2}{1-|\varphi(z)|^2}\left|g'(z)\varphi'(z)\right| \\
&\quad< \infty.
\end{aligned} \tag{55}$$

(iii) *If $1 < \alpha < 2$, then $T_gC_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if $g'' \in H^\infty_{v_\beta}$ and*

$$\sup_{n\geq 0}(n+1)^{\alpha-1}\left\|\left(g'\varphi'\right)\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}\left|g'(z)\varphi'(z)\right| < \infty. \tag{56}$$

(iv) *If $\alpha = 2$, then $T_gC_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if*

$$\begin{aligned}&\sup_{n\geq 0}(n+1)\left\|\left(g'\varphi'\right)\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)}\left|g'(z)\varphi'(z)\right| < \infty,\\ &\sup_{n\geq 0}(\log n)\left\|\left(g''\right)\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\left(1-|z|^2\right)^\beta\log\frac{2}{\left(1-|\varphi(z)|^2\right)}\left|g''(z)\right| < \infty.\end{aligned} \tag{57}$$

(v) *If $\alpha > 2$, then $T_gC_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator if and only if (56) holds and*

$$\sup_{n\geq 0}(n+1)^{\alpha-2}\left\|\left(g''\right)\varphi^n\right\|_{v_\beta} \asymp \sup_{z\in D}\frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^{\alpha-2}}\left|g''(z)\right| < \infty. \tag{58}$$

The proof is similar to that of Theorem 8, and the details are omitted.

## 3. Essential Norms

In this section we estimate the essential norms of these integral type operators on *Zygmund* type spaces in terms of $g, \varphi$, their derivatives, and the $n$th power $\varphi^n$ of $\varphi$.

Let $\widetilde{\mathcal{Z}^\alpha} = \{f \in \mathcal{Z}^\alpha : f(0) = f'(0) = 0\}$ and $\widetilde{\mathcal{B}^\alpha} = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$. We note that every compact operator $T \in \mathcal{K}(\widetilde{\mathcal{Z}^\alpha}, \mathcal{Z}^\beta)$ can be extended to a compact operator $K \in \mathcal{K}(\mathcal{Z}^\alpha, \mathcal{Z}^\beta)$. In fact, for every $f \in \mathcal{Z}^\alpha$, $f - f(0) - f'(0)z \in \widetilde{\mathcal{Z}^\alpha}$, and we can define $K(f) = T(f - f(0) - f'(0)z) + f(0) + f'(0)z$.

For $r \in (0,1)$, we consider the compact operator $K_r : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ defined by $K_r f(z) = f(rz)$.

**Lemma 10.** *If $X(I_gC_\varphi, C_\varphi I_g, C_\varphi T_g, T_gC_\varphi)$ is a bounded operator from $\mathcal{Z}^\alpha$ to $\mathcal{Z}^\beta$ space, then*

$$\|X\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} = \|X\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}. \tag{59}$$

*Proof.* Clearly $\|X\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \geq \|X\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}$. Then we prove $\|X\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \leq \|X\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}$.

Let $T \in K(\mathcal{Z}^\alpha, \mathcal{Z}^\beta)$ be given. Let $\{r_n\}$ be an increasing sequence in $(0,1)$ converging to 1 and $\mathcal{A} = \{h \mid h = a + bz\}$, the closed subspace of $\mathcal{Z}^\alpha$. Then

$$\begin{aligned}\|X - T\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} &= \sup_{f\in\mathcal{Z}^\alpha,\|f\|_{\mathcal{Z}^\alpha}\leq 1}\left\|X(f) - T(f)\right\|_{\mathcal{Z}^\beta}\\ &\leq \sup_{f\in\mathcal{Z}^\alpha,\|f\|_{\mathcal{Z}^\alpha}\leq 1}\left\|X\left(f - f(0) - f'(0)z\right)\right.\\ &\quad\left.- T\left(f - f(0) - f'(0)z\right)\right\|_{\mathcal{Z}^\beta}\\ &\quad+ \sup_{f\in\mathcal{Z}^\alpha,\|f\|_{\mathcal{Z}^\alpha}\leq 1}\left\|X\left(f(0) + f'(0)z\right)\right.\\ &\quad\left.- T\left(f(0) + f'(0)z\right)\right\|_{\mathcal{Z}^\beta} \leq \sup_{g\in\widetilde{\mathcal{Z}^\alpha},\|g\|_{\mathcal{Z}^\alpha}\leq 1}\left\|X(g)\right.\\ &\quad\left.- T|_{\widetilde{\mathcal{Z}^\alpha}}(g)\right\|_{\mathcal{Z}^\beta} + \sup_{h\in\mathcal{A},\|h\|_{\mathcal{Z}^\alpha}\leq 1}\left\|X(h) - T|_\alpha(h)\right\|_{\mathcal{Z}^\beta}.\end{aligned} \tag{60}$$

Hence

$$\begin{aligned}&\inf_{T\in K(\mathcal{Z}^\alpha,\mathcal{Z}^\beta)}\|X - T\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\\ &\quad\leq \inf_{T\in K(\mathcal{Z}^\alpha,\mathcal{Z}^\beta)}\left\|X - T|_{\widetilde{\mathcal{Z}^\alpha}}\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\\ &\qquad+ \inf_{T\in K(\mathcal{Z}^\alpha,\mathcal{Z}^\beta)}\left\|X - T|_\alpha\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\\ &\quad\leq \|X\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta} + \lim_{n\to\infty}\left\|X\left(I - K_{r_n}\right)\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}.\end{aligned} \tag{61}$$

Since $\lim_{n\to\infty}\|X(I - K_{r_n})\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} = 0$, we have $\|X\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \leq \|X\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}$, and the proof is finished. □

Let $D_\alpha : \mathcal{Z}^\alpha \to \mathcal{B}_\alpha$ and $S_\alpha : \mathcal{B}^\alpha \to H^\infty_{v_\alpha}$ be the derivative operators. Then clearly $D_\alpha$ and $S_\alpha$ are linear isometries on $\widetilde{\mathcal{Z}^\alpha}$ and $\widetilde{\mathcal{B}^\alpha}$, respectively, and

$$S_\beta D_\beta I_g C_\varphi D_\alpha^{-1} S_\alpha^{-1} = g\left(\varphi'\right)C_\varphi + g'C_\varphi S_\alpha^{-1}. \tag{62}$$

Therefore

$$\begin{aligned}\left\|I_gC_\varphi\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta} &\leq \left\|g'C_\varphi\right\|_{e,\widetilde{\mathcal{B}^\alpha}\to H^\infty_{v_\beta}}\\ &\quad+ \left\|g\left(\varphi'\right)C_\varphi\right\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\beta}}.\end{aligned} \tag{63}$$

Similarly,

$$\begin{aligned}&S_\beta D_\beta C_\varphi I_g D_\alpha^{-1} S_\alpha^{-1}\\ &\quad= \left(g\circ\varphi\left(\varphi'\right)^2\right)C_\varphi\\ &\qquad+ \left(g'\circ\varphi\left(\varphi'\right)^2 + g\circ\varphi\left(\varphi''\right)\right)C_\varphi S_\alpha^{-1},\end{aligned} \tag{64}$$

$$\begin{aligned}&\left\|C_\varphi I_g\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}\\&\quad\le\left\|\left(\left(g'\circ\varphi\left(\varphi'\right)^2\right)\right)C_\varphi\right\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\beta}}\\&\qquad+\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2+g\circ\varphi\left(\varphi''\right)\right)C_\varphi\right\|_{e,\widetilde{\mathcal{B}^\alpha}\to H^\infty_{v_\beta}},\end{aligned}\tag{65}$$

$$\begin{aligned}&S_\beta D_\beta C_\varphi T_g D_\alpha^{-1}S_\alpha^{-1}\\&\quad=\left(g''\circ\varphi\left(\varphi'\right)^2+g'\circ\varphi\left(\varphi''\right)\right)C_\varphi D_\alpha^{-1}S_\alpha^{-1}\\&\qquad+\left(g'\circ\varphi\left(\varphi'\right)^2\right)C_\varphi S_\alpha^{-1},\end{aligned}\tag{66}$$

$$\begin{aligned}&\left\|C_\varphi T_g\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}\\&\quad\le\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)C_\varphi\right\|_{e,\widetilde{\mathcal{B}^\alpha}\to H^\infty_{v_\beta}}\\&\qquad+\left\|\left(\left(g''\circ\varphi\right)\left(\varphi'\right)^2+g'\circ\varphi\left(\varphi''\right)\right)C_\varphi\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to H^\infty_{v_\beta}},\end{aligned}\tag{67}$$

$$S_\beta D_\beta T_g C_\varphi D_\alpha^{-1}S_\alpha^{-1}=g''C_\varphi D_\alpha^{-1}S_\alpha^{-1}+g'\left(\varphi'\right)C_\varphi S_\alpha^{-1},\tag{68}$$

$$\begin{aligned}&\left\|T_g C_\varphi\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to\mathcal{Z}^\beta}\\&\quad\le\left\|g'\left(\varphi'\right)\right\|_{e,\widetilde{\mathcal{B}^\alpha}\to H^\infty_{v_\beta}}+\left\|g''C_\varphi\right\|_{e,\widetilde{\mathcal{Z}^\alpha}\to H^\infty_{v_\beta}}.\end{aligned}\tag{69}$$

**Theorem 11.** *Let $I_gC_\varphi$ be a bounded operator from $\mathcal{Z}^\alpha$ to $\mathcal{Z}^\beta$ space.*

(i) *If $0<\alpha<1$, then*

$$\left\|I_gC_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\asymp\limsup_{n\to\infty}(n+1)^\alpha\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}.\tag{70}$$

(ii) *If $\alpha=1$, then*

$$\begin{aligned}\left\|I_gC_\varphi\right\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta}\asymp\max\Bigg\{&\limsup_{n\to\infty}(n+1)\\&\cdot\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(\log n)\left\|\left(g'\right)\varphi^n\right\|_{v_\beta}\Bigg\}.\end{aligned}\tag{71}$$

(iii) *If $\alpha>1$, then*

$$\begin{aligned}\left\|I_gC_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\asymp\max\Bigg\{&\limsup_{n\to\infty}(n+1)^\alpha\\&\cdot\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(n+1)^{\alpha-1}\left\|\left(g'\right)\varphi^n\right\|_{v_\beta}\Bigg\}.\end{aligned}\tag{72}$$

*Proof.* (i) We start with the upper bound. First we show that $g'C_\varphi$ is a compact weighted composition operator for $\mathcal{B}^\alpha$ into $H^\infty_{v_\beta}$. Suppose that $\{f_n\}$ is bounded sequence in $\mathcal{B}^\alpha$. From Lemma 3.6 in [27], $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges uniformly on $D$ to a function, which we can assume to be identically zero. Then it follows from Theorem 6 and Lemma 1 that

$$\begin{aligned}&\lim_{k\to\infty}\sup_{z\in D}\left(1-|z|^2\right)^\beta\left|g'(z)\right|\left|f_{n_k}(\varphi(z))\right|\\&\quad\le C\lim_{k\to\infty}\sup_{z\in D}\left|f_{n_k}(z)\right|=0,\end{aligned}\tag{73}$$

which shows that $g'C_\varphi:\mathcal{B}^\alpha\to H^\infty_{v_\beta}$ is a compact operator and $\|g'C_\varphi\|_{e,\mathcal{B}^\alpha\to H^\infty_{v_\beta}}=0$. Applying (63), Lemmas 4, 5, and 10, we get that

$$\begin{aligned}\left\|I_gC_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}&\le\left\|g\left(\varphi'\right)C_\varphi\right\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\beta}}\\&=\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^\alpha}\left|g(z)\varphi'(z)\right|\\&=\limsup_{n\to\infty}\frac{\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}}{\|z^n\|_{v_\alpha}}\\&=\left(\frac{e}{2\alpha}\right)^\alpha\limsup_{n\to\infty}(n+1)^\alpha\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}.\end{aligned}\tag{74}$$

For the lower bound, let $\{z_n\}\subseteq D$ with $|\varphi(z_n)|>1/2$ and $|\varphi(z_n)|\to1$ as $n\to\infty$. Taking $g_n=g_{\varphi(z_n)}$ defined in (25), we obtain that $\{g_n\}$ is bounded sequence in $\mathcal{Z}_0^\alpha$ converging to 0 uniformly on compact subset of $D$ and $\sup_{n\in N}\|g_n\|_{\mathcal{Z}^\alpha}\le C$. For every compact operator $T:\mathcal{Z}^\alpha\to\mathcal{Z}^\beta$,

$$\begin{aligned}&C\left\|I_gC_\varphi-T\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\\&\quad\ge\limsup_{n\to\infty}\left\|I_gC_\varphi g_n\right\|_{\mathcal{Z}^\beta}-\limsup_{n\to\infty}\left\|T(g_n)\right\|_{\mathcal{Z}^\beta}\\&\quad\ge\alpha\limsup_{n\to\infty}\left(1-|z_n|^2\right)^\beta\frac{\left|g(z_n)\varphi'(z_n)\right|}{\left(1-|\varphi(z_n)|^2\right)^\alpha}.\end{aligned}\tag{75}$$

Now we use (14) and Lemma 4 to obtain that

$$\begin{aligned}\left\|I_gC_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}&\ge\left\|I_gC_\varphi-T\right\|_{\mathcal{Z}^\alpha\to\mathcal{Z}^\beta}\\&\ge\frac{\alpha}{C}\limsup_{|\varphi(z_n)|\to1}\left|g(z_n)\varphi'(z_n)\right|\frac{\left(1-|z_n|^2\right)^\beta}{\left(1-|\varphi(z_n)|^2\right)^\alpha}\\&=\frac{\alpha}{C}\limsup_{n\to\infty}\frac{\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}}{\|z^n\|_{v_\alpha}}\\&=C\left(\frac{e}{2\alpha}\right)^\alpha\limsup_{n\to\infty}(n+1)^\alpha\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}.\end{aligned}\tag{76}$$

Hence (70) holds.

(ii) The boundedness of $I_gC_\varphi$ guarantees that $(g\varphi')C_\varphi : H^\infty_{v_1} \to H^\infty_{v_\beta}$ and $(g')C_\varphi : \mathscr{B} \to H^\infty_{v_\beta}$ are bounded weighted composition operators. Theorem 3.4 in [28] ensures that

$$\begin{aligned}&\left\|g'C_\varphi\right\|_{e,\mathscr{B}\to H^\infty_{v_\beta}}\\&\quad\asymp \lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)^\beta\left|g'(z)\right|\log\frac{2}{1-|\varphi(z)|^2}.\end{aligned}\tag{77}$$

Now we use Lemmas 4, 5 and (63) to conclude that

$$\begin{aligned}&\left\|I_gC_\varphi\right\|_{e,\mathscr{Z}\to\mathscr{Z}^\beta}\le\left\|g'C_\varphi\right\|_{e,\widetilde{\mathscr{B}}\to H^\infty_{v_\beta}}+\left\|g\left(\varphi'\right)\right.\\&\cdot\left.C_\varphi\right\|_{e,H^\infty_{v_1}\to H^\infty_{v_\beta}}\le C\limsup_{n\to\infty}\frac{\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}}{\|z^n\|_{v_1}}+C\\&\cdot\limsup_{n\to\infty}\frac{\left\|g'\varphi^n\right\|_{v_\beta}}{\|z^n\|_{v_{\log}}}=C\limsup_{n\to\infty}(\log n)\left\|g'\varphi^n\right\|_{v_\beta}\\&+\frac{Ce}{2}\limsup_{n\to\infty}(n+1)\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}\le C\\&\cdot\max\left\{\limsup_{n\to\infty}(n+1)\right.\\&\left.\cdot\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(\log n)\left\|\left(g'\right)\varphi^n\right\|_{v_\beta}\right\}.\end{aligned}\tag{78}$$

On the other hand, let $\{z_n\}$ be a sequence in $D$ such that $|\varphi(z_n)|>1/2$ and $|\varphi(z_n)|\to 1$ as $n\to\infty$. Given

$$\begin{aligned}h_n(z)=&\frac{h\left(\overline{\varphi(z_n)}z\right)}{\overline{\varphi(z_n)}}\left(\log\frac{2}{1-|\varphi(z_n)|^2}\right)^{-1}\\&-\left(\int_0^z\log^3\frac{2}{1-\overline{\varphi(z_n)}\omega}d\omega\right)\\&\cdot\left(\log\frac{2}{1-|\varphi(z_n)|^2}\right)^{-2},\end{aligned}\tag{79}$$

where $h(z)=(z-1)((1+\log(2/(1-z)))^2+1)$, from [11] we know that $\{h_n\}$ is a bounded sequence in $\mathscr{Z}^0_1$ which converges to zero uniformly on compact subsets of $D$, and

$$\begin{aligned}&h_n''(\varphi(z_n))=\frac{-\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2},\\&h_n'(\varphi(z_n))=0,\\&\sup_n\|h_n\|_{\mathscr{Z}}<+\infty.\end{aligned}\tag{80}$$

For every compact operator $T:\mathscr{Z}\to\mathscr{Z}^\beta$, we have $\|T(h_n)\|_{\mathscr{Z}^\beta}\to 0$ as $n\to\infty$. By Lemmas 4 and 5, we obtain that

$$\begin{aligned}C\left\|I_gC_\varphi-T\right\|_{\mathscr{Z}\to\mathscr{Z}^\beta}&\ge\left\|I_gC_\varphi h_n\right\|_{\mathscr{Z}\to\mathscr{Z}^\beta}\\&\ge\limsup_{n\to\infty}\left(1-|z_n|^2\right)^\beta\left|g(z_n)\varphi'(z_n)\right|\frac{|\varphi(z_n)|}{1-|\varphi(z_n)|^2}\\&\ge\limsup_{n\to\infty}\frac{\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}}{\|z^n\|_{v_1}}\\&=\frac{e}{2}\limsup_{n\to\infty}(n+1)\left\|\left(g\varphi'\right)\varphi^n\right\|_{v_\beta}.\end{aligned}\tag{81}$$

Now we take another function

$$f_n=\frac{h\left(\overline{\varphi(z_n)}z\right)}{\overline{\varphi(z_n)}}\left(\log\frac{2}{1-|\varphi(z_n)|^2}\right)^{-1}.\tag{82}$$

From [11] we know that $\{f_n\}$ is a bounded sequence in $\mathscr{Z}^0_1$ which converges to zero uniformly on compact subsets of $D$, and $\sup_{n\ge1}\|f_n\|_{\mathscr{Z}}<+\infty$. It follows from Lemmas 4 and 5 that

$$\begin{aligned}C\left\|I_gC_\varphi\right\|_{e,\mathscr{Z}\to\mathscr{Z}^\beta}&\ge\lim_{n\to\infty}\sup\left\|I_gC_\varphi f_n\right\|_{\mathscr{Z}^\beta}\\&\ge\limsup_{n\to\infty}\left(1-|z_n|^2\right)^\beta\left|g'(z_n)\right|\log\frac{2}{1-|\varphi(z_n)|^2}\\&-\limsup_{n\to\infty}\left(1-|z_n|^2\right)^\beta\left|g(z_n)\varphi'(z_n)\right|\frac{|\varphi(z_n)|}{1-|\varphi(z_n)|^2}.\end{aligned}\tag{83}$$

Noting that $\limsup_{n\to\infty}(n+1)\|(g\varphi')\varphi^n\|_{v_\beta}\le(2C/e)\|I_gC_\varphi\|_{e,\mathscr{Z}\to\mathscr{Z}^\beta}$, we obtain

$$\begin{aligned}&\left(C+\frac{2C}{e}\right)\left\|I_gC_\varphi\right\|_{e,\mathscr{Z}\to\mathscr{Z}^\beta}\\&\quad\ge\limsup_{|\varphi(z_n)|\to1}\left(1-|z_n|^2\right)^\beta\left|g'(z_n)\right|\log\frac{2}{1-|\varphi(z_n)|^2}\\&\quad=\limsup_{n\to\infty}(\log n)\left\|\left(g'\right)\varphi^n\right\|_{v_\beta}.\end{aligned}\tag{84}$$

Hence we have $\|I_gC_\varphi\|_{e,\mathscr{Z}\to\mathscr{Z}^\beta}\ge C\max\{\limsup_{n\to\infty}(n+1)\|(g\varphi')\varphi^n\|_{v_\beta},\ \limsup_{n\to\infty}(\log n)\|(g')\varphi^n\|_{v_\beta}\}$.

(iii) Let $\alpha>1$. The proof of the upper bound is similar to that of (ii). From the proof of (i), we get that, for some constant $C$,

$$\begin{aligned}&C\left\|I_gC_\varphi\right\|_{e,\mathscr{Z}^\alpha\to\mathscr{Z}^\beta}\\&\quad\ge\limsup_{|\varphi(z)|\to1}\left|g(z)\varphi'(z)\right|\frac{\left(1-|z_n|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^\alpha}.\end{aligned}\tag{85}$$

Now, let $\{z_n\}$ be as before and note that the function $f_n=f_{\varphi(z_n)}$ given in (23). Then $\{f_n\}$ is bounded sequence in

$\mathcal{Z}_0^\alpha$ converging to zero uniformly on compact subsets of $D$; therefore

$$\begin{aligned} C\left\|I_g C_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} &\geq \lim_{n\to\infty}\left\|I_g C_\varphi\left(f_n\right)\right\|_{\mathcal{Z}^\beta} \\ &\geq 2\alpha \limsup_{n\to\infty}\frac{\left(1-\left|z_n\right|^2\right)^\beta}{\left(1-\left|\varphi\left(z_n\right)\right|^2\right)^{\alpha-1}}\left|g'\left(z_n\right)\right| \\ &\quad - \limsup_{n\to\infty}\frac{\left(1-\left|z_n\right|^2\right)^\beta\left|g\left(z_n\right)\varphi'\left(z_n\right)\right|}{\left|\varphi\left(z_n\right)\right|\left(1-\left|\varphi\left(z_n\right)\right|^2\right)^\alpha}. \end{aligned} \tag{86}$$

By (85), we have

$$\begin{aligned} &C\left\|I_g C_\varphi\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \\ &\quad\geq \limsup_{n\to\infty}\frac{\left(1-\left|z_n\right|^2\right)^\beta}{\left(1-\left|\varphi\left(z_n\right)\right|^2\right)^{\alpha-1}}\left|g'\left(z_n\right)\right|, \end{aligned} \tag{87}$$

and the rest of the proof is similar to that of the previous, and we omit it. □

**Theorem 12.** *Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$, and $C_\varphi T_g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ is a bounded operator.*

(i) *If $0 < \alpha < 1$, then*

$$\left\|C_\varphi T_g\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} = 0. \tag{88}$$

(ii) *If $\alpha = 1$, then*

$$\begin{aligned} &\left\|C_\varphi T_g\right\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta} \\ &\quad\asymp \lim_{n\to\infty}\sup\left(\log n\right)\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{89}$$

(iii) *If $1 < \alpha < 2$, then*

$$\begin{aligned} &\left\|C_\varphi T_g\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \\ &\quad\asymp \lim_{n\to\infty}\sup\left(n+1\right)^{\alpha-1}\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{90}$$

(iv) *If $\alpha = 2$, then*

$$\begin{aligned} \left\|C_\varphi T_g\right\|_{e,\mathcal{Z}^2\to\mathcal{Z}^\beta} &\asymp \max\left\{\lim_{n\to\infty}\sup\left(n+1\right)\right. \\ &\quad\cdot\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}, \lim_{n\to\infty}\sup\left(\log n\right) \\ &\quad\left.\cdot\left\|\left(\left(g''\circ\varphi\right)\left(\varphi'\right)^2+\left(g'\circ\varphi\right)\left(\varphi''\right)\right)\varphi^n\right\|_{v_\beta}\right\}. \end{aligned} \tag{91}$$

(v) *If $\alpha > 2$, then*

$$\begin{aligned} \left\|C_\varphi T_g\right\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} &\asymp \max\left\{\lim_{n\to\infty}\sup\left(n+1\right)^{\alpha-1}\right. \\ &\quad\cdot\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}, \lim_{n\to\infty}\sup\left(n+1\right)^{\alpha-2} \\ &\quad\left.\cdot\left\|\left(\left(g''\circ\varphi\right)\left(\varphi'\right)^2+\left(g'\circ\varphi\right)\left(\varphi''\right)\right)\varphi^n\right\|_{v_\beta}\right\}. \end{aligned} \tag{92}$$

*Proof.* (i) For the compactness of $((g'\circ\varphi)(\varphi')^2)C_\varphi$, the argument is similar to the proof of Theorem 11(i); then we have $\|((g'\circ\varphi)(\varphi')^2)C_\varphi\|_{e,\mathcal{B}^\alpha\to H_{v_\beta}^\infty} = 0$. Hence by (67) and Lemma 3, we get that $\|C_\varphi T_g\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} = 0$.

Next we will prove (ii). The boundedness of $C_\varphi T_g$ guarantees that $((g'\circ\varphi)(\varphi')^2)C_\varphi : \mathcal{B} \to H_{v_\beta}^\infty$ and $((g''\circ\varphi)(\varphi')^2+(g'\circ\varphi)(\varphi''))C_\varphi : \mathcal{Z} \to H_{v_\beta}^\infty$ are bounded weighted composition operators. We know that if $uC_\varphi : \mathcal{Z} \to H_{v_\beta}^\infty$ is a bounded operator, then $uC_\varphi$ is a compact operator by Lemma 3. Hence we consider the boundedness of $C_\varphi T_g : \mathcal{Z} \to \mathcal{Z}^\beta$ and just consider that $((g'\circ\varphi)(\varphi')^2)C_\varphi : \mathcal{B} \to H_{v_\beta}^\infty$ is a bounded operator.

Theorem 3.4 in [28] ensures that

$$\begin{aligned} &\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)C_\varphi\right\|_{e,\mathcal{B}\to H_{v_\beta}^\infty} \asymp \lim_{|\varphi|\to 1}\left(1-|z|^2\right)^\beta \\ &\quad\cdot\left|g'\left(\varphi\left(z\right)\right)\left(\varphi'\left(z\right)\right)^2\right|\log\frac{2}{1-\left|\varphi\left(z\right)\right|^2}. \end{aligned} \tag{93}$$

From (67) and Lemmas 4 and 5, we have

$$\begin{aligned} \left\|C_\varphi T_g\right\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta} &\leq C\limsup_{n\to\infty}\frac{\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}}{\left\|z^n\right\|_{v_{\log}}} \\ &= C\limsup_{n\to\infty}\left(\log n\right)\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{94}$$

In order to prove $\|C_\varphi T_g\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta} \geq \limsup_{n\to\infty}(\log n)\|((g'\circ\varphi)(\varphi')^2)\varphi^n\|_{v_\beta}$, we take the function

$$\begin{aligned} g_n\left(z\right) &= \frac{\overline{\varphi\left(z_n\right)}z-1}{\overline{\varphi\left(z_n\right)}}\left(\left(1+\log\frac{1}{1-\overline{\varphi\left(z_n\right)}z}\right)^2+1\right) \\ &\quad\cdot\left(\log\frac{1}{1-\left|\varphi\left(z_n\right)\right|^2}\right)^{-1}-a_n, \end{aligned} \tag{95}$$

where

$$\begin{aligned} a_n &= \frac{\left|\varphi\left(z_n\right)\right|^2-1}{\overline{\varphi\left(z_n\right)}}\left(\left(1+\log\frac{1}{1-\left|\varphi\left(z_n\right)\right|^2}\right)^2+1\right) \\ &\quad\cdot\left(\log\frac{1}{1-\left|\varphi\left(z_n\right)\right|^2}\right)^{-1}, \end{aligned} \tag{96}$$

and $\lim_{n\to\infty}a_n = 0$. From [16] we obtain that $\{g_n\}$ is a bounded sequence in $\mathcal{Z}_1^0$ which converges to zero uniformly on compact subsets of $D$. By a direct calculation, we have

$$\begin{aligned} g_n\left(\varphi\left(z_n\right)\right) &= 0, \\ g_n'\left(\varphi\left(z_n\right)\right) &= \log\frac{1}{1-\left|\varphi\left(z_n\right)\right|^2}. \end{aligned} \tag{97}$$

For every compact operator $T : \mathcal{Z} \to \mathcal{Z}^{\beta}$, we have $\|T(h_n)\|_{\mathcal{Z}^{\beta}} \to 0$ as $n \to \infty$. Let $M = \sup_{n\geq 1}\|g_n\|_{\mathcal{Z}^{\beta}}$. It follows from Lemma 5 that

$$\begin{aligned} M\left\|C_{\varphi}T_g - T\right\|_{e,\mathcal{Z}\to\mathcal{Z}^{\beta}} &\geq \left\|C_{\varphi}T_g f_n\right\|_{e,\mathcal{Z}\to\mathcal{Z}^{\beta}} \\ &\geq \limsup_{n\to\infty}\left(1-|z_n|^2\right)^{\beta}\left|g'(\varphi(z_n))\left(\varphi'(z_n)\right)^2\right| \\ &\quad\cdot\log\frac{1}{1-|\varphi(z_n)|^2} \\ &\geq \limsup_{n\to\infty}\frac{\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}}{\|z^n\|_v} \\ &\geq \limsup_{n\to\infty}(\log n)\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{98}$$

This completes the proof.

The proof of (iii) is the same as that of Theorem 11 (iii); we do not prove it again.

(iv) Let $\alpha = 2$. Applying Lemma 3 (ii) and Theorem 3.2 in [29], we get that

$$\begin{aligned} &\left\|\left(g'\circ\varphi\right)\left(\varphi'\right)^2 C_{\varphi}\right\|_{e,\mathcal{B}^2\to H^{\infty}_{v_\beta}} \asymp \limsup_{n\to\infty}(n+1) \\ &\quad\cdot\left\|\left(g'\circ\varphi\right)\left(\varphi'\right)^2\varphi^n\right\|_{v_\beta}, \\ &\left\|\left(\left(g''\circ\varphi\right)\left(\varphi'\right)^2+\left(g'\circ\varphi\right)\varphi''\right)C_{\varphi}\right\|_{e,\mathcal{Z}_2\to H^{\infty}_{v_\beta}} \\ &\quad\asymp \limsup_{n\to\infty}(\log n) \\ &\quad\cdot\left\|\left(\left(g''\circ\varphi\right)\left(\varphi'\right)^2+\left(g'\circ\varphi\right)\varphi''\right)\varphi^n\right\|_{v_\beta}, \end{aligned} \tag{99}$$

which yields the upper bound by (67).

With the same arguments as in the proof of Theorems 8 and 11, for some constant $C$, we have

$$\begin{aligned} &C\left\|C_{\varphi}T_g\right\|_{e,\mathcal{Z}^2\to\mathcal{Z}^{\beta}} \\ &\quad\geq \limsup_{n\to\infty}(n+1)\left\|\left(\left(g'\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{100}$$

On the other hand, let $\{z_n\} \subseteq D$ with $|\varphi(z_n)| > 1/2$ and $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let the test function

$$\begin{aligned} O_n(z) &= \left(1+\left(\log\frac{2}{1-\overline{\varphi(z_n)}z}\right)^2\right) \\ &\quad\cdot\left(\log\frac{2}{1-|\varphi(z_n)|^2}\right)^{-1}. \end{aligned} \tag{101}$$

From [8] we obtain that $\{O_n\}$ is a bounded sequence in $\mathcal{Z}_0^2$ which converges to zero uniformly on compact subsets of $D$, and

$$\begin{aligned} &\lim_{n\to\infty}\left(1-|z_n|^2\right)^{\beta}\left|g'(\varphi(z_n))\left(\varphi'(z_n)\right)^2\right|\left|O_n'(\varphi(z_n))\right| \\ &\quad= 2\lim_{n\to\infty}\frac{\left(1-|z_n|^2\right)^{\beta}}{1-|\varphi(z_n)|^2}\left|g'(\varphi(z_n))\left(\varphi'(z_n)\right)^2\right| \\ &\quad\cdot|\varphi(z_n)| \leq C\left\|C_{\varphi}T_g\right\|_{e,\mathcal{Z}^2\to\mathcal{Z}^{\beta}}. \end{aligned} \tag{102}$$

Applying Theorem 8 we get

$$\begin{aligned} &C\left\|C_{\varphi}T_g\right\|_{e,\mathcal{Z}_2\to\mathcal{Z}^{\beta}} \geq \left\|C_{\varphi}T_g(O_n)\right\|_{\mathcal{Z}^{\beta}} \\ &\quad\geq \lim_{n\to\infty}\left(1-|z_n|^2\right)^{\beta} \\ &\quad\cdot\left|g''(\varphi(z_n))\left(\varphi'(z_n)\right)^2+g'(\varphi(z_n))\varphi''(z_n)\right| \\ &\quad\cdot\left|O_n(\varphi(z_n))\right| \\ &\quad-\lim_{n\to\infty}\left(1-|z_n|^2\right)^{\beta}\left|g'(\varphi(z_n))\left(\varphi'(z_n)\right)^2\right| \\ &\quad\cdot\left|O_n'(\varphi(z_n))\right| \geq \lim_{n\to\infty}\left(1-|z_n|^2\right)^{\beta} \\ &\quad\cdot\left|g''(\varphi(z_n))\left(\varphi'(z_n)\right)^2+g'(\varphi(z_n))\varphi''(z_n)\right| \\ &\quad\cdot\log\frac{2}{1-|\varphi(z_n)|^2}-2C\left\|C_{\varphi}T_g\right\|_{e,\mathcal{Z}^2\to\mathcal{Z}^{\beta}}. \end{aligned} \tag{103}$$

Hence

$$\begin{aligned} &\lim_{n\to\infty}\left(1-|z_n|^2\right)^{\beta} \\ &\quad\cdot\left|g''(\varphi(z_n))\left(\varphi'(z_n)\right)^2+g'(\varphi(z_n))\varphi''(z_n)\right| \\ &\quad\cdot\log\frac{2}{1-|\varphi(z_n)|^2} \leq C\left\|C_{\varphi}T_g\right\|_{e,\mathcal{Z}^2\to\mathcal{Z}^{\beta}}. \end{aligned} \tag{104}$$

On the other hand, the lower bound can be easily proved by Lemmas 4 and 5.

If $\alpha > 2$, the proof is similar to that of (iv) except that we now choose the test function $t_n(z) = (1-|\varphi(z_n)|^2)^2/(1-\overline{\varphi(z_n)}z)^{\alpha}$ instead of $O_n(z)$. This completes the proof of Theorem 12. □

Using the same methods of Theorems 11 and 12, we can have the following results.

**Theorem 13.** *Let $C_{\varphi}I_g$ be a bounded operator from $\mathcal{Z}^{\alpha}$ to $\mathcal{Z}^{\beta}$ space.*

(i) *If $0 < \alpha < 1$, then*

$$\begin{aligned} &\left\|C_{\varphi}I_g\right\|_{e,\mathcal{Z}^{\alpha}\to\mathcal{Z}^{\beta}} \\ &\quad\asymp \limsup_{n\to\infty}(n+1)^{\alpha}\left\|\left(\left(g\circ\varphi\right)\left(\varphi'\right)^2\right)\varphi^n\right\|_{v_\beta}. \end{aligned} \tag{105}$$

(ii) *If* $\alpha = 1$, *then*

$$\|C_\varphi I_g\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta} \asymp \max\left\{\limsup_{n\to\infty}(n+1)\right.$$
$$\cdot\left\|\left((g\circ\varphi)(\varphi')^2\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(\log n) \tag{106}$$
$$\left.\cdot\left\|\left((g'\circ\varphi)(\varphi')^2+(g\circ\varphi)(\varphi'')\right)\varphi^n\right\|_{v_\beta}\right\}.$$

(iii) *If* $\alpha > 1$, *then*

$$\|C_\varphi I_g\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \asymp \max\left\{\limsup_{n\to\infty}(n+1)^\alpha\right.$$
$$\cdot\left\|\left((g\circ\varphi)(\varphi')^2\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(n+1)^{\alpha-1} \tag{107}$$
$$\left.\cdot\left\|\left((g'\circ\varphi)(\varphi')^2+(g\circ\varphi)(\varphi'')\right)\varphi^n\right\|_{v_\beta}\right\}.$$

**Theorem 14.** *Let* $\varphi$ *be an analytic self-map of* $D$ *and* $g \in H(D)$, *and* $T_gC_\varphi : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta$ *is a bounded operator.*

(i) *If* $0 < \alpha < 1$, *then*

$$\|T_gC_\varphi\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} = 0. \tag{108}$$

(ii) *If* $\alpha = 1$, *then*

$$\|T_gC_\varphi\|_{e,\mathcal{Z}\to\mathcal{Z}^\beta} \asymp \limsup_{n\to\infty}(\log n)\left\|(g'\varphi')\varphi^n\right\|_{v_\beta}. \tag{109}$$

(iii) *If* $1 < \alpha < 2$, *then*

$$\|T_gC_\varphi\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \asymp \limsup_{n\to\infty}(n+1)^{\alpha-1}\left\|(g'\varphi')\varphi^n\right\|_{v_\beta}. \tag{110}$$

(iv) *If* $\alpha = 2$, *then*

$$\|T_gC_\varphi\|_{e,\mathcal{Z}^2\to\mathcal{Z}^\beta} \asymp \max\left\{\limsup_{n\to\infty}(n+1)\right.$$
$$\left.\cdot\left\|\left((g'\varphi')\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(\log n)\left\|(g'')\varphi^n\right\|_{v_\beta}\right\}. \tag{111}$$

(v) *If* $\alpha > 2$, *then*

$$\|T_gC_\varphi\|_{e,\mathcal{Z}^\alpha\to\mathcal{Z}^\beta} \asymp \max\left\{\limsup_{n\to\infty}(n+1)^{\alpha-1}\right.$$
$$\cdot\left\|\left((g'\varphi')\right)\varphi^n\right\|_{v_\beta},\ \limsup_{n\to\infty}(n+1)^{\alpha-2} \tag{112}$$
$$\left.\cdot\left\|(g'')\varphi^n\right\|_{v_\beta}\right\}.$$

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Acknowledgments

The first author was partially supported by the National Natural Science Foundation of China (Grant nos. 11671357, 11571217) and the Natural Science Foundation of Fujian Province, China (Grant no. 2015J01005).

## References

[1] R. Yoneda, "The composition operators on weighted Bloch space," *Archiv der Mathematik*, vol. 78, no. 4, pp. 310–317, 2002.

[2] S. Stevic, "On a new operator from the logarithmic BLOch space to the BLOch-type space on the unit ball," *Applied Mathematics and Computation*, vol. 206, no. 1, pp. 313–320, 2008.

[3] S. Stevic, "Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces," *Utilitas Mathematica*, vol. 77, pp. 167–172, 2008.

[4] S. Ye, "Weighted composition operators from $F(p,q,s)$ into logarithmic Bloch space," *Journal of the Korean Mathematical Society*, vol. 45, no. 4, pp. 977–991, 2008.

[5] S. Ye, "A weighted composition operator between different weighted Bloch-type spaces," *Acta Mathematica Sinica. Chinese Series A*, vol. 50, pp. 927–942, 2007.

[6] S. Ye, "Multipliers and cyclic vectors on the weighted Bloch space," *Mathematical Journal of Okayama University*, vol. 48, pp. 135–143, 2006.

[7] S. Ye, "Weighted composition operator between the little $\alpha$-Bloch spaces and the logarithmic Bloch," *Journal of Computational Analysis and Applications*, vol. 10, no. 2, pp. 243–252, 2008.

[8] K. Esmaeili and M. Lindström, "Weighted composition operators between Zygmund type spaces and their essential norms," *Integral Equations and Operator Theory*, vol. 75, no. 4, pp. 473–490, 2013.

[9] S. Li and S. Stevic, "Weighted composition operators from Zygmund spaces into Bloch spaces," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 825–831, 2008.

[10] S. Li and S. Stevic, "Products of Volterra type operator and composition operator from H∞ and Bloch spaces to Zygmund spaces," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 40–52, 2008.

[11] S. Li and S. Stevic, "Generalized composition operators on Zygmund spaces and Bloch type spaces," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1282–1295, 2008.

[12] S. Stevic and A. K. Sharma, "Composition operators from weighted Bergman-Privalov spaces to Zygmund type spaces on the unit disk," *Annales Polonici Mathematici*, vol. 105, no. 1, pp. 77–86, 2012.

[13] S. Stevic, "On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball," *Abstract and Applied Analysis*, vol. 2010, Article ID 198608, 7 pages, 2010.

[14] S. Stevic, "Composition operators from the Hardy space to the Zygmund-type space on the upper half-plane," *Abstract and Applied Analysis*, vol. 2009, Article ID 161528, 8 pages, 2009.

[15] S. Stevic, "On an integral operator from the Zygmund space to the Bloch-type space on the unit ball," *Glasgow Mathematical Journal*, vol. 51, no. 2, pp. 275–287, 2009.

[16] S. Ye and Q. Hu, "Weighted composition operators on the Zygmund space," *Abstract and Applied Analysis*, vol. 2012, Article ID 462482, 18 pages, 2012.

[17] S. Ye and Z. Zhuo, "Weighted composition operators from Hardy to Zygmund type spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 365286, 10 pages, 2013.

[18] S. Ye and C. Lin, "Composition followed by differentiation on the Zygmund space," *Acta Mathematica Sinica. Chinese Series*, vol. 59, no. 1, pp. 11–20, 2016.

[19] Z. Zhuo and S. Ye, "Volterra-type operators from analytic Morrey spaces to Bloch space," *Journal of Integral Equations and Applications*, vol. 27, no. 2, pp. 289–309, 2015.

[20] C. Pommerenke, "Schlichte FUNktionen und analytische FUNktionen von beschrankter mittlerer Oszillation," *Commentarii Mathematici Helvetici*, vol. 52, no. 4, pp. 591–602, 1977.

[21] E. Wolf, "Products of Volterra type operators and composition operators between weighted Bergman spaces of infinite order and weighted BLOch type spaces," *Georgian Mathematical Journal*, vol. 17, no. 3, pp. 621–627, 2010.

[22] S. Ye, "Products of volterra-type operators and composition operators on logarithmic Bloch space," *WSEAS Transactions on Mathematics*, vol. 12, no. 2, pp. 180–188, 2013.

[23] S. Li and S. Stevic, "Products of integral-type operators and composition operators between Bloch-type spaces," *Journal of Mathematical Analysis and Applications*, vol. 349, no. 2, pp. 596–610, 2009.

[24] O. Hyväarinen, M. Kemppainen, M. Lindström, A. Rautio, and E. Saukko, "The essential norm of weighted composition operators on weighted Banach spaces of analytic functions," *Integral Equations and Operator Theory*, vol. 72, no. 2, pp. 151–157, 2012.

[25] A. Montes-Rodríguez, "Weighted composition operators on weighted Banach spaces of analytic functions," *Journal of the London Mathematical Society. Second Series*, vol. 61, no. 3, pp. 872–884, 2000.

[26] O. Hyväarinen and M. Lindström, "Estimates of essential norms of weighted composition operators between Bloch-type spaces," *Journal of Mathematical Analysis and Applications*, vol. 393, no. 1, pp. 38–44, 2012.

[27] S. Ohno, K. Stroethoff, and R. Zhao, "Weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 191–215, 2003.

[28] S. Stevic, "Essential norms of weighted composition operators from the $\alpha$-Bloch space to a weighted-type space on the unit ball," *Abstract and Applied Analysis*, vol. 2008, Article ID 279691, 11 pages, 2008.

[29] S. Stevic, "Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 1939–1943, 2010.