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Research Article

Diffusion Convection Equation with Variable Nonlinearities

Huashui Zhan

School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China

Correspondence should be addressed to Huashui Zhan; huashuizhan@163.com

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The paper studies diffusion convection equation with variable nonlinearities and degeneracy on the boundary. Unlike the usual Dirichlet boundary value, only a partial boundary value condition is imposed. If there are some restrictions in the diffusion coefficient, the stability of the weak solution based on the partial boundary value condition is obtained. In general, we may obtain a local stability of the weak solutions without any boundary value condition.

1. Introduction

Consider the following diffusion convection equation with variable nonlinearities:

$$u_t = \operatorname{div}\left(a(x)|u|^{\alpha(x)}|\nabla u|^{p(x)-2}\nabla u\right) + \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $p(x)$ is a measurable function. Equation (1) comes from the so-called electrorheological fluids, comes from a motion of an ideal barotropic gas through a porous medium, and comes from the flows in fractured media, and so on (see [1, 2]). If $a(x) \geq a^- > 0$, one can impose the following initial-boundary value conditions:

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

and there are many references recently. We would like to suggest that the first paper where parabolic equations with variable growth exponent are considered is Acerbi et al. [3]; some other interesting works are listed as [4–11] in our references. If $a(x) = 0$, $x \in \partial\Omega$, then the equation is degenerate on the boundary; the author had shown that, in [12], besides the initial value condition (2), only a partial boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{4}$$

is imposed, where $\Sigma_p \subseteq \partial\Omega$ is a relatively open subset. In some cases, $\Sigma_p = \emptyset$; then the solutions are determined by the initial value completely.

Throughout the paper, we assume that $1 < p(x) \in C^1(\overline{\Omega})$ and denote

$$p^+ = \max_{\overline{\Omega}} p(x), \qquad 1 < p^- = \min_{\overline{\Omega}} p(x). \tag{5}$$

The main aim of our paper is to study the stability based on the partial boundary value condition (4).

Theorem 1. *Suppose that $\alpha^- \geq 1$, $a(x) > 0$ when $x \in \Omega$, and $a(x) = 0$ when $x \in \partial\Omega$. Let u, v be two solutions of (1) with the same partial boundary value condition (4) and with the different initial values u_0, v_0, respectively. If*

$$\int_\Omega a^{-1/(p(x)-1)}(x) < \infty, \qquad \eta^{1/p^+-1}\left(\int_{\Omega\setminus\Omega_\eta} |\nabla a|^{p(x)}\,dx\right)^{1/p^+} \leq c, \tag{6}$$

for small positive η, and

$$\int_\Omega a(x)|\nabla u|^{p(x)}\,dx \leq c, \qquad \int_\Omega a(x)|\nabla v|^{p(x)}\,dx \leq c, \tag{7}$$

then

$$\int_\Omega |u(x,t) - v(x,t)|\,dx \le \int_\Omega |u_0 - v_0|\,dx + c \sup_{(x,t)\in\Sigma_p\times(0,T)} |u - v|. \tag{8}$$

Here, $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$.

If $p(x) \equiv p$, $a(x) = d^\gamma(x)$, $d(x) = \operatorname{dist}(x, \partial\Omega)$, the two restrictions of $a(x)$ in (6) are incompatible, and $\int_\Omega d^{-\gamma/(p(x)-1)}(x) < \infty$ implies $\gamma < p - 1$, while $\eta^{1/p^+-1}(\int_{\Omega\setminus\Omega_\eta} |\nabla a|^{p(x)} dx)^{1/p^+} \le c$ implies that $\gamma \ge p - 1$. So, if $p(x) \equiv p$, it is impossible to obtain Theorem 1 (also Theorem 2). However, $p(x)$ is a continuous function; when $p^+ > p^-$, the two restrictions of $a(x)$ in (6) are compatible; for example, if $a(x) = d^\gamma(x)$, by (6), γ satisfies

$$p^- - 1 > \gamma > 1 + \frac{1}{p^-}\left(1 - \frac{2}{p^+}\right). \tag{9}$$

One can see that only if $p^- > 3$, p^+ is large enough, and (9) is true.

If $\alpha = 0$ in Theorem 1, without condition (7), conclusion (8) is true. In other words, we have the following important result.

Theorem 2. *Suppose that $a(x) > 0$ when $x \in \Omega$ and $a(x) = 0$ when $x \in \partial\Omega$. Let u, v be two solutions of equation*

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) + \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{10}$$

with the same partial boundary value condition (4) and with the different initial values u_0, v_0, respectively. Then stability (8) is true only if $a(x)$ satisfies (6).

Theorem 2 (also Theorem 1) has shown an essential difference between (1) and the usual evolutionary p-Laplacian equation. For the usual evolutionary p-Laplacian equation, to obtain the stability of the weak solutions, the Dirichlet boundary value condition (3) is necessary.

In general, if $a(x)$ does not satisfy conditions (6), we have the following local stability.

Theorem 3. *Suppose that $\alpha > 0$, $a(x) > 0$ when $x \in \Omega$, and $a(x) = 0$ when $x \in \partial\Omega$. Let u, v be two solutions of (1) with the initial values u_0, v_0, respectively. If*

$$a(x)|\nabla u|^{p(x)} \le c, \quad a(x)|\nabla v|^{p(x)} \le c, \tag{11}$$

then there exists a constant $\beta \ge 2$ such that

$$\int_\Omega a^\beta |u(x,t) - v(x,t)|^2\,dx \le \int_\Omega a^\beta |u_0 - v_0|^2\,dx \tag{12}$$

which implies that (1) with the initial value (3) is unique.

2. The Definition of the Weak Solutions

Here, the basic function spaces with variable exponents are quoted; for more details, see [13–16] et al. Set

$$C_+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\right\}. \tag{13}$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x), \quad h^- = \inf_{x\in\Omega} h(x). \tag{14}$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue spaces and the variable exponent Sobolev space.

(1) $L^{p(x)}(\Omega)$ Space

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}, \tag{15}$$

and it is equipped with the following Luxemburg's norm:

$$|u|_{L^{p(x)}}(\Omega) = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{16}$$

The space $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$ is a separable, uniformly convex Banach space.

(2) $W^{1,p(x)}(\Omega)$ Space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{17}$$

and it is endowed with the following norm:

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{18}$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$.

Some properties of the function spaces $W^{1,p(x)}(\Omega)$ are quoted in the following lemma.

Lemma 4. *(i) The space $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), |\cdot|_{W^{1,p(x)}(\Omega)})$, and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.*

(ii) $p(x)$-Hölder's inequality: let $q_1(x)$ and $q_2(x)$ be real functions with $1/q_1(x) + 1/q_2(x) = 1$ and $q_1(x) > 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. And, for any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\,dx\right| \le 2|u|_{L^{q_1(x)}(\Omega)}|v|_{L^{q_2(x)}(\Omega)}. \tag{19}$$

(iii)

$$\begin{aligned}
&\text{If } |u|_{L^{p(x)}(\Omega)} = 1, \quad \text{then } \int_\Omega |u|^{p(x)}\,dx = 1,\\
&\text{If } |u|_{L^{p(x)}}(\Omega) > 1,\\
&\text{then } |u|_{L^{p(x)}}^{p^-} \le \int_\Omega |u|^{p(x)}\,dx \le |u|_{L^{p(x)}}^{p^+},\\
&\text{If } |u|_{L^{p(x)}}(\Omega) < 1,\\
&\text{then } |u|_{L^{p(x)}}^{p^+} \le \int_\Omega |u|^{p(x)}\,dx \le |u|_{L^{p(x)}}^{p^-}.
\end{aligned} \tag{20}$$

(iv) If $p_1(x) \le p_2(x)$, then

$$L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega). \tag{21}$$

(v) If $p_1(x) \le p_2(x)$, then

$$W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega). \tag{22}$$

(vi) $p(x)$-Poincarés inequality: if $p(x) \in C(\Omega)$, then there is a constant $C > 0$, such that

$$|u|_{L^{p(x)}}(\Omega) \le C\,|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{23}$$

This implies that $|\nabla u|_{L^{p(x)}}(\Omega)$ and $|u|_{W^{1,p(x)}(\Omega)}$ are equivalent norms of $W_0^{1,p(x)}$.

In [14], Zhikov showed that

$$\begin{aligned}
W_0^{1,p(x)}(\Omega) &\ne \left\{v \in W_0^{1,p(x)}(\Omega)\,|v|_{\partial\Omega} = 0\right\}\\
&= \overset{\circ}{W}{}^{1,p(x)}(\Omega).
\end{aligned} \tag{24}$$

Hence, the property of the space is different from the case when p is a constant. This fact can make the general methods used in studying the well-posedness of the solutions to the evolutionary p-Laplacian equation not be used directly.

If the exponent $p(x)$ is required to satisfy logarithmic Hölder continuity condition,

$$\begin{aligned}
&|p(x) - p(y)| \le \omega(|x-y|),\\
&\qquad \forall x, y \in \Omega,\ |x-y| < \frac{1}{2},
\end{aligned} \tag{25}$$

with

$$\overline{\lim_{s\to 0^+}}\,\omega(s)\ln\left(\frac{1}{s}\right) = C < \infty, \tag{26}$$

then

$$W_0^{1,p(x)}(\Omega) = \overset{\circ}{W}{}^{1,p(x)}(\Omega). \tag{27}$$

Lemma 5. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and $p \in C_+(\overline{\Omega})$ with $1 < p^- \le p^+ < N$ satisfy the log-Hölder continuity (25). If $r \in L^\infty(\Omega)$ with $r_- > 1$ satisfies*

$$r(x) \leqslant p^*(x) := \frac{Np(x)}{N - p(x)} \quad \forall x \in \Omega, \tag{28}$$

then we have

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \tag{29}$$

and the embedding is compact if $\inf_{x\in\Omega}(p^*(x) - r(x)) > 0$.

Remark 6. Furthermore, under the same assumptions as in the above lemma, if we remove the log-Hölder continuity condition (25), then there is also a continuous and compact embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \quad \forall x \in \Omega, \tag{30}$$

where $p, r \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$.

Definition 7. A function $u(x,t)$ is said to be a weak solution of (1) with the initial value (2) and the partial boundary value condition (4), if

$$\begin{aligned}
&u \in L^\infty(Q_T),\\
&\frac{\partial u}{\partial t} \in L^1(Q_T),\\
&a(x)\,|u|^{\alpha(x)}\,|\nabla u|^{p(x)} \in L^1(Q_T),
\end{aligned} \tag{31}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\begin{aligned}
&\iint_{Q_T}\left(\frac{\partial u}{\partial t}\varphi + a(x)\,|u|^{\alpha(x)}\,|\nabla u|^{p(x,t)-2}\,\nabla u\cdot\nabla\varphi\right.\\
&\qquad \left. + b_i(u)\,\varphi_{x_i}\right)dx\,dt = 0.
\end{aligned} \tag{32}$$

The initial value (2) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega |u(x,t) - u(x,0)|\,dx = 0. \tag{33}$$

The partial boundary value condition (4) is satisfied in the sense of the trace.

If u_0 satisfies

$$\begin{aligned}
&u_0 \in L^\infty(Q_T),\\
&a(x)\,|u_0|^{\alpha(x)}\,|\nabla u|^{p(x)} \in L^1(Q_T),
\end{aligned} \tag{34}$$

it is not difficult to prove there exists a weak solution in the sense of Definition 7.

Definition 8. A function $u(x,t)$ is said to be a solution of (1) with the initial value (2), if u satisfies (31) and

$$\begin{aligned}
&\iint_{Q_T}\left[u_t(\varphi_1\varphi_2) + a(x)\,|u|^{\alpha(x)}\,|\nabla u|^{p(x)-2}\,\nabla u\right.\\
&\qquad \left.\cdot\nabla(\varphi_1\varphi_2) + b_i(u)\,(\varphi_1\varphi_2)_{x_i}\right]dx\,dt = 0,
\end{aligned} \tag{35}$$

where $\varphi_2 \in C_0^1(Q_T)$, and if we denote that

$$W_\alpha^{1,p(x)} = \{u : u \text{ satisfies } (31)\}, \tag{36}$$

then, for any given t, $\varphi_1(x,t) \in W_\alpha^{1,p(x)}$ and, for any given x, $|\varphi_1(x,t)| \le c$. The initial value (2) is satisfied in the sense of (33).

Based on the existence of the weak solution in the sense of Definition 7, one also can be able to prove the existence of the weak solution in the sense of Definition 8. Since we mainly are concerned with the stability of the weak solutions, we are not ready to give the proof of the existence of the weak solutions in what follows.

3. The Proofs of Theorems

Proof of Theorem 1. Let u, v be two solutions of (1) with the partial homogeneous boundary values (4) and with the initial values u_0, v_0, respectively. From the definition of the weak solution in the sense of Definition 7, for all $\varphi \in C_0^1(Q_T)$,

$$\begin{aligned}\int_\Omega \varphi \frac{\partial (u-v)}{\partial t} dx = &- \int_\Omega a(x) \\ &\cdot \left(|u|^{\alpha(x)} |\nabla u|^{p(x)-2} \nabla u - |v|^\alpha |\nabla v|^{p(x)-2} \nabla v\right) \\ &\cdot \nabla \varphi \, dx.\end{aligned} \tag{37}$$

For small $\eta > 0$, let

$$\begin{aligned}S_\eta(s) &= \int_0^s h_\eta(\tau)\, d\tau, \\ h_\eta(s) &= \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+.\end{aligned} \tag{38}$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{aligned}h_\eta(s) &\geq 0, \\ \left|s h_\eta(s)\right| &\leq 1, \\ \left|S_\eta(s)\right| &\leq 1; \\ \lim_{\eta\to 0} S_\eta(s) &= \operatorname{sgn} s, \\ \lim_{\eta\to 0} s S_\eta'(s) &= 0.\end{aligned} \tag{39}$$

Let

$$a_\eta(x) = \begin{cases} \dfrac{a(x)}{\eta}, & a(x) < \eta, \\ 1, & a(x) \geq \eta. \end{cases} \tag{40}$$

By a process of limit, we can choose $a_\eta S_\eta(u-v)$ as the test function; then

$$\begin{aligned}&\int_\Omega a_\eta S_\eta(u-v) \frac{\partial (u-v)}{\partial t} dx \\ &\quad + \int_\Omega a(x) a_\eta |u|^\alpha \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v\right) \\ &\quad \cdot \nabla (u-v) S_\eta'(u-v)\, dx \\ &\quad + \int_\Omega a(x) a_\eta \left(|u|^\alpha - |v|^\alpha\right) |\nabla v|^{p(x)-2} \nabla v \\ &\quad \cdot \nabla (u-v) S_\eta'(u-v)\, dx \\ &\quad + \int_\Omega a(x) |u|^\alpha \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v\right) \\ &\quad \cdot \nabla a_\eta S_\eta(u-v)\, dx \\ &\quad + \int_\Omega a(x) \left(|u|^\alpha - |v|^\alpha\right) |\nabla v|^{p(x)-2} \nabla v \\ &\quad \cdot \nabla a_\eta S_\eta(u-v)\, dx + \int_\Omega \left(b_i(u) - b_i(v)\right) \\ &\quad \cdot (u-v)_{x_i} S_\eta'(u-v) a_\eta(x)\, dx \\ &\quad + \int_\Omega \left(b_i(u) - b_i(v)\right) \cdot S_\eta(u-v) a_{\eta x_i}(x)\, dx = 0.\end{aligned} \tag{41}$$

Thus

$$\begin{aligned}&\lim_{\eta\to 0} \int_\Omega a_\eta S_\eta(u-v) \frac{\partial (u-v)}{\partial t} dx = \frac{d}{dt} \|u-v\|_1, \\ &\int_\Omega a_\eta a(x) |u|^\alpha \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v\right) \\ &\quad \cdot \nabla (u-v) S_\eta'(u-v)\, dx \geqslant 0.\end{aligned} \tag{42}$$

Thus, by (ii) and (iii) in Lemma 4,

$$\begin{aligned}&\left|\int_\Omega a_\eta a(x) \left(|u|^\alpha - |v|^\alpha\right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla (u-v) \right. \\ &\quad \left. \cdot S_\eta'(u-v)\, dx\right| \leq \int_\Omega a(x) \left||u|^\alpha - |v|^\alpha\right| |\nabla v|^{p(x)-1} \\ &\quad \cdot (|\nabla u| + |\nabla v|) S_\eta'(u-v)\, dx \leq c \int_\Omega a(x) \left||u|^\alpha \right. \\ &\quad \left. - |v|^\alpha\right| |\nabla v|^{p(x)-1} |\nabla u| S_\eta'(u-v)\, dx + \int_\Omega a(x) \\ &\quad \cdot \left||u|^\alpha - |v|^\alpha\right| |\nabla v|^{p(x)} S_\eta'(u-v)\, dx \leq c \left(\int_\Omega a(x) \right. \\ &\quad \left. \cdot \left||u|^\alpha - |v|^\alpha\right| |\nabla v|^{p(x)} S_\eta'(u-v)\, dx\right)^{1/q_1} \left(\int_\Omega a(x) \right. \\ &\quad \left. \cdot \left||u|^\alpha - |v|^\alpha\right| |\nabla u|^{p(x)} S_\eta'(u-v)\, dx\right)^{1/p_1} + \int_\Omega a(x) \\ &\quad \cdot \left||\nabla v|^{p(x)} \left||u|^\alpha - |v|^\alpha\right|\right| S_\eta'(u-v)\, dx.\end{aligned} \tag{43}$$

Here, $p_1 = p^+$ or p^- according to (iii) of Lemma 4; also q_1 has the same meaning if we denote that $q(x) = p(x)/(p(x)-1)$.

Using the mean theorem, by (7) and $\alpha^- \geq 1$,

$$\begin{aligned}&\int_\Omega a(x)\, a_\eta \left||u|^\alpha - |v|^\alpha\right| |\nabla v|^{p(x)} S'_\eta(u-v)\, dx \\ &= \int_\Omega a(x)\, a_\eta \left|\alpha(x) |\xi|^{\alpha-1} |u-v|\right| |\nabla v|^{p(x)} \\ &\cdot S'_\eta(u-v)\, dx \leq c \int_\Omega a(x) |\nabla v|^{p(x)} |u-v| \\ &\cdot S'_\eta(u-v)\, dx\end{aligned} \tag{44}$$

which goes to zero as $\eta \to 0$ by the Lebesgue dominated convergence theorem. Then, letting $\eta \to 0$ in (43), we have

$$\begin{aligned}&\lim_{\eta\to 0}\left|\int_\Omega a_\eta a(x)\left(|u|^\alpha - |v|^\alpha\right)|\nabla v|^{p(x)-2}\nabla v \right. \\ &\left. \cdot \nabla(u-v) S'_\eta(u-v)\, dx\right| = 0.\end{aligned} \tag{45}$$

Let $\Omega_\eta = \{x \in \Omega : a(x) > \eta\}$.

$$\begin{aligned}&\left|\int_\Omega a(x)|u|^\alpha\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\right. \\ &\left.\cdot \nabla a_\eta S_\eta(u-v)\, dx\right| \\ &= \left|\int_{\Omega\setminus\Omega_\eta} a(x)|u|^\alpha\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\right. \\ &\left.\cdot \nabla a_\eta S_\eta(u-v)\, dx\right| \leq c\left\|a^{(p(x)-1)/p(x)}\left(|\nabla u|^{p(x)-1}\right.\right. \\ &\left.\left. + |\nabla v|^{p(x)-1}\right)\right\|_{L^{q(x)}(\Omega\setminus\Omega_\eta)}\left\|\frac{1}{\eta}a^{1/p(x)}\nabla a\right\|_{L^{p(x)}(\Omega\setminus\Omega_\eta)} \\ &\leq c\left[\left(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla u|^{p(x)}\right)^{1/q^+}\right. \\ &\left. + \left(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla v|^{p(x)}\right)^{1/q^+}\right]\end{aligned} \tag{46}$$

which goes to 0 as $\eta \to 0$ by

$$\begin{aligned}&\int_\Omega a(x)|\nabla u|^{p(x)}\, dx < \infty, \\ &\int_\Omega a(x)|\nabla v|^{p(x)}\, dx < \infty.\end{aligned} \tag{47}$$

Here, we had used the fact

$$\begin{aligned}&\left\|\frac{1}{\eta}a^{1/p(x)}\nabla a\right\|_{L^{p(x)}(\Omega\setminus\Omega_\eta)} \\ &\leq \frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla a|^{p(x)}\, dx\right)^{1/p^+} \\ &\leq \eta^{1/p^+-1}\left(\int_{\Omega\setminus\Omega_\eta}|\nabla a|^{p(x)}\, dx\right)^{1/p^+} \leq c,\end{aligned} \tag{48}$$

by (6).

For any given $t \in (0,T)$, let $D_\eta = \{x \in \Omega : |u-v| < \eta\}$. Then

$$\begin{aligned}&\left|\int_\Omega \left(b_i(u) - b_i(v)\right)\cdot(u-v)_{x_i} S'_\eta(u-v)\, a_\eta(x)\, dx\right| \\ &= \left|\int_{\{x\in\Omega:|u-v|<\eta\}}\left(b_i(u) - b_i(v)\right)\right. \\ &\left.\cdot (u-v)_{x_i} S'_\eta(u-v)\, a_\eta(x)\, dx\right| \leq \left\|a^{1/p(x)}\left(b_i(u)\right.\right. \\ &\left.\left. - b_i(v)\right) S'_\eta(u-v)\, a_\eta(x)\right\|_{L^{q(x)}(D_\eta)}\left\|a^{1/p(x)}(u\right. \\ &\left. - v)_{x_i}\right\|_{L^{p(x)}(D_\eta)},\end{aligned} \tag{49}$$

which goes to zero when $\eta \to 0$ by the assumption that

$$\int_\Omega a^{-1/(p(x)-1)}(x)\, dx < \infty. \tag{50}$$

At the same time,

$$\begin{aligned}&\lim_{\eta\to 0}\left|\int_\Omega \left(b_i(u) - b_i(v)\right)\cdot S_\eta(u-v)\, a_{\eta x_i}(x)\, dx\right| \\ &\leq c\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}\left|\left(b_i(u) - b_i(v)\right) S_\eta(u-v)\, a_{x_i}(x)\right| dx \\ &\leq c\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}\left|(u-v) S_\eta(u-v)\, a_{x_i}(x)\right| dx \\ &\leq c\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}|u-v|\, dx \leq c\sup_{(x,t)\in\Sigma_p\times(0,T)}|u-v|.\end{aligned} \tag{51}$$

Now, let $\eta \to 0$ in (41). Then

$$\frac{d}{dt}\|u-v\|_1 \leq c\sup_{(x,t)\in\Sigma_p\times(0,T)}|u-v|. \tag{52}$$

It implies that

$$\begin{aligned}&\int_\Omega |u(x,t) - v(x,t)|\, dx \\ &\leqslant \int_\Omega \left|u_0 - v_0\right| dx + c\sup_{x\in\Sigma_p}|u-v|, \quad \forall t \in [0,T).\end{aligned} \tag{53}$$

□

Proof of Theorem 2. From Definition 7, if $\alpha = 0$, one can see that condition (7) is naturally true. In other words, condition (7) is not necessary. □

Proof of Theorem 3. Let u, v be two solutions of (1) with the initial values $u_0(x)$, $v_0(x)$, respectively. For a small positive constant, $\beta \geq 2$, we may choose $\chi_{[\tau,s]}(u-v)a^\beta$ as a test function. Then

$$\begin{aligned}\iint_{Q_{\tau s}} (u-v)\,\xi_\lambda \frac{\partial (u-v)}{\partial t} dx\,dt = &-\iint_{Q_{\tau s}} a(x)\,|u|^{\alpha(x)} \\ &\cdot \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla\left[(u-v)\right. \\ &\left.\cdot a^\beta\right] dx\,dt - \iint_{Q_{\tau s}} a(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right) \\ &\cdot |\nabla v|^{p(x)-2}\nabla v \nabla\left[(u-v)\,a^\beta\right] \\ &- \iint_{Q_{\tau s}} \left[b_i(u) - b_i(v)\right]\left[(u-v)\,a^\beta\right]_{x_i} dx\,dt.\end{aligned} \tag{54}$$

We have

$$\begin{aligned}&\iint_{Q_{\tau s}} a^{1+\beta}(x)\,|u|^{\alpha(x)} \\ &\quad\cdot\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v)\,dx\,dt \\ &\quad\geq 0, \\ &\iint_{Q_{\tau s}} a(x)\,|u|^{\alpha(x)}\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \\ &\quad\cdot(u-v)\,\nabla a^\beta dx\,dt \leq \iint_{Q_{\tau s}} |u-v|\,a(x)\,|u|^{\alpha(x)} \\ &\quad\cdot\left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)\left|\nabla a^\beta\right| dx\,dt \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a(x)\left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right)dx\,dt\right)^{1/q_1} \\ &\quad\cdot\left(\int_\tau^s\int_\Omega a^{1+p(x)(\beta-1)}\,|u-v|^{p(x)}\,dx\,dt\right)^{1/p_1} \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a^{1+p(x)(\beta-1)}\,|u-v|^{p(x)}\,dx\,dt\right)^{1/p_1}.\end{aligned} \tag{55}$$

Here, we have used the fact that $|\nabla a| \leq c$. Now, since $\beta \geq 2$, by Hölder inequality, we have

$$\begin{aligned}&\left|\iint_{Q_{\tau s}} (u-v)\,a(x)\,|u|^{\alpha(x)}\right. \\ &\quad\left.\cdot\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla a^\beta dx\,dt\right| \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a^\beta\,|u-v|^{p(x)}\,dx\,dt\right)^{1/p_1}.\end{aligned} \tag{56}$$

Let $\Omega_1 = \{x \in \Omega : p(x) \geq 2\}$, $\Omega_2 = \{x \in \Omega : 1 < p(x) < 2\}$; then

$$\begin{aligned}&\left(\int_\tau^s\int_{\Omega_1} a^\beta\,|u-v|^{p(x)}\,dx\,dt\right)^{1/p_1} \\ &\quad\leq c\left(\int_\tau^s\int_{\Omega_1} a^\beta\,|u-v|^2\,dx\,dt\right)^{1/p_1}.\end{aligned} \tag{57}$$

By Hölder inequality

$$\begin{aligned}&\left(\int_\tau^s\int_{\Omega_2} a^\beta\,|u-v|^{p(x)}\,dx\,dt\right)^{1/p_1} \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a^\beta\,|u-v|^2\,dx\,dt\right)^{(1/p_1)(1/p_{12})},\end{aligned} \tag{58}$$

where $p_{12} = (2/p(x))^+$ or $p_{12} = (2/p(x))^-$. By (54)–(58), we have

$$\begin{aligned}&\left|\iint_{Q_{\tau s}} (u-v)\,a(x)\,|u|^{\alpha(x)}\right. \\ &\quad\left.\cdot\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla a^\beta dx\,dt\right| \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a^\beta\,|u-v|^2\,dx\,dt\right)^{1/l},\end{aligned} \tag{59}$$

where $l > 1$.

At the same time, we have

$$\begin{aligned}&\iint_{Q_{\tau s}} a(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right) \\ &\quad\cdot|\nabla v|^{p(x)-2}\nabla v\nabla\left[(u-v)\,a^\beta\right]dx\,dt \\ &\quad= \iint_{Q_{\tau s}} a(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right) \\ &\quad\cdot|\nabla v|^{p(x)-2}\nabla v\nabla(u-v)\,a^\beta dx\,dt + \iint_{Q_{\tau s}} a(x,t) \\ &\quad\cdot\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right)|\nabla v|^{p(x)-2}\nabla v\nabla a\,(u-v) \\ &\quad\cdot\beta a^{\beta-1}dx\,dt.\end{aligned} \tag{60}$$

By condition (11), $a(x)|\nabla u|^{p(x)} \leq c$, $a(x)|\nabla v|^{p(x)} \leq c$, using Young inequality; by (60), it is easy to show that

$$\begin{aligned}&\iint_{Q_{\tau s}} a(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right) \\ &\quad\cdot|\nabla v|^{p-2}\nabla v\nabla\left[(u-v)\,a^\beta\right]dx\,dt \\ &\quad\leq c\left(\int_\tau^s\int_\Omega a^\beta\,|u-v|^2\,dx\,dt\right)^{1/k},\end{aligned} \tag{61}$$

where $k > 1$.

At last

$$\begin{aligned}
&\iint_{Q_{\tau s}} [b_i(u) - b_i(v)] \left[(u-v)a^{\beta}\right]_{x_i} dx\,dt \\
&\quad = \iint_{Q_{\tau s}} [b_i(u) - b_i(v)](u-v)a^{\beta}_{x_i} dx\,dt \\
&\qquad + \iint_{Q_s} [b_i(u) - b_i(v)](u-v)_{x_i} a^{\beta} dx\,dt,
\end{aligned} \tag{62}$$

and since b_i is a Lipschitz function, $u, v \in L^{\infty}(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_{\tau s}} [b_i(u) - b_i(v)](u-v)a^{\beta}_{x_i} dx\,dt \\
&\quad = \beta \int_{\tau}^{s}\int_{\Omega} [b_i(u) - b_i(v)](u-v)a^{\beta-1}\left|a_{x_i}\right| dx \\
&\quad \le c\int_{\tau}^{s}\int_{\Omega} |u-v|^2 a^{\beta-1} dx\,dt \\
&\quad \le c\int_{\tau}^{s}\int_{\Omega} a^{\beta-1}|u-v|\,dx\,dt \\
&\quad = c\int_{\tau}^{s}\int_{\Omega} a^{\beta/2}|u-v|\,a^{\beta/2-1} dx\,dt \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{\beta}|u-v|^2 dx\right)^{1/2}\left(\int_{\tau}^{s}\int_{\Omega} a^{\beta-2} dx\right)^{1/2} \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{\beta}|u-v|^2 dx\right)^{1/2},
\end{aligned} \tag{63}$$

only if $\beta \ge 2$.

$$\begin{aligned}
&\iint_{Q_{\tau s}} [b_i(u) - b_i(v)](u-v)_{x_i} a^{\beta} dx\,dt \\
&\quad \le \int_{\tau}^{s} \left\|a^{(\beta-\alpha/p(x))}\left(\left|b_i(u) - b_i(v)\right|\right)\right\|_{L^{q(x)}(\Omega)} \\
&\quad \cdot \left\|a^{1/p(x)}\left|(u-v)_{x_i}\right|\right\|_{L^{p(x)}(\Omega)} dt \\
&\quad \le \int_{\tau}^{s}\left(\int_{\Omega} a^{(\beta-1/p(x))q(x)}\left(\left|b_i(u) - b_i(v)\right|\right)^{q(x)} dx\right)^{1/q_1} \\
&\quad \cdot \left(\int_{\Omega} a(x)\left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right) dx\right)^{1/p_1} dt \\
&\quad \le \int_{\tau}^{s}\left(\int_{\Omega} a^{(\beta-1/p(x))q(x)}\left(\left|b_i(u) - b_i(v)\right|\right)^{q(x)} dx\right)^{1/q_1} dt \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{(\beta-1/p(x))q(x)}\left(\left|b_i(u) - b_i(v)\right|\right)^{q(x)} dx\,dt\right)^{1/q_1} \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{(\beta-1/p(x))q(x)}|u-v|^{q(x)-1}|u-v|\,dx\,dt\right)^{1/q_1} \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{k}|u-v|\,dx\,dt\right)^{1/q_1} \\
&\quad \le c\left(\int_{\tau}^{s}\int_{\Omega} a^{\beta}|u-v|^2 dx\,dt\right)^{1/2q_1}.
\end{aligned} \tag{64}$$

Here

$$k = \left(\beta - \frac{1}{p(x)} + 1\right)\frac{p(x)}{p(x)-1} - 1 = \beta\frac{p(x)}{p(x)}, \tag{65}$$

and clearly $2k \ge \beta$.

Since

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u-v)a^{\beta}\frac{\partial(u-v)}{\partial t} dx\,dt \\
&\quad = \int_{\Omega} a^{\beta}[u(x,s) - v(x,s)]^2 dx \\
&\quad - \int_{\Omega} a^{\beta}[u(x,\tau) - v(x,\tau)]^2 dx,
\end{aligned} \tag{66}$$

then, by (54)–(66), we have

$$\begin{aligned}
&\int_{\Omega} a^{\beta}[u(x,s) - v(x,s)]^2 dx \\
&\quad - \int_{\Omega} a^{\beta}[u(x,\tau) - v(x,\tau)]^2 dx \\
&\quad \le c\left(\int_{0}^{s}\int_{\Omega} a^{\beta}|u(x,t) - v(x,t)|^2 dx\,dt\right)^{q},
\end{aligned} \tag{67}$$

where $q < 1$. By (67), it is easy to obtain the local stability (12), and we omit the details here. □

4. Conclusions

The equation considered in the paper comes from electrorheological fluids, which may be double degenerate or singular. Moreover, the diffusion coefficient is degenerate on the boundary; then the solutions generally lack the regularity to define the trace on the boundary. The facts make it difficult to obtain the stability of the weak solutions. By introducing a new kind of the weak solution, the paper successfully overcomes the difficulty. Moreover, importantly, the main result (Theorem 1) shows that the electrorheological fluid theory must be complicated compared to the non-Newtonian fluid theory.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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