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Research Article

Global Hölder Estimates via Morrey Norms for Hypoelliptic Operators with Drift

Yuexia Hou and Pengcheng Niu

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, China

Correspondence should be addressed to Pengcheng Niu; pengchengniu@nwpu.edu.cn

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Suppose that $X_0, X_1, \ldots, X_m$ are left invariant real vector fields on the homogeneous group G with X_0 being homogeneous of degree two and $X_1, \ldots, X_m$ homogeneous of degree one. In the paper we study the hypoelliptic operator with drift of the kind $L = \sum_{i,j=1}^m a_{ij} X_i X_j + a_0 X_0$, where $a_0 \neq 0$ and (a_{ij}) is a constant matrix satisfying the elliptic condition on $\mathbb{R}^m$. By proving the boundedness of two integral operators on the Morrey spaces with two weights, we obtain global Hölder estimates for L.

1. Introduction and the Main Results

Let G be a homogeneous group on $\mathbb{R}^N$ and let $X_0, X_1, \ldots, X_m$ $(m < N)$ be left invariant real vector fields on G, where X_0 is homogeneous of degree two and $X_1, \ldots, X_m$ are homogeneous of degree one satisfying Hörmander's condition

$$\operatorname{rank} \mathscr{L}(X_0, X_1, \ldots, X_m)(x) = N, \quad x \in G, \tag{1}$$

$\mathscr{L}(X_0, X_1, \ldots, X_m)$ denotes the Lie algebra generated by $X_0, X_1, \ldots, X_m$. The purpose of this paper is to study the following hypoelliptic operator with drift:

$$L = \sum_{i,j=1}^m a_{ij} X_i X_j + a_0 X_0, \tag{2}$$

where $a_0 \neq 0$ and $(a_{ij})_{i,j=1}^m$ is a constant coefficients matrix and there exists a constant $\mu > 0$ such that

$$\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^m a_{ij} \xi_i \xi_j \leq \mu |\xi|^2, \quad \xi \in \mathbb{R}^m$$
$$\mu^{-1} \leq a_0 \leq \mu. \tag{3}$$

Since Hörmander put forward the operator of sum of squares in [1], many authors paid attention to regularity of hypoelliptic operators constructed by Hörmander's vector fields. Folland [2] concluded that any left invariant homogeneous differential operator of second order possesses a unique homogeneous fundamental solution. Bramanti and Brandolini [3] investigated further the related properties of the fundamental solutions. Recently, the a priori estimates for the operator L in (2) have been considered by several researchers. A priori L^p estimates, C^α estimates, and Sobolev-Morrey estimates for L especially were proved in [3–5], respectively. We mention that the operator L contains Laplacian and parabolic operators in the Euclidean space. When $X_0 = \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t$, $X_i = \partial_{x_i}$, $i = 1, \ldots, m$, $m < N$, the operator L becomes

$$L_1 u = \sum_{i,j=1}^m a_{ij} \partial^2_{x_i x_j} u + \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} u - \partial_t u, \tag{4}$$

where $(x,t) \in \mathbb{R}^{n+1}$, $(a_{ij})_{i,j=1}^m$ is a positive definite matrix in $\mathbb{R}^m$, and (b_{ij}) is a constant coefficients matrix with a suitable upper triangular structure. Clearly, L_1 is a class of Kolmogorov-Fokker-Planck ultraparabolic operators and appears in many research ranges, for example, stochastic processes and kinetic models [6, 7] and mathematical finance theory [8, 9]. After the previous study on L_1 in [10, 11], the authors of [12–14] established an invariant Harnack inequality for the nonnegative solution of $L_1 u = 0$ by using the mean value formula. Based on the theory of singular integral, Polidoro and Ragusa in [15] demonstrated Morrey-type imbedding results and gave a local Hölder continuity of the solution.

Komori and Shirai in [16] defined weighted Morrey spaces in the Euclidean space, which are the extension of Morrey spaces (see [17]) and showed the boundedness in these spaces of some important operators in harmonic analysis. The authors of [18] established the boundedness of commutators of fractional integral operators with BMO functions on Morrey spaces with two weights. In the framework of homogeneous groups, we proved in [19] similar results to [16] and a priori estimates for L on Morrey spaces with two weights. In this paper, we try to study the global Hölder estimates for L. More precisely, motivated by [3, 16], we will establish the boundedness of two integral operators on the Morrey spaces with two weights and then prove global Hölder estimates for L.

Before stating the main results, we first introduce $A_{p,q}$ classes and the Morrey space with two weights on the homogeneous group G. Let us recall that a weight is a nonnegative locally integrable function on G. Given a weight w and a measurable set $E \subset G$, we set

$$w(E) = \int_E w(y)\,dy. \tag{5}$$

For $1 < p < q < \infty$ and the weight w, if there exists $c > 1$ such that, for any ball B in G,

$$\left(\frac{1}{|B|}\int_B w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p'}\,dx\right)^{1/p'} \le c, \tag{6}$$

where $1/p + 1/p' = 1$, then w is said to be in the class $A_{p,q}$. The infimum of these constants is denoted by $[w]_{A_{p,q}}$.

For $p \in (1,\infty)$ and $\kappa \in (0,1)$, the Morrey space with two weights μ and ν on G is defined by

$$L^{p,\kappa}(\mu,\nu,G) = \left\{g \in L^p_{\mathrm{loc}}(w,G) : \|g\|_{L^{p,\kappa}(\mu,\nu,G)} < \infty\right\}, \tag{7}$$

where

$$\|g\|_{L^{p,\kappa}(\mu,\nu,G)} = \sup_B\left(\frac{1}{\nu(B)^\kappa}\int_B |g(y)|^p\,\mu(y)\,dy\right)^{1/p}, \tag{8}$$

and the supremum is taken over all balls in G.

Observe that if $\mu = \nu = 1$ and $\kappa = \lambda/Q$ in (7), $0 < \lambda < Q$, Q is the homogeneous dimension of G, thus $L^{p,\kappa}(\mu,\nu,G) = L^{p,\lambda}(G)$ which is the usual Morrey space.

We next state the main results of this paper.

Theorem 1. *If $1 < p < q < \infty$, $1/Q < 1/p - 1/q < 2/Q$, $w \in A_{p,q}$, and $Lu \in L^{p,p/q}(w^p, w^q, G)$, then there exists $c > 0$ such that, for any test function u and every $x, z \in G$, $x \ne z$,*

$$\frac{|u(x) - u(z)|}{\|z^{-1}\circ x\|^\theta} \le c\,\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}, \tag{9}$$

where $\theta = 2 - Q(1/p - 1/q)$.

Theorem 2. *If $1 < p < q < \infty$, $0 < 1/p - 1/q < 1/Q$, $w \in A_{p,q}$, and $Lu \in L^{p,p/q}(w^p, w^q, G)$, then there exists $c > 0$ such that, for any test function u and every $x, z \in G$, $x \ne z$,*

$$\frac{|X_i u(x) - X_i u(z)|}{\|z^{-1}\circ x\|^\delta} \le c\,\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}, \quad i = 1,\ldots,m, \tag{10}$$

where $\delta = 1 - Q(1/p - 1/q)$.

Remark 3. Authors in [19] have proved Morrey estimates with two weights for L: if $1 < p < \infty$, $1/q = 1/p - 1/Q$, $0 < \kappa < p/q$, and $w \in A_{p,q}$, there exists a constant $c > 0$ such that, for every $Lu \in L^{p,\kappa}(w^p, w^q, G)$, we have

$$\|X_i u\|_{L^{q,\kappa q/p}(w^q,G)} \le c\,\|Lu\|_{L^{p,\kappa}(w^p,w^q,G)}, \quad i = 1,2,\ldots,m. \tag{11}$$

Our results reflect the relations between the weighted Morrey norms of Lu and Hölder exponents of u and $X_i u$. These statements are new even to elliptic operators, parabolic operators, and some ultraparabolic operators.

Since the second and higher order vector fields derivatives of a test function u are determined by Calderón-Zygmund operators (see [3]), we cannot use the method here to give Hölder estimates for higher order derivatives of u.

The paper is organized as follows. In Section 2 we present some preliminaries about homogeneous groups and fundamental solutions of L. Furthermore, we establish pointwise estimates for the two integral operators on the Morrey spaces with two weights. Section 3 is devoted to the proofs of the main results.

2. Preliminaries and Two Integral Operators

Given two smooth mappings

$$\left[(x,y) \longmapsto x\circ y\right] : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}^N; \qquad \left[x \longmapsto x^{-1}\right] : \mathbb{R}^N \longmapsto \mathbb{R}^N, \tag{12}$$

the space $\mathbb{R}^N$ with these mappings forms a group and the identity is the origin. If there exist $0 < \omega_1 \le \omega_2 \le \cdots \le \omega_N$, such that the dilations

$$D(\lambda) : (x_1,\ldots,x_N) \longmapsto (\lambda^{\omega_1}x_1,\ldots,\lambda^{\omega_N}x_N), \quad \lambda > 0 \tag{13}$$

are group automorphisms, then the group with this structure is called a homogeneous group denoted by G. Homogeneous groups include the Euclidean space, the Heisenberg group, and the Carnot group; see [20, 21].

Definition 4. A homogeneous norm $\|\cdot\|$ on G is defined as follows: for any $x \in G$, $x \ne 0$,

$$\|x\| = \rho \Longleftrightarrow \left|\tau\left(\frac{1}{\rho}\right)x\right| = 1, \tag{14}$$

where $|\cdot|$ is the Euclidean norm. Also, define $\|0\| = 0$.

It is not difficult to verify that the following properties hold for the homogeneous norm:

(1) $\|\tau(\lambda)x\| = \lambda\|x\|$, $x \in G$, $\lambda > 0$;

(2) there exists $c_1, c_2 \geq 1$, such that, for $x, y \in G$,

$$\left\|x^{-1}\right\| \leq c_1 \|x\|, \qquad \|x \circ y\| \leq c_2 (\|x\| + \|y\|). \tag{15}$$

By virtue of these properties, it is natural to define the quasi-distance d by

$$d(x,y) = \left\|y^{-1} \circ x\right\|. \tag{16}$$

Furthermore, the ball in G with respect to d is defined by

$$B(x,r) \equiv B_r(x) = \{y \in G : d(x,y) < r\}. \tag{17}$$

Observe that $B(0,r) = \tau(r)B(0,1)$; then

$$|B(x,r)| = r^Q |B(0,1)|, \quad x \in G, \ r > 0, \tag{18}$$

where

$$Q = \omega_1 + \cdots + \omega_N \tag{19}$$

is the homogeneous dimension of G. By (18) the doubling condition holds on G; that is,

$$|B(x,2r)| \leq c|B(x,r)|, \quad \text{for } x \in G, \ r > 0, \tag{20}$$

where c is a positive constant, and therefore (G, dx, d) is a space of homogeneous type.

Definition 5. A differential operator Y on G is said to be homogeneous operator of degree β ($\beta > 0$), if, for every test function φ,

$$Y(\varphi(D(\lambda)x)) = \lambda^\beta (Y\varphi)(D(\lambda)x), \quad \lambda > 0, \ x \in G; \tag{21}$$

a function f is said to be homogeneous operator of degree α, if

$$f((D(\lambda)x)) = \lambda^\alpha f(x), \quad \lambda > 0, \ x \in G. \tag{22}$$

Obviously, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous operator of degree $\alpha - \beta$.

Lemma 6 (see [3]). *The operator L is a homogeneous left invariant differential operator of degree two on G and there is a unique fundamental solution $\Gamma(\cdot)$ such that, for any test function u and every $x \in G$,*

(1) $\Gamma(\cdot) \in C^\infty(G \setminus \{0\})$;

(2) $\Gamma(\cdot)$ *is homogeneous operator of degree* $2 - Q$;

(3) $u(x) = (Lu * \Gamma)(x) = \int_G \Gamma(y^{-1} \circ x) Lu(y)\,dy$;

(4) $X_i u(x) = \int_G X_i \Gamma(y^{-1} \circ x) Lu(y)\,dy$.

If we set $\Gamma_i = X_i\Gamma$, $i = 1, \ldots, m$, then it is obvious from Definition 5 that $\Gamma_i(\cdot)$ is homogeneous of degree $1 - Q$.

Lemma 7 (see [22]). *For any $x, y, z \in G$, the following hold:*

(1) *there exists a constant $c > 0$, such that*

$$\Gamma(y^{-1} \circ x) \leq \frac{c}{\left\|y^{-1} \circ x\right\|^{Q-2}}, \qquad \Gamma_i(y^{-1} \circ x) \leq \frac{c}{\left\|y^{-1} \circ x\right\|^{Q-1}}; \tag{23}$$

(2) *there exist two constants $c > 0$ and $M > 1$, such that if $\|y^{-1} \circ x\| \geq M\|x^{-1} \circ z\|$, then*

$$\left|\Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z)\right| \leq \frac{c\left\|x^{-1} \circ z\right\|}{\left\|y^{-1} \circ x\right\|^{Q-1}}, \qquad \left|\Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z)\right| \leq \frac{c\left\|x^{-1} \circ z\right\|}{\left\|y^{-1} \circ x\right\|^{Q}}. \tag{24}$$

Now we introduce two integral operators. For $1 < p < q < \infty$ and $\sigma > 0$, fixed $z \in G$, we define for every $g \in L^{p,p/q}(w^p, w^q, G)$, $x \in G$, and $x \neq z$,

$$T_\alpha g(x) = \int_{\|y^{-1}\circ x\| \geq \sigma \|z^{-1} \circ x\|} \frac{g(y)}{\left\|y^{-1} \circ x\right\|^{Q-\alpha}} dy, \quad \alpha \in [0, Q),$$
$$T_\beta g(x) = \int_{\|y^{-1}\circ x\| < \sigma \|z^{-1} \circ x\|} \frac{g(y)}{\left\|y^{-1} \circ x\right\|^{Q-\beta}} dy, \quad \beta \in (0, Q). \tag{25}$$

Lemma 8. *For $1 < p < q < \infty$, $w \in A_{p,q}$, $x, z \in G$, and $x \neq z$, if $\alpha/Q < 1/p - 1/q$, then there exists $c > 0$ such that*

$$\left|T_\alpha g(x)\right| \leq c\|g\|_{L^{p,p/q}(w^p,w^q,G)} \left\|z^{-1} \circ x\right\|^{\alpha - Q(1/p-1/q)}; \tag{26}$$

if $\beta/Q > 1/p - 1/q$, then there exists $c > 0$ such that

$$\left|T_\beta g(x)\right| \leq c\|g\|_{L^{p,p/q}(w^p,w^q,G)} \left\|z^{-1} \circ x\right\|^{\beta - Q(1/p-1/q)}. \tag{27}$$

Proof. By decomposing the domain of integration and applying the Hölder inequality, it is shown that

$$\begin{aligned}
&\left|T_\alpha g(x)\right| \\
&\leq \sum_{k=1}^{\infty} \int_{2^{k-1}\sigma\|z^{-1}\circ x\| \leq \|y^{-1}\circ x\| < 2^k \sigma \|z^{-1}\circ x\|} \frac{|g(y)|}{\left\|y^{-1} \circ x\right\|^{Q-\alpha}} dy \\
&\leq \sum_{k=1}^{\infty} \frac{1}{\left(2^{k-1}\sigma \left\|x^{-1} \circ x_0\right\|\right)^{Q-\alpha}} \int_{B_{2^k\sigma\|z^{-1}\circ x\|}(x)} |g(y)|\,dy \\
&\leq \sum_{k=1}^{\infty} \frac{1}{\left(2^{k-1}\sigma \left\|x^{-1} \circ x_0\right\|\right)^{Q-\alpha}} \left(\int_{B_{2^k\sigma\|z^{-1}\circ x\|}(x)} |g(y)|^p \right. \\
&\left. \cdot\, w(y)^p\,dy\right)^{1/p} \cdot \left(\int_{B_{2^k\sigma\|z^{-1}\circ x\|}(x)} w(y)^{-p'}\,dy\right)^{1/p'}.
\end{aligned} \tag{28}$$

Due to $w \in A_{p,q}$, we get

$$\left(\int_B w(x)^{-p'}\,dx\right)^{1/p'} \le c\frac{|B|^{1/p'+1/q}}{w^q(B)^{1/q}}; \qquad (29)$$

then

$$\begin{aligned}
&\left|T_\alpha g(x)\right| \\
&\le \sum_{k=1}^{\infty}\frac{1}{\left(2^{k-1}\sigma\left\|z^{-1}\circ x\right\|\right)^{Q-\alpha}}\left(\int_{B_{2^k\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|^p\right. \\
&\left.\cdot w(y)^p\,dy\right)^{1/p}\cdot\frac{\left|B_{2^k\sigma\|z^{-1}\circ x\|}(x)\right|^{1/p'+1/q}}{w^q\left(B_{2^k\sigma\|z^{-1}\circ x\|}(x)\right)^{1/q}} \\
&= c\sum_{k=1}^{\infty}\left(\frac{1}{w^q\left(B_{2^k\sigma\|z^{-1}\circ x\|}(x)\right)^{p/q}}\int_{B_{2^k\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|^p\right. \\
&\left.\cdot w(y)^p\,dy\right)^{1/p}\cdot\frac{\left(2^k\sigma\left\|z^{-1}\circ x\right\|\right)^{Q(1/p'+1/q)}}{\left(2^{k-1}\sigma\left\|z^{-1}\circ x\right\|\right)^{Q-\alpha}} \\
&\le c\left\|g\right\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{\alpha-Q(1/p-1/q)} \\
&\cdot\sum_{k=1}^{\infty}\left(2^{\alpha-Q(1/p-1/q)}\right)^k.
\end{aligned} \qquad (30)$$

If $\alpha/Q < 1/p - 1/q$, then the series in the right hand side in (31) is convergent, and (26) is proved. Analogously, it yields

$$\begin{aligned}
&\left|T_\beta g(x)\right| \\
&\le \sum_{k=1}^{\infty}\int_{2^{-k}\sigma\|z^{-1}\circ x\|\le\|y^{-1}\circ x\|<2^{1-k}\sigma\|z^{-1}\circ x\|}\frac{\left|g(y)\right|}{\left\|y^{-1}\circ x\right\|^{Q-\beta}}dy \\
&\le \sum_{k=1}^{\infty}\frac{1}{\left(2^{-k}\sigma\left\|x^{-1}\circ x_0\right\|\right)^{Q-\beta}}\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|dy \\
&\le \sum_{k=1}^{\infty}\frac{1}{\left(2^{-k}\sigma\left\|x^{-1}\circ x_0\right\|\right)^{Q-\beta}}\left(\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|^p\right. \\
&\left.\cdot w(y)^p\,dy\right)^{1/p}\cdot\left(\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}w(y)^{-p'}\,dy\right)^{1/p'}.
\end{aligned} \qquad (31)$$

It follows from (29) that

$$\begin{aligned}
&\left|T_\beta g(x)\right| \\
&\le \sum_{k=1}^{\infty}\frac{1}{\left(2^{-k}\sigma\left\|z^{-1}\circ x\right\|\right)^{Q-\beta}}\left(\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|^p\right. \\
&\left.\cdot w(y)^p\,dy\right)^{1/p}\cdot\frac{\left|B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)\right|^{1/p'+1/q}}{w^q\left(B_{2^{1-k}\sigma\|x^{-1}\circ x_0\|}(x)\right)^{1/q}} \\
&= c\sum_{k=1}^{\infty}\left(\frac{1}{w^q\left(B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)\right)^{p/q}}\right. \\
&\left.\cdot\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}\left|g(y)\right|^p w(y)^p\,dy\right)^{1/p} \\
&\cdot\frac{\left(2^{1-k}\sigma\left\|z^{-1}\circ x\right\|\right)^{Q(1/p'+1/q)}}{\left(2^{-k}\sigma\left\|z^{-1}\circ x\right\|\right)^{Q-\beta}} \le c\left\|g\right\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\right. \\
&\left.\circ x\right\|^{\beta-Q(1/p-1/q)}\sum_{k=1}^{\infty}\left(2^{\beta-Q(1/p-1/q)}\right)^{-k}.
\end{aligned} \qquad (32)$$

If $\beta/Q > 1/p - 1/q$, the series in (32) is convergent and the proof of (27) is ended. □

3. Proof of the Main Theorems

Proof of Theorem 1. For any test function u and every $x, z \in G$, $x \ne z$, applying Lemma 6, there exists $M > 1$ such that

$$\begin{aligned}
&\left|u(x) - u(z)\right| \le \int_G\left|\Gamma\left(y^{-1}\circ x\right) - \Gamma\left(y^{-1}\circ z\right)\right| \\
&\cdot\left|Lu(y)\right|dy \\
&= \int_{\|y^{-1}\circ x\|\ge M\|x^{-1}\circ z\|}\left|\Gamma\left(y^{-1}\circ x\right) - \Gamma\left(y^{-1}\circ z\right)\right| \\
&\cdot\left|Lu(y)\right|dy \\
&+ \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma\left(y^{-1}\circ x\right) - \Gamma\left(y^{-1}\circ z\right)\right| \\
&\cdot\left|Lu(y)\right|dy \\
&\le \int_{\|y^{-1}\circ x\|\ge M\|x^{-1}\circ z\|}\left|\Gamma\left(y^{-1}\circ x\right) - \Gamma\left(y^{-1}\circ z\right)\right| \\
&\cdot\left|Lu(y)\right|dy + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma\left(y^{-1}\circ x\right)\right| \\
&\cdot\left|Lu(y)\right|dy + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma\left(y^{-1}\circ z\right)\right| \\
&\cdot\left|Lu(y)\right|dy.
\end{aligned} \qquad (33)$$

It follows by Lemma 7 that

$$\begin{aligned}
&\left|u(x) - u(z)\right| \\
&\le \int_{\|y^{-1}\circ x\|\ge M\|x^{-1}\circ z\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}\left|Lu(y)\right|dy \\
&+ \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}\left|Lu(y)\right|dy \\
&+ \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}\left|Lu(y)\right|dy.
\end{aligned} \qquad (34)$$

Note that if $\|y^{-1}\circ x\| \geq M\|x^{-1}\circ z\|$, then

$$\left\|y^{-1}\circ x\right\| \geq M\left\|x^{-1}\circ z\right\| \geq \frac{M}{c_1}\left\|z^{-1}\circ x\right\|; \quad (35)$$

if $\|y^{-1}\circ x\| < M\|x^{-1}\circ z\|$, then

$$\begin{aligned}\left\|y^{-1}\circ x\right\| &< M\left\|x^{-1}\circ z\right\| < Mc_1\left\|z^{-1}\circ x\right\|,\\ \left\|y^{-1}\circ z\right\| &< c_2\left(\left\|y^{-1}\circ x\right\| + \left\|x^{-1}\circ z\right\|\right)\\ &< c_2\left(M\left\|x^{-1}\circ z\right\| + \left\|x^{-1}\circ z\right\|\right)\\ &= c_2(1+M)\left\|x^{-1}\circ z\right\|.\end{aligned} \quad (36)$$

It immediately derives by (34) that

$$\begin{aligned}&|u(x)-u(z)|\\ &\leq \int_{\|y^{-1}\circ x\|\geq (M/c_1)\|z^{-1}\circ x\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ x\|<Mc_1\|z^{-1}\circ x\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ z\|<c_2(1+M)\|z^{-1}\circ x\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}|Lu(y)|\,dy\\ &\doteq I_1 + I_2 + I_3.\end{aligned} \quad (37)$$

If $1/Q < 1/p - 1/q$, it is shown by choosing $\alpha = 1$, $\sigma = M/c_1$ in Lemma 8 that there exists $c = c(p,\lambda,\sigma) > 0$ such that

$$\begin{aligned}I_1 &\leq c\left\|z^{-1}\circ x\right\|\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{1-Q(1/p-1/q)}\\ &= c\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{2-Q(1/p-1/q)};\end{aligned} \quad (38)$$

if $2/Q > 1/p - 1/q$, we get from Lemma 8 ($\beta = 2$, $\sigma = Mc_1$ and $\beta = 2$, $\sigma = c_2(1+M)$, resp.) that

$$\begin{aligned}I_2 &\leq c\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{2-Q(1/p-1/q)};\\ I_3 &\leq c\|Lu\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{2-Q(1/p-1/q)}.\end{aligned} \quad (39)$$

Putting (38) and (39) in (37), we have (9) and this finishes the proof. □

Proof of Theorem 2. For $i = 1,\ldots,m$, we have from Lemma 6 that there exists $M > 1$ such that for any test function u and every $x, z \in G$, $x \neq z$,

$$\begin{aligned}&\left|X_iu(x) - X_iu(z)\right| \leq \int_G\left|\Gamma_i\left(y^{-1}\circ x\right) - \Gamma_i\left(y^{-1}\circ z\right)\right|\\ &\quad\cdot|Lu(y)|\,dy\\ &= \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\left|\Gamma_i\left(y^{-1}\circ x\right) - \Gamma_i\left(y^{-1}\circ z\right)\right|\\ &\quad\cdot|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma_i\left(y^{-1}\circ x\right) - \Gamma_i\left(y^{-1}\circ z\right)\right|\\ &\quad\cdot|Lu(y)|\,dy\\ &\leq \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\left|\Gamma_i\left(y^{-1}\circ x\right) - \Gamma_i\left(y^{-1}\circ z\right)\right|\\ &\quad\cdot|Lu(y)|\,dy + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma_i\left(y^{-1}\circ x\right)\right|\\ &\quad\cdot|Lu(y)|\,dy + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma_i\left(y^{-1}\circ z\right)\right|\\ &\quad\cdot|Lu(y)|\,dy.\end{aligned} \quad (40)$$

Summarizing (35) and (36) and using Lemma 7, it yields

$$\begin{aligned}&\left|X_iu(x) - X_iu(z)\right|\\ &\leq \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-1}}|Lu(y)|\,dy\\ &\leq \int_{\|y^{-1}\circ x\|\geq (M/c_1)\|z^{-1}\circ x\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ x\|<Mc_1\|z^{-1}\circ x\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|\,dy\\ &\quad + \int_{\|y^{-1}\circ z\|<c_2(1+M)\|z^{-1}\circ x\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-1}}|Lu(y)|\,dy\\ &\doteq I_4 + I_5 + I_6.\end{aligned} \quad (41)$$

If $0 < 1/p - 1/q$, we have by using Lemma 8 with $\alpha = 0$ and $\sigma = M/c_1$ that there exists $c = c(p,\lambda,\sigma) > 0$ such that

$$\begin{aligned}I_4 &\leq c\left\|z^{-1}\circ x\right\|\|g\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{-Q(1/p-1/q)}\\ &= c\|g\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{1-Q(1/p-1/q)};\end{aligned} \quad (42)$$

if $1/Q > 1/p - 1/q$, it is shown by applying Lemma 8 ($\beta = 1$, $\sigma = Mc_1$ and $\beta = 1$, $\sigma = c_2(1+M)$, resp.) that

$$\begin{aligned}I_5 &\leq c\|g\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{1-Q(1/p-1/q)};\\ I_6 &\leq c\|g\|_{L^{p,p/q}(w^p,w^q,G)}\left\|z^{-1}\circ x\right\|^{1-Q(1/p-1/q)}.\end{aligned} \quad (43)$$

Substituting (42) and (43) into (41), we get (10). □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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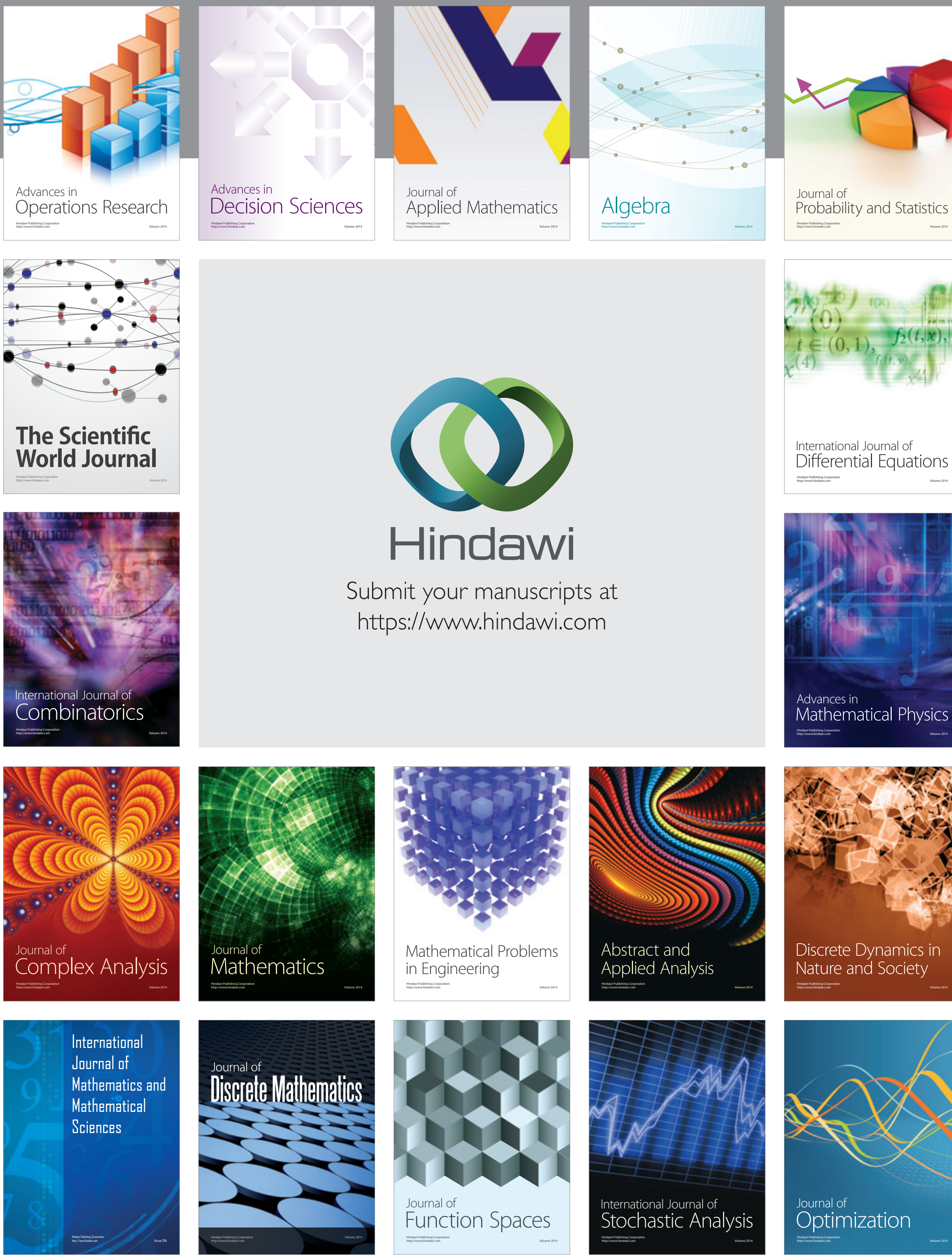
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