Hindawi
Journal of Function Spaces
Volume 2017, Article ID 3295376, 9 pages
https://doi.org/10.1155/2017/3295376

Research Article

Boundedness of Fractional Oscillatory Integral Operators and Their Commutators in Vanishing Generalized Weighted Morrey Spaces

Bilal Çekiç[1] and Ayşegül Çelik Alabalık[2]

[1] *Department of Mathematics, Faculty of Science, Dicle University, 21280 Diyarbakir, Turkey*
[2] *Department of Mathematics, Institute of Natural and Applied Sciences, Dicle University, 21280 Diyarbakir, Turkey*

Correspondence should be addressed to Bilal Çekiç; bilalcekic@gmail.com

Received 12 December 2016; Accepted 26 January 2017; Published 16 February 2017

Academic Editor: Yoshihiro Sawano

In this article, we give the boundedness conditions in terms of Zygmund-type integral inequalities for oscillatory integral operators and fractional oscillatory integral operators on the vanishing generalized weighted Morrey spaces. Moreover, we investigate corresponding commutators.

1. Introduction

The classical Morrey spaces $M^{p,\lambda}$ that play important role in the theory of partial differential equations were introduced by Morrey [1] in 1938. Since then Morrey spaces have been studied by various authors. We refer readers to the survey [2] and to the elegant book [3] for further information about these spaces and references on recent developments in this field.

So far various generalizations of Morrey spaces have been defined. Mizuhara [4] introduced the generalized Morrey space $M^{p,\varphi}$ and Komori and Shirai [5] defined the weighted Morrey spaces $L^{p,\kappa}(w)$. Guliyev [6] gave the notion of generalized weighted Morrey space $M^{p,\varphi}(w)$ which can be accepted as an extension of $M^{p,\varphi}$ and $L^{p,\kappa}(w)$. Eroglu [7] proved the boundedness of oscillatory integral operators, fractional oscillatory integral operators, and the corresponding commutators on $M^{p,\varphi}$ and Shi et al. [8] proved the boundedness of these operators and commutators on $L^{p,\kappa}(w)$. In [9], Lu et al. obtained the boundedness of sublinear operators with rough kernels on $M^{p,\varphi}$ and in [10], Shi and Fu showed the boundedness of these operators on $L^{p,\kappa}(w)$. The boundedness of some sublinear operators and their commutators on $L^{p,\kappa}(w)$ was obtained by Shi et al. [11] and the boundedness of sublinear operators on $M^{p,\varphi}(w)$ was proved by Mustafayev [12].

Vanishing Morrey spaces $VM^{p,\lambda}(\mathbb{R}^n)$ are subspaces of functions in Morrey spaces which were introduced by Vitanza [13] satisfying the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} t^{-\lambda/p} \|f\|_{L^p(B(x,t))} = 0. \tag{1}$$

The properties and applications of vanishing Morrey Spaces were given in [14]. On vanishing Morrey spaces, the boundedness of commutators of the multidimensional Hardy type operators was proved in [15]. The vanishing generalized Morrey spaces $VM^{p,\varphi}(\mathbb{R}^n)$ were introduced and studied by Samko in [16].

In this study, as distinct from [8], we focus on vanishing generalized weighted Morrey spaces and give Zygmund-type conditions to prove the boundedness of oscillatory and the fractional oscillatory integral operators and their commutators in these spaces. In Section 2 we recall some definitions and necessary preliminaries and in Section 3 we give our main results; namely, we prove our theorems on vanishing generalized weighted Morrey spaces.

Throughout this paper, C, c, c_i and so on are used as positive constant that can change from one line to another. $A \lesssim B$ means that $A \leq cB$ with some positive constant c. If $A \lesssim B$ and $B \lesssim A$, then we say $A \approx B$ which means A and B are equivalent.

2. Preliminaries

Let w be a weight function on $\mathbb{R}^n$, such that $w(x) > 0$ for almost every $x \in \mathbb{R}^n$. E is a measurable set with Lebesgue measure notated by $|E|$ and we define $w(E) = \int_E w(x)dx$.

For $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ we denote the weighted Lebesgue space by $L^{p,w}(\Omega)$ with the norm

$$\|f\|_{L^{p,w}(\Omega)} = \left(\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p} < \infty. \quad (2)$$

Let $1 \leq p < \infty$, $\varphi(x, r)$ be a positive continuous function on $\mathbb{R}^n \times (0, \infty)$ and let w be a weight function on $\mathbb{R}^n$. We show the generalized weighted Morrey space by $M^{p,\varphi}(\mathbb{R}^n; w) = M^{p,\varphi}(w)$, which is space of all functions $f \in L^{loc}_{p,w}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))}, \quad (3)$$

where $B(x, r)$ is the open ball centered at x of radius r.

The fractional integral operator (Riesz potential) I_α and fractional Maximal operator M_α, which play important roles in real and harmonic analysis, are defined by

$$\begin{aligned} I_\alpha f(x) &= \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \\ M_\alpha f(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} f(y)\,dy, \quad 0 \leq \alpha < n, \end{aligned} \quad (4)$$

where $f \in L^{loc}_1(\mathbb{R}^n)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The class of A_p weights was introduced by Muckenhoupt in [17] to show that Hardy-Littlewood maximal function M is bounded on weighted Lebesgue spaces $L^{p,w}(\mathbb{R}^n)$ if and only if $w \in A_p$, $1 < p < \infty$.

Now we define Muckenhoupt class A_p. Let $1 \leq p < \infty$. A weight w is said to be an A_p weight, if there exists a positive constant c_p such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\int_B w(x)\,dx\right)\left(\int_B w(x)^{1-p'}\,dx\right)^{p-1} \leq c_p |B|^p, \quad (5)$$

when $1 < p < \infty$, and for $p = 1$

$$\int_B w(y)\,dy \leq c_1 w(x)|B| \quad (6)$$

for almost everywhere $x \in B$. The smallest c_p is shown by $[w]_{A_p}$. We define $A_\infty = \bigcup_{p\geq1} A_p$.

A weight w belongs to $A_{p,q}$ for $1 < p < q < \infty$ if there exists $C > 1$ such that

$$\begin{aligned} &\left(\int_B w(x)^q\,dx\right)^{1/q}\left(\int_B w(x)^{-p'}\,dx\right)^{1/p'} \\ &\quad \leq C|B|^{1+1/q-1/p}, \end{aligned} \quad (7)$$

where $1/p + 1/p' = 1$. $A_{p,q}$ class was introduced by Muckenhoupt and Wheeden [18] to study weighted norm inequalities for fractional integral operators.

2.1. Vanishing Generalized Weighted Morrey Spaces. Let Ω be an open set in $\mathbb{R}^n$ and let Π be an arbitrary subset of Ω. Let also $\varphi(x, r)$ be a measurable nonnegative function on $\Pi \times [0, l)$ ($l = \operatorname{diam}\Omega$) and positive for all $(x, t) \in \Pi \times (0, l)$. Let w be a weight function on Ω; then we notate by $VM^{p,\varphi}_\Pi(\Omega; w) := VM^{p,\varphi}_\Pi(w)$ the vanishing weighted Morrey spaces which are defined as the spaces of functions $f \in L^{loc}_{p,w}(\Omega)$ with finite quasi norm:

$$\|f\|_{VM^{p,\varphi}_\Pi(w)} = \sup_{x\in\Pi, r>0} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(\widetilde{B}(x,r))} \quad (8)$$

such that

$$\lim_{r\to0} \sup_{x\in\Pi} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(\widetilde{B}(x,r))} = 0, \quad (9)$$

where $\widetilde{B}(x, r) = B(x, r) \cap \Omega$ and $1 \leq p \leq \infty$.

Naturally, it is suitable to impose the function $\varphi(x, r)$ on the following conditions:

$$\begin{aligned} &\lim_{r\to0} \sup_{x\in\Pi} \frac{\|w\|_{L_1(\widetilde{B}(x,r))}}{\varphi(x,r)} = 0, \\ &\inf_{r>1} \sup_{x\in\Pi} \varphi(x,r) > 0. \end{aligned} \quad (10)$$

which makes $VM^{p,\varphi}_\Pi(w)$ nontrivial, since bounded functions which have compact support belong to this space.

Henceforth we denote by $\varphi \in \mathfrak{B}(w)$ if $\varphi(x, r)$ is a nonnegative measurable function on $\Pi \times [0, l)$ and positive for all $(x, t) \in \Pi \times (0, l)$ and satisfies conditions (10).

2.2. Oscillatory Integral Operators and Fractional Oscillatory Integral Operators. Oscillatory integral operators appear in many fields of mathematics and physics. Furthermore oscillatory integrals have been an essential part of harmonic analysis. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as the Fourier transform, Bochner-Riesz means, and Radon transform. Properties of oscillatory integral operators have been studied by Stein in [19].

A distribution kernel K is called a standard Calderón-Zygmund (in short C-Z) kernel when it satisfies the following conditions:

$$\begin{aligned} |K(x,y)| &\leq \frac{C}{|x-y|^n}, \quad x \neq y \\ |\nabla_x K(x,y)| + |\nabla_y K(x,y)| &\leq \frac{C}{|x-y|^{n+1}}, \quad x \neq y. \end{aligned} \quad (11)$$

C-Z integral operator T and the oscillatory integral operator S are defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y) f(y)\,dy,$$
$$Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y)\,dy, \tag{12}$$

where $P(x, y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. Lu and Zhang [20] used L^2-boundedness of T to get L^p-boundedness of S with $1 < p < \infty$.

Ricci and Stein [21] introduced the standard fractional C-Z kernel, K_α with $0 < \alpha < n$, where

$$\left|K_\alpha(x,y)\right| \le \frac{C}{|x-y|^{n-\alpha}},$$
$$\left|\nabla_x K_\alpha(x,y)\right| + \left|\nabla_y K_\alpha(x,y)\right| \le \frac{C}{|x-y|^{n+1-\alpha}}, \tag{13}$$
$$x \ne y.$$

The fractional oscillatory integral operator is defined in [22] as

$$S_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K_\alpha(x,y) f(y)\,dy, \tag{14}$$

where $P(x, y)$ is a real valued polynomial and defined on $\mathbb{R}^n \times \mathbb{R}^n$ and K_α is a standard fractional C-Z kernel; note that when $\alpha = 0$, $S_\alpha = S$ and $K_\alpha = K$.

Lemma 1 (see [23]). *If K is a standard C-Z kernel and the C-Z singular integral operator T is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$, then for any real polynomial $P(x, y)$ and $w \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that $\|Sf\|_{L^{p,w}(\mathbb{R}^n)} \le C\|f\|_{L^{p,w}(\mathbb{R}^n)}$.*

Lemma 2 (see [24]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w \in A_{p,q}$, $\varphi \in \mathfrak{B}(w^p)$, and $\psi \in \mathfrak{B}(w^q)$. Then the operators M_α and I_α are bounded from $VM_\Pi^{p,\varphi}(\mathbb{R}^n; w^p)$ to $VM_\Pi^{q,\psi}(\mathbb{R}^n; w^q)$, if*

$$\int_\delta^\infty \sup_{x\in\mathbb{R}^n} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} < \infty \tag{15}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \le C \frac{\psi^{1/q}(x,r)}{\|w\|_{L^q(B(x,r))}}, \tag{16}$$

where C does not depend on $x \in \mathbb{R}^n$ and r.

2.3. The Commutator Operators S_b and $S_{\alpha,b}$. Let $f \in L^1_{loc}(\mathbb{R}^n)$; we define

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| dy < \infty, \tag{17}$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy. \tag{18}$$

In harmonic analysis a function of bounded mean oscillation, also known as a BMO function, is a real valued function whose mean oscillation is bounded. BMO$(\mathbb{R}^n)$ is the set of all locally integrable functions f on $\mathbb{R}^n$ with $\|f\|_* < \infty$.

Next we shall introduce the commutators of oscillatory integral operators and fractional oscillatory integral operators.

Let b be a locally integrable function; the commutator operators S_b and $S_{\alpha,b}$ which are formed by S and b are defined by

$$S_b f(x) = Sf(x)\,b(x) - S(bf)(x),$$
$$S_{\alpha,b} f(x) = S_\alpha f(x)\,b(x) - S_\alpha(bf)(x). \tag{19}$$

Lemma 3 (see [25]). *Suppose that K is a standard C-Z kernel, $w \in A_p$ $(1 < p < \infty)$, and the operator T is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$. Then for any $b \in BMO(\mathbb{R}^n)$, there exists constants $C > 0$ independent on the coefficients of P such that*

$$\left\|S_b f\right\|_{L^{p,w}(\mathbb{R}^n)} \le C \|b\|_* \left\|f\right\|_{L^{p,w}(\mathbb{R}^n)}. \tag{20}$$

Lemma 4 (see [18]). *Let $w \in A_\infty$. Then the norm of $BMO(\mathbb{R}^n)$ is equivalent to the norm of $BMO(w)$, where*

$$BMO(w) = \left\{b : \|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \cdot \int_{B(x,r)} \left|b(y) - b_{B(x,r),w}\right| w(y)\,dy < \infty\right\},$$
$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y)\,w(y)\,dy. \tag{21}$$

Remark 5 (the John-Nirenberg inequality). There are possible constants $C_1, C_2 > 0$ such that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$\left|\left\{x \in B : \left|b(x) - b_B\right| > \beta\right\}\right| \le C_1 |B| e^{-C_2\beta/\|b\|_*}, \tag{22}$$

for all $B \subset \mathbb{R}^n$. The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{w(B(x,r))} \cdot \int_{B(x,r)} \left|b(y) - b_{B(x,r)}\right|^p w(y)\,dy\right)^{1/p} \tag{23}$$

for $1 \le p < \infty$ and $w \in A_\infty$.

Lemma 6 (see [26]). *(i) Let $w \in A_\infty$ and let b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w(B(x,r_1))} \cdot \int_{B(x,r_1)} \left|b(y) - b_{B(x,r_2),w}\right|^p w(y)\,dy\right)^{1/p} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*, \tag{24}$$

where $C > 0$ is independent of f, x, r_1, and r_2.

(ii) Let $w \in A_p$ and let b be a function in $BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then

$$\left(\frac{1}{w^{1-p'}(B(x,r_1))} \cdot \int_{B(x,r_1)} \left|b(y) - b_{B(x,r_2),w}\right|^{p'} w(y)^{1-p'}\,dy\right)^{1/p'} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*, \tag{25}$$

where $C > 0$ is independent of f, x, r_1, and r_2.

Lemma 7 (see [24]). *Let $1 < p < \infty, b \in BMO(\mathbb{R}^n), 0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $w \in A_{p,q}$, $\varphi \in \mathfrak{B}(w^p)$, and $\psi \in \mathfrak{B}(w^q)$. Then $M_{\alpha,b}$ and $I_{\alpha,b}$ are bounded from $VM_\Pi^{p,\varphi}(\mathbb{R}^n; w^p)$ to $VM_\Pi^{q,\psi}(\mathbb{R}^n; w^q)$ if*

$$\sup_{0<r<\delta}\int_\delta^\infty \ln\left(e + \frac{t}{r}\right)\sup_{x\in\mathbb{R}^n}\frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}}\frac{dt}{t} < \infty \tag{26}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}}\frac{dt}{t} \le C\frac{\psi^{1/q}(x,r)}{\|w\|_{L^q(B(x,r))}}, \tag{27}$$

where C does not depend on x and r.

3. Main Results

3.1. Boundedness of S and S_α in $VM_\Pi^{p,\varphi}(\mathbb{R}^n; w)$

Lemma 8. *Let $1 < p < \infty$, $w \in A_p$ and K is a standard C-Z kernel and C-Z singular operator T is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$. Then for any ball $B = B(x_0, r)$ in $\mathbb{R}^n$ and any real polynomial $P(x,y)$ the following inequality holds:*

$$\|Sf\|_{L^{p,w}(B)} \le c\,\|b\|_*\,\|w\|_{L_1(B)}^{1/p}\int_r^\infty \|f\|_{L^{p,w}(B(x_0,t))}\|w\|_{L_1(B(x_0,t))}^{-1/p}\frac{dt}{t}, \tag{28}$$

where the constant $c > 0$ does not depend on B and f.

Proof. We split f into two parts in a neighbourhood of point x_0 as follows:

$$f = f_1 + f_2,\ f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{\mathbb{R}^n\setminus(2B)}(y). \tag{29}$$

By linearity of the operator S we have

$$\|Sf\|_{L^{p,w}(B)} \le \|Sf_1\|_{L^{p,w}(B)} + \|Sf_2\|_{L^{p,w}(B)}. \tag{30}$$

From the boundedness of Sf on $L^{p,w}(\mathbb{R}^n)$ (see Lemma 1), it follows that

$$\|Sf_1\|_{L^{p,w}(B)} \le \|Sf_1\|_{L^{p,w}(\mathbb{R}^n)} \le c\,\|f_1\|_{L^{p,w}(\mathbb{R}^n)} = c\,\|f\|_{L^{p,w}(2B)}, \tag{31}$$

where constant $c > 0$ is independent of B and f.

Since $|K(x,y)| \le C/|x-y|^n$, $x \ne y$, we have

$$\left|Sf_2(x)\right| \le \left|\int_{\mathbb{R}^n\setminus(2B)} e^{iP(x,y)}K(x,y)\,f_2(y)\,dy\right| \le C\int_{\mathbb{R}^n}\frac{|f_2(y)|\,dy}{|x-y|^n} \tag{32}$$

for $x \in B(x_0, r)$. It is obvious that $x \in B$ and $y \in \mathbb{R}^n \setminus (2B)$ implies that $|x_0 - y| < |x - y|$. Thus we get

$$\left|Sf_2(x)\right| \le C\int_{\mathbb{R}^n\setminus(2B)}\frac{|f(y)|\,dy}{|x_0-y|^n}. \tag{33}$$

Since the right-hand side does not depend on x we get

$$\|Sf_2\|_{L^{p,w}(B)} \lesssim \|w\|_{L_1(B)}^{1/p}\int_{\mathbb{R}^n\setminus(2B)}\frac{|f(y)|}{|x_0-y|^n}dy. \tag{34}$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus(2B)}\frac{|f(y)|\,dy}{|x_0-y|^n} &\approx \int_{\mathbb{R}^n\setminus(2B)}|f(y)|\left(\int_{|x_0-y|}^\infty\frac{dt}{t^{n+1}}\right)dy\\ &\approx \int_{2r}^\infty\left(\int_{2r\le|x_0-y|\le t}|f(y)|\,dy\right)\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^\infty\left(\int_{B(x_0,t)}|f(y)|\,dy\right)\frac{dt}{t^{n+1}}.\end{aligned} \tag{35}$$

By Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n\setminus(2B)}\frac{|f(y)|\,dy}{|x_0-y|^n} &\lesssim \int_{2r}^\infty\|f\|_{L^{p,w}(B(x_0,t))}\left\|w^{-1/p}\right\|_{L^{p'}(B(x_0,t))}\frac{dt}{t^{n+1}}\\ &\lesssim [w]_{A_p}^{1/p}\int_{2r}^\infty\|f\|_{L^{p,w}(B(x_0,t))}\|w\|_{L_1(B(x_0,t))}^{-1/p}\frac{dt}{t}.\end{aligned} \tag{36}$$

Thus,

$$\|Sf_2\|_{L^{p,w}(B)} \lesssim \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-1/p} \frac{dt}{t}. \quad (37)$$

On the other hand,

$$\begin{aligned} \|f\|_{L^{p,w}(2B)} &\approx r^n \|f\|_{L^{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\ &\lesssim r^n \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L^{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}. \end{aligned} \quad (38)$$

By (31) and (38) we write

$$\begin{aligned} \|Sf_1\|_{L^{p,w}(B)} &\lesssim \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L^{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\approx \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \|w\|_{L_1(B(x_0,t))}^{-1/p} \frac{dt}{t}. \end{aligned} \quad (39)$$

Then by (37) and (39) we get the inequality (29). □

Remark 9. Note that Lemma 8 was proved in [7] in the case $w \equiv 1$.

Theorem 10. *Let $1 < p < \infty$, $w \in A_p$, and $\varphi, \psi \in \mathfrak{B}(w)$. The oscillatory integral operator S is bounded from $VM_{\Pi}^{p,\varphi}(\mathbb{R}^n; w)$ to $VM_{\Pi}^{p,\psi}(\mathbb{R}^n; w)$ for any real polynomial $P(x,y)$ if K is a standard C-Z kernel, the C-Z singular operator T is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$:*

$$c_\delta := \int_\delta^\infty \sup_{x\in\Pi} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \frac{dt}{t} < \infty \quad (40)$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi^{1/p}(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \frac{dt}{t} \le c_0 \frac{\psi^{1/p}(x,r)}{\|w\|_{L_1(B(x,r))}^{1/p}}, \quad (41)$$

where c_0 does not depend on $x \in \Pi$ and $r > 0$.

Proof. To estimate the norm of the operator we use (28) and get

$$\begin{aligned} \|Sf\|_{VM_{\Pi}^{p,\psi}(w)} &= \sup_{x\in\Pi, r>0} \frac{1}{\psi^{1/p}(x,r)} \|Sf\|_{L^{p,w}(B(x,r))} \\ &\le c \sup_{x\in\Pi, r>0} \frac{1}{\psi^{1/p}(x,r)} \|w\|_{L_1}^{1/p} \\ &\quad \cdot \int_r^\infty t^{-1} \|f\|_{L^{p,w}(B(x,t))} \|w\|_{L_1(B(x,t))}^{-1/p} dt. \end{aligned} \quad (42)$$

By using (41) we can write

$$\|Sf\|_{VM_{\Pi}^{p,\varphi}(w)} \le c \sup_{x\in\Pi, r>0} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))}. \quad (43)$$

Thus,

$$\|Sf\|_{VM_{\Pi}^{p,\varphi}(w)} \le c \|f\|_{VM_{\Pi}^{p,\varphi}(w)}. \quad (44)$$

Now we prove that it belongs to $VM_{\Pi}^{p,\psi}(w)$; that is,

$$\begin{aligned} &\lim_{r\to 0} \sup_{x\in\Pi} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))} = 0 \Longrightarrow \\ &\lim_{r\to 0} \sup_{x\in\Pi} \frac{1}{\psi^{1/p}(x,r)} \|Sf\|_{L^{p,w}(B(x,r))} = 0. \end{aligned} \quad (45)$$

To show that $\sup_{x\in\Pi}(1/\psi^{1/p}(x,r))\|Sf\|_{L^{p,w}(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (28):

$$\frac{1}{\psi^{1/p}(x,r)} \|Sf\|_{L^{p,w}(B(x,r))} \le C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right], \quad (46)$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$) and

$$\begin{aligned} I_{\delta_0}(x,r) &:= \frac{\|w\|_{L_1(B(x,r))}^{1/p}}{\psi^{1/p}(x,r)} \left(\int_r^{\delta_0} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \sup_{0<r<t} \frac{1}{\varphi^{1/p}(x,r)} \right. \\ &\quad \left. \cdot \|f\|_{L^{p,w}(B(x,r))} \frac{dt}{t} \right), \\ J_{\delta_0}(x,r) &:= \frac{\|w\|_{L_1(B(x,r))}^{1/p}}{\psi^{1/p}(x,r)} \left(\int_{\delta_0}^{\infty} \frac{\varphi(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \sup_{0<r<t} \frac{1}{\varphi^{1/p}(x,r)} \right. \\ &\quad \left. \cdot \|f\|_{L^{p,w}(B(x,r))} \frac{dt}{t} \right) \end{aligned} \quad (47)$$

and it is supposed that $r < \delta_0$.

Now we fix $\delta_0 > 0$ such that $\sup_{x\in\Pi}\sup_{0<r<t}(1/\varphi^{1/p}(x,r))\|f\|_{L^{p,w}(B(x,r))} < \varepsilon/2Cc_0$, where c_0 and C are constants from (41) and (46), respectively. Then we can write

$$\sup_{x\in\Pi} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0. \quad (48)$$

By choosing r sufficiently small and considering (40) we have

$$J_{\delta_0}(x,r) \le c_{\delta_0} \frac{\|w\|_{L_1(B(x,r))}^{1/p}}{\psi^{1/p}(x,r)} \|f\|_{VM_{\Pi}^{p,\varphi}(w)}, \quad (49)$$

where c_{δ_0} is the constant from (40). Then by (10) we choose r small enough such that

$$\sup_{x\in\Pi} \frac{\|w\|_{L_1(B(x,r))}}{\psi(x,r)} < \left(\frac{\varepsilon}{2Cc_{\delta_0} \|f\|_{VM_{\Pi}^{p,\varphi}(w)}} \right)^p \quad (50)$$

which completes the proof. □

Theorem 11. *Let $1 < p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $P(x,y)$ be real polynomial; $w \in A_{p,q}$, $\varphi \in \mathfrak{B}(w^p)$, and $\psi \in \mathfrak{B}(w^q)$. Then S_α is bounded from $VM_\Pi^{p,\varphi}(w^p)$ to $VM_\Pi^{q,\psi}(w^q)$ if*

$$\int_\delta^\infty \sup_{x\in\Pi} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} < \infty \tag{51}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \le C \frac{\psi^{1/q}(x,r)}{\|w\|_{L^q(B(x,r))}}, \tag{52}$$

where C is not depending on $x \in \Pi$ and r.

Proof. Since $|S_\alpha f(x)| \le I_\alpha(|f|)(x)$ and thanks to Lemma 2 proof is completed. □

3.2. Boundedness of S_b and $S_{\alpha,b}$ in the Spaces $VM_\Pi^{p,\varphi}(w)$

Lemma 12. *Let $1 \le p < \infty$, $b \in BMO(\mathbb{R}^n)$, $w \in A_p$, K is a standard C-Z kernel, and the C-Z integral operator T is type of $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$. Then for $1 < p < \infty$ and polynomial $P(x,y)$ the inequality*

$$\begin{aligned} \|S_b f\|_{L_{p,w}(B)} &\le C\|b\|_* \|w\|_{L_1(B)}^{1/p} \\ &\cdot \int_r^\infty \ln\left(e + \frac{t}{r}\right) \|f\|_{L^{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t} \end{aligned} \tag{53}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_p^{loc}(w)$.

Proof. Let $1 < p < \infty$. For any $x_0 \in \mathbb{R}^n$, we split f into two parts in a neighbourhood of point x_0 such that

$$\begin{aligned} f &= f_1 + f_2, \\ f_1(y) &= f(y)\chi_{2B}(y), \\ f_2(y) &= f(y)\chi_{\mathbb{R}^n\setminus(2B)}(y), \end{aligned} \tag{54}$$

where $r > 0$, and by linearity of the operator S_b we have

$$\|S_b f\|_{L^{p,w}(B)} \le \|S_b f_1\|_{L^{p,w}(B)} + \|S_b f_2\|_{L^{p,w}(B)}. \tag{55}$$

Moreover, from the boundedness of S_b in $L^{p,w}(\mathbb{R}^n)$ (see Lemma 3), it follows that

$$\begin{aligned} \|S_b f_1\|_{L^{p,w}(B)} &\le \|S_b f_1\|_{L^{p,w}(\mathbb{R}^n)} \le C\|b\|_* \|f_1\|_{L^{p,w}(\mathbb{R}^n)} \\ &= C\|b\|_* \|f\|_{L^{p,w}(2B)}, \end{aligned} \tag{56}$$

where constant $C > 0$ is independent of f.

For $x \in B(x_0,r)$ we have

$$\begin{aligned} |S_b f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y)-b(x)|}{|x-y|^n} |f_2(y)|\,dy \\ &\approx \int_{\mathbb{R}^n\setminus(2B)} \frac{|b(y)-b(x)|}{|x_0-y|^n} |f(y)|\,dy. \end{aligned} \tag{57}$$

Then

$$\begin{aligned} &\|S_b f_2\|_{L^{p,w}(B)} \\ &\le C\left(\int_B \left(\int_{\mathbb{R}^n\setminus(2B)} \frac{|b(y)-b(x)|}{|x_0-y|^n}|f(y)|\,dy\right)^p \cdot w(x)\,dx\right)^{1/p} \\ &\lesssim \left(\int_B \left(\int_{\mathbb{R}^n\setminus(2B)} \frac{|b(y)-b_{B,w}|}{|x_0-y|^n}|f(y)|\,dy\right)^p \cdot w(x)\,dx\right)^{1/p} \\ &+ \left(\int_B \left(\int_{\mathbb{R}^n\setminus(2B)} \frac{|b(x)-b_{B,w}|}{|x_0-y|^n}|f(y)|\,dy\right)^p \cdot w(x)\,dx\right)^{1/p} = I + II. \end{aligned} \tag{58}$$

Let us estimate I

$$\begin{aligned} I &= C\|w\|_{L_1(B)}^{1/p} \int_{\mathbb{R}^n\setminus(2B)} \frac{|b(y)-b_{B,w}|}{|x_0-y|^n}|f(y)|\,dy \\ &= C\|w\|_{L_1(B)}^{1/p} \cdot \int_{\mathbb{R}^n\setminus(2B)} |b(y)-b_{B,w}||f(y)| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1}}\,dy \\ &= C\|w\|_{L_1(B)}^{1/p} \cdot \int_{2r}^\infty \int_{2r\le|x_0-y|\le t} |b(y)-b_{B,w}||f(y)|\,dy \frac{dt}{t^{n+1}} \\ &\le C\|w\|_{L_1(B)}^{1/p} \cdot \int_{2r}^\infty \int_{B(x_0,t)} |b(y)-b_{B,w}||f(y)|\,dy \frac{dt}{t^{n+1}}. \end{aligned} \tag{59}$$

Using Hölder's inequality and (25), we have

$$\begin{aligned} I &\lesssim \|w\|_{L_1(B)}^{1/p} \int_{2r}^\infty \left(\int_{B(x_0,t)} |b(y)-b_{B(x_0,r),w}|^{p'} \cdot w(y)^{1-p'}\,dy\right)^{1/p'} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim [w]_{A_p}^{1/p} \|b\|_* \|w\|_{L_1(B)}^{1/p} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right) \cdot \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \|f\|_{L^{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \end{aligned}$$

$$\lesssim [w]_{A_p}^{1/p} \|b\|_* \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \left(e + \ln \frac{t}{r}\right) \cdot w\left(B\left(x_0, t\right)\right)^{-1/p} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t}. \tag{60}$$

In order to estimate II note that

$$II = \left(\int_B \left|b(x) - b_{B,w}\right|^p w(x)\, dx\right)^{1/p} \cdot \int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{\left|x_0 - y\right|^n} dy. \tag{61}$$

By (23) we get

$$II \lesssim \|b\|_* \|w\|_{L_1(B)}^{1/p} \int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{\left|x_0 - y\right|^n} dy. \tag{62}$$

Applying Hölder's inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{\left|x_0 - y\right|^n} dy \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq [w]_{A_p}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} w\left(B\left(x_0, t\right)\right)^{-1/p} \frac{dt}{t}. \end{aligned} \tag{63}$$

Thus by (63) we write

$$II \lesssim \|b\|_* \|w\|_{L_1(B)}^{1/p} \cdot \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} w\left(B\left(x_0, t\right)\right)^{-1/p} \frac{dt}{t}. \tag{64}$$

Summing up I and II, for all $1 < p < \infty$ we get

$$\left\|S_b f_2\right\|_{L_{p,w}(B(x,r))} \lesssim \|b\|_* \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \cdot \|f\|_{L^{p,w}(B(x_0,t))} w\left(B\left(x_0, t\right)\right)^{-1/p} \frac{dt}{t}. \tag{65}$$

On the other hand,

$$\begin{aligned} &\|f\|_{L^{p,w}(2B)} \approx |B| \|f\|_{L^{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \leq |B| \\ &\cdot \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \leq \|w\|_{L_1(B)}^{1/p} \left\|w^{-1/p}\right\|_{L^{p'}(B)} \\ &\cdot \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \leq \|w\|_{L_1(B)}^{1/p} \\ &\cdot \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L^{p'}(B)} \frac{dt}{t^{n+1}} \leq [w]_{A_p}^{1/p} \\ &\cdot \|w\|_{L_1(B)}^{1/p} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x_0,t))} w\left(B\left(x_0, t\right)\right)^{-1/p} \frac{dt}{t}. \end{aligned} \tag{66}$$

Thus,

$$\begin{aligned} \left\|S_b f\right\|_{L_{p,w}(B)} &\lesssim \|b\|_* \|f\|_{L^{p,w}(2B)} + \|b\|_* \|w\|_{L_1(B)}^{1/p} \\ &\cdot \int_{2r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L^{p,w}(B(x_0,t))} \\ &\cdot w\left(B\left(x_0, t\right)\right)^{-1/p} \frac{dt}{t}, \end{aligned} \tag{67}$$

which along with (66) and (67) lead us to (53). □

Theorem 13. *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n), w \in A_p$ and $\varphi, \psi \in \mathfrak{B}(w)$. Then S_b is bounded from $VM_{\Pi}^{p,\varphi}(\mathbb{R}^n; w)$ to $VM_{\Pi}^{p,\psi}(\mathbb{R}^n; w)$ if*

$$c_\delta := \sup_{0<r<\delta} \int_{\delta}^{\infty} \ln\left(e + \frac{t}{r}\right) \sup_{x \in \mathbb{R}^n} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \frac{dt}{t} < \infty \tag{68}$$

for every $\delta > 0$, and

$$\int_r^{\infty} \ln\left(e + \frac{t}{r}\right) \frac{\varphi^{1/p}(x,t)}{\|w\|_{L_1(B(x,t))}^{1/p}} \frac{dt}{t} \leq c_0 \frac{\psi^{1/p}(x,r)}{\|w\|_{L_1(B(x,r))}^{1/p}}, \tag{69}$$

where c_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Proof. Boundedness follows from Lemma 12 and the same procedure is argued in the proof of Theorem 10. We have to prove that

$$\begin{aligned} &\lim_{r\to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))} = 0 \Longrightarrow \\ &\lim_{r\to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\psi^{1/p}(x,r)} \left\|S_b f\right\|_{L^{p,w}(B(x,r))} = 0. \end{aligned} \tag{70}$$

We suppose that $0 < r < \delta_0$. In view of (53) we can write

$$\begin{aligned} \frac{1}{\psi^{1/p}(x,r)} \left\|S_b f\right\|_{L^{p,w}(B(x,r))} &\leq C \frac{\|b\|_* \|w\|_{L_1(B(x,r))}^{1/p}}{\psi^{1/p}(x,r)} \\ &\cdot \int_r^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L^{p,w}(B(x,t))} \|w\|_{L^1(B(x,t))}^{-1/p} \frac{dt}{t}. \end{aligned} \tag{71}$$

To show that $\sup_{x\in\mathbb{R}^n}(1/\psi^{1/p}(x,r))\|S_b f\|_{L^{p,w}(B(x,r))} < \varepsilon$ for small r, we split the right side of (71)

$$\begin{aligned} &\frac{1}{\psi^{1/p}(x,r)} \left\|S_b f\right\|_{L^{p,w}(B(x,r))} \\ &\leq C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right], \end{aligned} \tag{72}$$

where

$$
\begin{aligned}
I_{\delta_0}(x,r) &:= \|b\|_* \frac{\|w\|_{L_1(B)}^{1/p}}{\psi^{1/p}(x,r)} \\
&\cdot \int_r^{\delta_0} \frac{\varphi^{1/p}(x,t)\ln(e+t/r)}{t\|w\|_{L^1(B(x,t))}^{1/p}} \sup_{0<r<t} \frac{1}{\varphi^{1/p}(x,r)} \\
&\cdot \|f\|_{L^{p,w}(B(x,r))}\, dt, \\
J_{\delta_0}(x,r) &:= \|b\|_* \frac{\|w\|_{L_1(B(x,t))}^{1/p}}{\psi^{1/p}(x,r)} \\
&\cdot \int_{\delta_0}^{\infty} \frac{\varphi^{1/p}(x,t)\ln(e+t/r)}{t\|w\|_{L^1(B(x,t))}^{1/p}} \sup_{0<r<t} \frac{1}{\varphi^{1/p}(x,r)} \\
&\cdot \|f\|_{L^{p,w}(B(x,r))}\, dt.
\end{aligned} \tag{73}
$$

Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \sup_{0<r<t} \frac{1}{\varphi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))} < \frac{\varepsilon}{2Cc_0\|b\|_*}, \tag{74}$$

where C and c_0 are constants appearing in (72) and (69), respectively. This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0. \tag{75}$$

The estimation of the second term can be obtained by choosing r sufficiently small. Indeed, by (10), we have

$$J_{\delta_0}(x,r) \le \|b\|_* c_{\delta_0} \|f\|_{VM_\Pi^{p,\varphi}(w)} \frac{\|w\|_{L_1(B)}^{1/p}}{\psi^{1/p}(x,r)}, \tag{76}$$

where c_{δ_0} is the constant from (68). Then by (10) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\|w\|_{L_1(B(x,t))}}{\psi(x,r)} \le \left(\frac{\varepsilon}{2Cc_{\delta_0}\|b\|_* \|f\|_{VM^{p,\varphi}(w)}}\right)^p \tag{77}$$

which completes the proof. □

Theorem 14. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $P(x,y)$ be a real valued polynomial, $\varphi \in \mathfrak{B}(w^p)$ and $\psi \in \mathfrak{B}(w^q)$. Then $S_{\alpha,b}$ is bounded from $VM_\Pi^{p,\varphi}(w^p)$ to $VM_\Pi^{q,\psi}(w^q)$ if*

$$\sup_{0<r<\delta} \int_\delta^\infty \ln\left(e+\frac{t}{r}\right) \sup_{x\in\Pi} \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} < \infty \tag{78}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi^{1/p}(x,t)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \le C \frac{\psi^{1/q}(x,r)}{\|w\|_{L^q(B(x,r))}}, \tag{79}$$

where C does not depend on x and r.

Proof. Since $|S_{\alpha,b}f(x)| \le I_{\alpha,b}(|f|)(x)$, from the boundedness of $I_{\alpha,b}$ (see Lemma 7) proof is completed. □

Competing Interests

The authors declare that they have no competing interests.

References

[1] J. Morrey, "On the solutions of quasi-linear elliptic partial differential equations," *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.

[2] H. Rafeiro, N. Samko, and S. Samko, "Morrey-Campanato spaces: an overview," in *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics*, vol. 228 of *Operator Theory: Advances and Applications*, pp. 293–324, Birkhuser, Basel, Switzerland, 2013.

[3] D. R. Adams, *Morrey Spaces*, Lecture Notes in Applied and Numerical Harmonic Analysis, Birkhuser, 2015.

[4] T. Mizuhara, "Boundedness of some classical operators on generalized Morrey spaces," in *Harmonic Analysis (Sendai, 1990)*, ICM-90 Satellite Conference Proceedings, pp. 183–189, Springer, Tokyo, Japan, 1991.

[5] Y. Komori and S. Shirai, "Weighted Morrey spaces and a singular integral operator," *Mathematische Nachrichten*, vol. 282, no. 2, pp. 219–231, 2009.

[6] V. S. Guliyev, "Generalized weighted Morrey spaces and higher order commutators of sublinear operators," *Eurasian Mathematical Journal*, vol. 3, no. 3, pp. 33–61, 2012.

[7] A. Eroglu, "Boundedness of fractional oscillatory integral operators and their commutators on generalized Morrey spaces," *Boundary Value Problems*, vol. 2013, article no. 70, 2013.

[8] S. G. Shi, Z. W. Fu, and S. Z. Lu, "Boundedness of oscillatory integral operators and their commutators on weighted Morrey spaces," *Scientia Sinica Mathematica*, vol. 43, no. 2, pp. 147–158, 2013 (Chinese).

[9] S. Lu, D. Yang, and Z. Zhou, "Sublinear operators with rough kernel on generalized Morrey spaces," *Hokkaido Mathematical Journal*, vol. 27, no. 1, pp. 219–232, 1998.

[10] S. Shi and Z. Fu, "Boundedness of sublinear operators with rough kernels on weighted Morrey spaces," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 784983, 9 pages, 2013.

[11] S. Shi, Z. Fu, and F. Zhao, "Estimates for operators on weighted Morrey spaces and their applications to nondivergence elliptic equations," *Journal of Inequalities and Applications*, 2013.

[12] R. C. Mustafayev, "On boundedness of sublinear operators in weighted Morrey spaces," *Azerbaijan Journal of Mathematics*, vol. 2, no. 1, pp. 66–79, 2012.

[13] C. Vitanza, "Functions with vanishing Morrey norm and elliptic partial differential equations," in *Proceedings of the Methods of Real Analysis and Partial Differential Equations*, pp. 147–150, Springer, Capri, Italy, 1990.

[14] M. A. Ragusa, "Commutators of fractional integral operators on vanishing-Morrey spaces," *Journal of Global Optimization*, vol. 40, no. 1, pp. 361–368, 2008.

[15] L.-E. Persson, M. A. Ragusa, N. Samko, and P. Wall, "Commutators of Hardy operators in vanishing Morrey spaces," in *Proceedings of the 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences (ICNPAA '12)*, pp. 859–866, Vienna, Austria, July 2012.

[16] N. Samko, "Maximal, potential and singular operators in vanishing generalized Morrey spaces," *Journal of Global Optimization*, vol. 57, no. 4, pp. 1385–1399, 2013.

[17] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.

[18] B. Muckenhoupt and R. L. Wheeden, "Weighted bounded mean oscillation and the Hilbert transform," *Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica*, vol. 54, no. 3, pp. 221–237, 1975.

[19] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1993.

[20] S. Z. Lu and Y. Zhang, "Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels," *Revista Matematica Iberoamericana*, vol. 8, no. 2, pp. 201–219, 1992.

[21] F. Ricci and E. M. Stein, "Harmonic analysis on nilpotent groups and singular integrals I: oscillatory integrals," *Journal of Functional Analysis*, vol. 73, no. 1, pp. 179–194, 1987.

[22] Y. Ding, "L^p-boundedness for fractional oscillatory integral operator with rough kernel," *Approximation Theory and Its Applications. New Series*, vol. 12, no. 2, pp. 70–79, 1996.

[23] S. Lu, Y. Ding, and D. Yan, *Singular Integrals and Related Topics*, World Scientific Publishing, Hackensack, NJ, USA, 2007.

[24] B. Çekiç and A. C. Alabalık, "Boundedness of fractional integral operators and their commutators in vanishing generalized weighted Morrey spaces," https://arxiv.org/abs/1701.07766.

[25] S. G. Shi, "Weighted boundedness for commutators of one class of oscillatory integral operators," *Journal of Beijing Normal University (Natural Science)*, no. 47, pp. 344–346, 2011.

[26] V. S. Guliyev, T. Karaman, R. Mustafayev, and A. Serbeti, "Commutators of sublinear operators generated by Calderon-Zygmund operator on generalized weighted Morrey spaces," *Czechoslovak Mathematical Journal*, vol. 64, no. 139, pp. 365–386, 2014.