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Research Article

On a Fourth-Order Boundary Value Problem at Resonance

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We investigate the spectrum structure of the eigenvalue problem $\{u^{(4)}(x)=\lambda u(x),\ x\in(0,1);\ u(0)=u(1)=u'(0)=u'(1)=0\}$. As for the application of the spectrum structure, we show the existence of solutions of the fourth-order boundary value problem at resonance $\{-u^{(4)}(x)+\lambda_1 u(x)+g(x,u(x))=h(x),\ x\in(0,1);\ u(0)=u(1)=u'(0)=u'(1)=0\}$, which models a statically elastic beam with both end-points being cantilevered or fixed, where λ_1 is the first eigenvalue of the corresponding eigenvalue problem and nonlinearity g may be unbounded.

1. Introduction

Starting from the seminal paper of Landesman and Lazer [1], the existence and multiplicity of solutions of nonlinear second-order boundary value problem at resonance,

$$\begin{gathered} u''(x)+\pi^2 u(x)+g(x,u(x))=e(x),\quad x\in(0,1),\\ u(0)=u(1)=0, \end{gathered}\tag{1}$$

and its general case have been extensively studied; see Gupta [2, 3], Iannacci and Nkashama [4, 5], Costa and Goncalves [6], Ambrosetti and Mancini [7], Fonda and Habets [8], Các [9], and Ahmad [10] and the references therein. Because of the linear operator $\mathscr{L}: D(\mathscr{L})\to L^2(0,1)$,

$$\begin{gathered} \mathscr{L}u := u''+\pi^2 u,\\ u\in D(\mathscr{L}) := \left\{y\in L^2(0,1): u(0)=u(1)=0\right\} \end{gathered}\tag{2}$$

is not reversible; this kind of problems as (1) is of problems *at resonance*.

In the past twenty years, the existence and multiplicity of solutions (or positive solutions) of nonlinear fourth-order boundary value problems at *nonresonance* case have been investigated by many authors. Especially, many works address the nonlinear fourth-order differential equation of the following form:

$$u^{(4)}(x)=g\left(x,u(x),u'(x),u''(x),u'''(x)\right),\quad x\in(0,1),\tag{3}$$

with one of the following sets of boundary conditions:

(i) Both end-points simply supported conditions:

$$u(0)=u(1)=u''(0)=u''(1)=0\quad (\text{Navier boundary condition}).\tag{4}$$

(ii) Both end-points cantilevered or fixed conditions:

$$u(0)=u(1)=u'(0)=u'(1)=0\quad (\text{Dirichlet boundary condition}).\tag{5}$$

(iii) One end simply supported and the other end sliding clamped conditions:

$$u(0)=u''(0)=u'(1)=u'''(1)=0.\tag{6}$$

See Rynne [11], Korman [12], Ma et al. [13–15], Cabada et al. [16, 17], Vrabel [18], Schröder [19], Drábek and Holubová

[20], Webb et al. [21], Chu and O'Regan [22], and Cid et al. [23] for references along this line.

However, relatively little is known about the fourth-order problem at resonance; see Gupta et al. [24, 25], Jurkiewicz [26], and Iannacci and Nkashama [27]. The likely reason is that the spectrum theory of fourth-order operators is not available.

The purpose of this paper is to show the existence of solutions of the fourth-order boundary value problem at resonance:

$$\begin{gathered} -u^{(4)}(x) + \lambda_1 u(x) + g(x, u(x)) = h(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{7}$$

which models a statically elastic beam with both end-points being cantilevered or fixed, where $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a L^2-Carathéodory function, $h \in L^2(0,1)$, and λ_1 is the first eigenvalue of the corresponding linear eigenvalue problem. More precisely, we provide a sufficient condition for the solvability of problem (7) in which nonlinearity g is not necessarily needed to satisfy the Landesman-Lazer type condition or the monotonicity assumption.

To do this, we investigate the spectrum structure of the linear eigenvalue problem:

$$\begin{gathered} u^{(4)}(x) = \lambda u(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{gathered} \tag{8}$$

We will show that the eigenvalues of (8) form a sequence:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \longrightarrow +\infty. \tag{9}$$

Moreover, for each $j \in \mathbb{N}$, λ_j ($\lambda_j = m_j^4$, m_j is the simple root of the equation $\cos m \cosh m - 1 = 0$) is simple, and the corresponding eigenfunction is φ_j, which forms (with a suitable normalization) an orthogonal system of $L^2(0,1)$. Since

$$\begin{gathered} u^{(4)}(x) = m_1^4 u(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{gathered} \tag{10}$$

has nontrivial solution $\sin m_1 x - \sinh m_1 x + ((\sin m_1 - \sinh m_1)/(\cos m_1 - \cosh m_1))(\cosh m_1 x - \cos m_1 x)$, problem (7) is a problem in resonance.

We refer, for motivations and results, to the classical papers of [4] for the second-order boundary value problems at resonance and [24] for the fourth-order boundary value problems which are simply supported at both ends and are at resonance. In this paper, we will use the classical spaces $C^k[0,1]$, $L^k(0,1)$, and $L^\infty(0,1)$; we shall make use, in what follows, of the Sobolev spaces $H^k(0,1)$ and $H_0^k(0,1)$; we refer the reader to see [28] for their definitions and properties.

The rest of the paper is arranged as follows. In Section 2, we investigate the spectrum structure of eigenvalue problem (8). In Section 3, we give some preliminary results that are needed to apply Leray-Schauder continuation method to obtain the existence of solutions for problem (7). Finally, Section 4 is devoted to stating and proving our main result.

2. The Eigenvalue Problem

In this section, we consider the linear eigenvalue problem:

$$\begin{gathered} u^{(4)}(x) = \lambda u(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{gathered} \tag{11}$$

Lemma 1. *The equation*

$$\cos m \cosh m - 1 = 0, \quad m \in \mathbb{R}^+ \tag{12}$$

has infinitely many simple roots

$$0 < m_1 < m_2 < m_3 \cdots \longrightarrow +\infty. \tag{13}$$

Moreover,

$$\begin{gathered} m_{2k-1} \in \left(\left(2k - \frac{1}{2}\right)\pi, 2k\pi\right), \\ m_{2k} \in \left(2k\pi, \left(2k + \frac{1}{2}\right)\pi\right) \end{gathered} \tag{14}$$

for $k \in \mathbb{N}$.

Proof. Let

$$\gamma(m) = \cos m \cosh m - 1, \quad m \in \mathbb{R}^+. \tag{15}$$

It is easy to check that, for $k \in \mathbb{N}$,

$$\begin{gathered} \gamma((2k-1)\pi) < 0, \\ \gamma(2k\pi) > 0. \end{gathered} \tag{16}$$

We claim that $\gamma(m)$ has exactly one root $m_j \in [j\pi, (j+1)\pi]$; moreover, for any $j \in \mathbb{N}$, m_j is simple. Assume that the claim is not true. Then, the following two cases must occur.

Case 1. There are three zeros in $(j_0\pi, (j_0+1)\pi)$ for some $j_0 \in \mathbb{N}$. In this case, we may find $\tau \in (j_0\pi, (j_0+1)\pi)$ such that

$$\gamma''(\tau) = 0. \tag{17}$$

However, this contradicts the fact that

$$\gamma''(m) = -2 \sin m \sinh m. \tag{18}$$

Case 2. There is a double zero $\hat{t} \in (j_0\pi, (j_0+1)\pi)$ for some $j_0 \in \mathbb{N}$. In this case, we only deal with case j_0 being odd. Case j_0 is even and can be treated similarly.

Since

$$\begin{gathered} \gamma(j_0\pi) < 0, \\ \gamma((j_0+1)\pi) > 0, \end{gathered} \tag{19}$$

we may assume that there exists $\hat{t} \in (j_0\pi, (j_0+1)\pi)$ such that

$$\begin{gathered} \gamma(m) < 0, \quad m \in (j\pi, \hat{t}), \\ \gamma(\hat{t}) = \gamma'(\hat{t}) = 0. \end{gathered} \tag{20}$$

Combining this with the fact $\gamma''(\hat{t}) > 0$, it concludes that $\gamma(m) > 0$ in some left neighborhood of $\hat{t}$. However, this is a contradiction. □

Lemma 2. *The linear eigenvalue problem (11) has infinitely many eigenvalues:*

$$\lambda_j = m_j^4 \quad j \in \mathbb{N}, \tag{21}$$

and the eigenfunction corresponding to λ_j is given by

$$\begin{aligned}\varphi_j(x) &= \sin m_j x - \sinh m_j x \\ &\quad + \frac{\sin m_j - \sinh m_j}{\cos m_j - \cosh m_j}\left(\cosh m_j x - \cos m_j x\right).\end{aligned} \tag{22}$$

Moreover, $\varphi_k \in S_{k,+}$, where $S_{k,+}$ denote the set of $u \in C^3[0,1]$ such that

(i) *u has only simple zeros in $(0,1)$ and has exactly $k-1$ such zeros;*

(ii) *$u''(0) > 0$ and $u''(1) \neq 0$.*

Proof. By [11, P. 308], we know problem (11) has a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ with $\lim_{k\to\infty}\lambda_k = +\infty$. For any given $k \in \mathbb{N}$, each eigenvalue λ_k is simple and has a corresponding eigenfunction φ_k satisfying (i) and (ii). By a direct calculation, we have (21) and (22). □

3. Preliminaries

Throughout the paper, we assume that

(H0) $p \in L^\infty(0,1)$ such that, for a.e. $x \in (0,1)$, $0 \le p(x) \le m_2^4 - m_1^4$; moreover $p(x) < m_2^4 - m_1^4$ on a subset of $(0,1)$ of positive measure.

Define a linear operator $L : D(L) \subset L^2(0,1) \to L^2(0,1)$ by

$$L(u) := u^{(4)} + \lambda_1 u, \tag{23}$$

where $D(L) = H^4(0,1) \cap H_0^2(0,1) := H$. Then L is the linear self-adjoint operator, and thus $L^2(0,1)$ admits the orthogonal direct sum decomposition $L^2(0,1) = \mathcal{N} \oplus \mathcal{R}$, where $\mathcal{N}$ is the one-dimensional null space of L and $\mathcal{R}$ is the range space of L, namely,

$$\begin{aligned}\mathcal{N} &= \left\{y \in L^2(0,1) \mid y = s\varphi_1 \text{ for some } s \in \mathbb{R}\right\}, \\ \mathcal{R} &= \left\{y \in L^2(0,1) \mid \int_0^1 y(x)\varphi_1(x)\,dx = 0\right\}.\end{aligned} \tag{24}$$

Therefore each $u \in H_0^2(0,1) \subset L^2(0,1)$ has a unique decomposition:

$$u = s\varphi_1 + w := \overline{u}(x) + \widetilde{u}(x), \tag{25}$$

where $s \in \mathbb{R}$, $w \in \mathcal{R}$, so that, with obvious notations, $H_0^2(0,1) = \overline{H}_0^2(0,1) + \widetilde{H}_0^2(0,1)$.

Since $u \in H_0^2(0,1) \subset L^2(0,1)$, it follows that u has the Fourier series expansion:

$$\begin{aligned}u(x) &= \sum_{j=1}^{\infty} s_j\varphi_j(x), \\ s_j &= \int_0^1 u(x)\varphi_j(x)\,dx.\end{aligned} \tag{26}$$

By (25), we observe that

$$\begin{aligned}\overline{u}(x) &= s_1\varphi_1(x), \\ \widetilde{u}(x) &= \sum_{j=2}^{\infty} s_j\varphi_j(x).\end{aligned} \tag{27}$$

Lemma 3. *Assume that p satisfies (H0). Let $\sigma > 0$ and $q \in L^\infty(0,1)$ satisfy*

$$0 \le q(x) \le p(x) + \sigma \quad \textit{for a.e. } x \in (0,1). \tag{28}$$

Then there exists a constant $\delta = \delta(p) > 0$ such that, for all $u \in H$, we have

$$\begin{aligned}&\int_0^1 \left[-u^{(4)}(x) + \lambda_1 u(x) + q(x)u(x)\right] \\ &\quad \cdot (\overline{u}(x) - \widetilde{u}(x))\,dx \ge (\delta - \sigma)\|\widetilde{u}\|_{H^2}^2.\end{aligned} \tag{29}$$

Proof. We will divide the proof into three steps.

Step 1. It follows from Lemma 2 that, for all $u \in H$,

$$\begin{aligned}\overline{u}^{(4)}(x) &= \lambda_1\overline{u}(x), \quad x \in (0,1), \\ u(0) &= u(1) = u'(0) = u'(1) = 0.\end{aligned} \tag{30}$$

Multiplying both sides of the equation in (30) by $\overline{u}$ and integrating from 0 to 1, we get that

$$\int_0^1 \left(\overline{u}''(x)\right)^2 dx - \lambda_1 \int_0^1 (\overline{u}(x))^2\,dx = 0. \tag{31}$$

This together with the orthogonality of $\overline{u}$ and $\widetilde{u}$ in $L^2(0,1)$ implies that

$$\begin{aligned}&\int_0^1 \left[-u^{(4)}(x) + \lambda_1 u(x) + q(x)u(x)\right] \\ &\quad \cdot (\overline{u}(x) - \widetilde{u}(x))\,dx = \int_0^1 q(x)(\overline{u}(x))^2\,dx \\ &\quad + \int_0^1 \left(\widetilde{u}''(x)\right)^2 dx - \int_0^1 \left(q(x) + \lambda_1\right)(\widetilde{u}(x))^2\,dx \\ &\quad \ge \int_0^1 \left(\widetilde{u}''(x)\right)^2 dx - \int_0^1 \left(q(x) + \lambda_1\right)(\widetilde{u}(x))^2\,dx \\ &\quad \equiv D_q(\widetilde{u}).\end{aligned} \tag{32}$$

Subsequently, by (28), we have

$$D_q(\widetilde{u}) \ge D_p(\widetilde{u}) - \sigma \int_0^1 (\widetilde{u}(x))^2\,dx. \tag{33}$$

Step 2. We show that if $u \in H$, then there exists a constant $\delta = \delta(p) > 0$ satisfying

$$D_p(\widetilde{u}) \ge \delta\,|\widetilde{u}|_{H^2}^2. \tag{34}$$

Firstly, by (27), we observe that

$$\widetilde{u}''(x) = \sum_{j=2}^{\infty} s_j \varphi_j''(x). \quad (35)$$

By Lemma 2, (m_j^4, φ_j) is a solution of (11). So that, substituting into (11) and multiplying both sides of the equation by φ_j and integrating from 0 to 1, we get that, for $x \in (0,1)$,

$$\int_0^1 \left(\varphi_j''(x)\right)^2 dx = m_j^4 \int_0^1 \left(\varphi_j(x)\right)^2 dx. \quad (36)$$

This fact together with (35) and using Parseval identity yields that

$$\begin{aligned} \int_0^1 (\widetilde{u}(x))^2 dx &= \sum_{j=2}^{\infty} s_j^2, \\ \int_0^1 \left(\widetilde{u}''(x)\right)^2 dx &= \sum_{j=2}^{\infty} m_j^4 s_j^2. \end{aligned} \quad (37)$$

Therefore, by (H0), we find that

$$\begin{aligned} D_p(\widetilde{u}) &\geq \int_0^1 \left[\left(\widetilde{u}''(x)\right)^2 - m_2^4 (\widetilde{u}(x))^2\right] dx \\ &\geq \sum_{j=2}^{\infty} s_j^2 \left(m_j^4 - m_2^4\right) \geq 0, \end{aligned} \quad (38)$$

with equality if and only if $s_j^2(m_j^4 - m_2^4) = 0$ for all $j \in \mathbb{N}$ and $j \geq 2$. Therefore, for $j > 2$, one has $s_j = 0$, and, by using the series expansion, $\widetilde{u}(x)$ reduces to $\widetilde{u}(x) = s_2\varphi_2(x)$. But then, we have

$$D_p(\widetilde{u}) = s_2^2 \int_0^1 \left(m_2^4 - m_1^4 - p(x)\right)\left(\varphi_2(x)\right)^2 dx = 0. \quad (39)$$

It follows from (H0) that $s_2 = 0$, and hence $\widetilde{u} = 0$.

Next we will prove that (34) is true. Suppose, on the contrary, that there exists a sequence $\{\widetilde{u}_n\} \subset \widetilde{H}_0^2(0,1)$ and $\widetilde{u} \subset \widetilde{H}_0^2(0,1)$ such that

$$\lim_{n\to\infty} D_p(\widetilde{u}) = 0 \quad \text{with } \left\|\widetilde{u}_n\right\|_{H^2} = 1. \quad (40)$$

It follows from the compact embedding of $\widetilde{H}_0^2(0,1)$ into $C^1[0,1]$ that

$$\begin{aligned} \widetilde{u}_n &\longrightarrow \widetilde{u} \quad \text{in } C^1[0,1], \\ \widetilde{u}_n &\rightharpoonup \widetilde{u} \quad \text{in } \widetilde{H}_0^2(0,1). \end{aligned} \quad (41)$$

Now (41) implies that

$$\|\widetilde{u}\|_{H^2}^2 \leq \liminf_{n\to\infty} \left\|\widetilde{u}_n\right\|_{H^2}^2. \quad (42)$$

At the same time by (40) and (41), we obtain

$$\lim_{n\to\infty} \left\|\widetilde{u}_n\right\|_{H^2}^2 = \int_0^1 \left(p(x) + \lambda_1\right)(\widetilde{u}(x))^2 dx. \quad (43)$$

This together with (42) implies that

$$\|\widetilde{u}\|_{H^2}^2 \leq \int_0^1 \left(p(x) + \lambda_1\right)(\widetilde{u}(x))^2 dx, \quad (44)$$

that is, $D_p(\widetilde{u}) \leq 0$. By the fact that $D_p(\widetilde{u}) \geq 0$ with equality if and only if $\widetilde{u} = 0$, we know $\widetilde{u} = 0$; this contradicts the fact that $\|\widetilde{u}_n\|_{H^2} = 1$.

Step 3. By a direct observation of (33) and (34), we obtain the desired results. □

Lemma 4. *Let $\xi \in (0, m_2^4 - m_1^4)$ be fixed constant. Define a linear operator $E : H \to L^2(0,1)$ by*

$$E(u) = u^{(4)} + \lambda_1 u + \xi u. \quad (45)$$

Then $E^{-1} : L^2(0,1) \to H$ is completely continuous.

Proof. By the theory of linear fourth-order differential equations, the operator $E : H \to L^2(0,1)$ defined by

$$E(u) = u^{(4)} + \lambda_1 u + \xi u \quad (46)$$

is one-to-one and continuous obviously. It follows that E^{-1} : $L^2(0,1) \to H$ is completely continuous. □

4. The Main Result and the Proof

The main result of the paper addresses the existence of solutions of fourth-order problem (7), when the nonlinearity is unbounded. For the sake of simplicity, we assume the following:

(H1) $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a L^2-Carathéodory function; namely, $g(\cdot, u)$ is measurable on $(0,1)$ for every $u \in \mathbb{R}$, $g(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in (0,1)$, for any constant $r > 0$, and there exists a function $\Gamma_r \in L^2(0,1)$ such that

$$\left|g(x,u)\right| \leq \Gamma_r(x) \quad (47)$$

for a.e. $x \in (0,1)$ and all $u \in \mathbb{R}$ with $|u| \leq r$.

(H2) $ug(x,u) \geq 0$ for a.e. $x \in (0,1)$ and all $u \in \mathbb{R}$.

(H3) For all constant $\sigma > 0$, there exist a constant $R = R(\sigma) > 0$ and a function $b = b(\sigma) \in L^{\infty}(0,1)$ such that

$$\left|g(x,u)\right| \leq \left(p(x) + \sigma\right)|u| + b(x) \quad (48)$$

for a.e. $x \in (0,1)$ and all $u \in \mathbb{R}$ with $|u| \geq R$, where $p \in L^{\infty}(0,1)$ has given by (H0).

Theorem 5. *Assume that (H0)–(H3) hold. Then problem (7) has at least one solution for any $h \in L^2(0,1)$ provided:*

$$\int_0^1 h(x)\varphi_1(x)\, dx = 0. \quad (49)$$

Proof. Let $\delta > 0$ be associated with function p and $\xi \in (0, m_2^4 - m_1^4)$ be a fixed constant with $\xi < \delta/2$. To study problem (7) using Leray-Schauder continuation method, we prove that each of the possible solutions of the homotopy

$$-u^{(4)} + \lambda_1 u + (1-\lambda)\xi u + \lambda g(x,u) = \lambda h \quad x \in (0,1), \quad (50)$$
$$u(0) = u(1) = u'(0) = u'(1) = 0$$

has a priori bound. Therefore, we claim that if $u \in H$ is a solution of (50), then there exists a constant $\rho > 0$ independently of $\lambda \in [0,1)$ such that

$$\|u\|_H < \rho. \quad (51)$$

If we assume on the contrary that there exists a sequence $\{\lambda_n\} \subset (0,1)$ and a sequence $\{u_n\} \subset H$ with $\|u_n\|_H \ge n$ for all $n \in \mathbb{N}$ such that

$$-u_n^{(4)} + \lambda_1 u_n + (1-\lambda_n)\xi u_n + \lambda_n g(x,u_n) = \lambda_n h, \quad x \in (0,1), \quad (52)$$
$$u_n(0) = u_n(1) = u_n'(0) = u_n'(1) = 0.$$

Let $v_n = u_n/\|u_n\|_H$. Then

$$-v_n^{(4)} + \lambda_1 v_n + \xi v_n = \lambda_n \xi v_n + \lambda_n \frac{h}{\|u_n\|_H} - \lambda_n \frac{g(x,u_n)}{\|u_n\|_H}, \quad x \in (0,1), \quad (53)$$
$$v_n(0) = v_n(1) = v_n'(0) = v_n'(1) = 0.$$

Obviously, by Lemma 4, (53) is equivalent to

$$v_n = E^{-1}\left[\lambda_n\left(\xi v_n + \frac{h}{\|u_n\|_H} - \frac{g(x,u_n)}{\|u_n\|_H}\right)\right]. \quad (54)$$

(47) together with (48) yields that there exists a function $c \in L^\infty(0,1)$ depending only on $R = R(\delta)$ such that

$$|g(x,u)| \le \left(p(x) + \frac{\delta}{2}\right)|u| + b(x) + c(x) \quad \text{for a.e. } x \in (0,1) \text{ and all } u \in \mathbb{R}. \quad (55)$$

Subsequently, the right-hand member of (54) is bounded in $L^2(0,1)$ independently of n. By Lemma 4, there exists $v \in H$ such that $\lim_{n\to\infty} v_n = v$ in H. Moreover, $\|v\|_H = 1$.

On the other hand, (H3) yields that there exist $R = R(\delta) > 0$ and $b = b(\delta) \in L^\infty(0,1)$ such that

$$|g(x,u)| \le \left(p(x) + \frac{\delta}{4}\right)|u| + b(x) \quad (56)$$

for a.e. $x \in (0,1)$ and all $u \in \mathbb{R}$ with $|u| \ge R$, where R is chosen such that $b(x)/|u| < \delta/4$. Let us define a function $\widetilde{p}: (0,1) \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{p}(x,u) = \begin{cases} \dfrac{g(x,u)}{u}, & \text{for } |u| \ge R, \\ \dfrac{g(x,R)}{R}\dfrac{u}{R} + \left(1 - \dfrac{u}{R}\right)p(x), & \text{for } 0 \le u \le R, \\ \dfrac{g(x,-R)}{R}\dfrac{u}{R} + \left(1 + \dfrac{u}{R}\right)p(x), & \text{for } -R \le u \le 0. \end{cases} \quad (57)$$

Then, this together with (H2) and (56) yields that

$$0 \le \widetilde{p}(x,u) \le p(x) + \frac{\delta}{2} \quad \text{for a.e. } x \in (0,1) \text{ and all } u \in \mathbb{R}. \quad (58)$$

Moreover, $\widetilde{p}(x,u)u$ is a L^2-Carathéodory function. Define $f: (0,1) \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,u) = g(x,u) - \widetilde{p}(x,u)u. \quad (59)$$

By (H1), it yields that, for a.e. $x \in (0,1)$ and all $u \in \mathbb{R}$, there exists $v \in L^2(0,1)$, such that

$$|f(x,u)| \le v(x). \quad (60)$$

Observe that v depend only on Γ and γ_R.

Thus, problem (50) is equivalent to

$$-u^{(4)}(x) + \lambda_1 u(x) + (1-\lambda)\xi u(x) + \lambda \widetilde{p}(x,u(x))u(x) + \lambda f(x,u(x)) = \lambda h(x), \quad x \in (0,1), \quad (61)$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

The fact $\xi \in (0, m_2^4 - m_1^4)$ with $\xi < \delta/2$ together with (58) yields that

$$0 \le (1-\lambda)\xi + \lambda \widetilde{p}(x,u) \le p(x) + \frac{\delta}{2} \quad \text{for a.e. } x \in (0,1) \text{ and all } u \in \mathbb{R}. \quad (62)$$

Therefore, by Lemma 3, (60), and the compact embedding of $H^2(0,1)$ into $L^2(0,1)$, we have

$$\begin{aligned} 0 &= \int_0^1 \left[-u^{(4)}(x) + \lambda_1 u(x) + (1-\lambda)\xi u(x) + \lambda \widetilde{p}(x,u(x))u(x)\right](\overline{u}(x) - \widetilde{u}(x))\,dx \\ &\quad + \int_0^1 (\lambda f(x,u(x)) - \lambda h(x))(\overline{u}(x) - \widetilde{u}(x))\,dx \\ &\ge \frac{\delta}{2}\|\widetilde{u}\|_{H^2}^2 - (\|\widetilde{u}\|_{L^2} + \|\overline{u}\|_{L^2})(\|f\|_{L^2} + \|h\|_{L^2}) \\ &\ge \frac{\delta}{2}\|\widetilde{u}\|_{H^2}^2 - C(\|\widetilde{u}\|_{H^2} + \|\overline{u}\|_{H^2}) \end{aligned} \quad (63)$$

for some constant $C > 0$.

By (63), we deduce immediately that $\lim_{n\to\infty}\tilde{v}_n = 0$ in $H^2(0,1)$. Therefore we can write $v_n = \overline{v}_n$. Since $\|v\|_H = 1$, we shall suppose that

$$v(x) = c\varphi_1(x) \quad \text{for some } c > 0. \tag{64}$$

Now, using Lemma 2, we can get that there exists N such that, for $n \geq N$, $v_n(x) > 0$ on $(0,1)$. So that, for $n \geq N$,

$$u_n(0) = u_n(1) = u_n'(0) = u_n'(1) = 0, \quad u_n(x) > 0. \tag{65}$$

Multiplying both sides of the equation in (53) by $\overline{v}_n$ and integrating from 0 to 1, by (31), (49), and the fact $\lambda_n \in (0,1)$, we have

$$(1-\lambda_n)\,\xi\int_0^1 (\overline{v}_n)^2\,dx = -\frac{\lambda_n}{\|u_n\|_H}\int_0^1 g(x,u_n)\,\overline{v}_n dx. \tag{66}$$

So that $\int_0^1 g(x,u_n)\overline{v}_n dx < 0$. By (H2) and (65), we conclude a contradiction. □

Conflicts of Interest

All of the authors of this article declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Man Xu and Ruyun Ma completed the main study together and Man Xu wrote the manuscript; Ruyun Ma checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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