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Research Article

General Holmstedt's Formulae for the K-Functional

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Explicit formulae for the K-functional for the general couple $((A_0, A_1)_{\Phi_0}, (A_0, A_1)_{\Phi_1})$, where (A_0, A_1) is a compatible couple of quasi-normed spaces, are proved. As a consequence, the corresponding reiteration theorems are derived.

1. Introduction

In recent papers [1–11] the classical Holmstedt formula for the K-functional [12] was extended to more general cases. See also [13] for more results about generalized Holmstedt's formula and reiteration theorems not only for the K-method, but for the J-method as well. Here we consider the K-method in the most general case for quasi-normed spaces. Namely, let (A_0, A_1) be a compatible couple of quasi-normed spaces, that is, both A_0 and A_1 are linearly and continuously embedded in some Hausdorff topological vector space. By definition, the K-interpolation space $(A_0, A_1)_\Phi$ has a quasi-norm

$$\|a\|_{(A_0,A_1)_\Phi} = \|K(t,a)\|_\Phi, \tag{1}$$

where $K(t,a) = K(t,a;A_0,A_1)$ is the K-functional of Peetre [13, 14], defined for $0 < t < \infty$, $a \in A_0 + A_1$ as follows:

$$K(t,a) = \inf_{a=a_0+a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\}, \tag{2}$$

and Φ is a quasi-normed space of Lebesgue measurable functions, defined on $(0,\infty)$, with monotone quasi norm as follows: $|g| \le |h|$ implies $\|g\|_\Phi \le \|h\|_\Phi$ such that $\min(1,t) \in \Phi$. If $\Phi = L^q_*(w)$, that is,

$$\|g\|_\Phi = \left(\int_0^\infty [w(t)|g(t)|]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < q \le \infty, \tag{3}$$

we write $(A_0, A_1)_{w(t),q}$ instead of $(A_0, A_1)_\Phi$; if $q = \infty$, the above integral has to be replaced by $\sup w(t)|g(t)|$. Here w is a nonnegative Lebesgue measurable function defined on $(0,\infty)$ and called weight.

In [1, 2] the case when $w(t) = t^{-\theta}v(t)$, $0 < \theta < 1$, and v-repeated logarithms was considered and this was extended to v-slowly varying in [3]. Certain limiting cases (when $\theta = 0$) are treated in [5, 8–10]. The case $\Phi = L^q_*(w)$ with arbitrary w was investigated in [6]. In [4, 7, 11] the case when $\Phi = E(t^{-\theta})$ and E is rearrangement invariant Banach space on $(0,\infty)$ with the Haar measure dt/t is treated in detail. Note that in the case $w(t) = t^{-\theta}v(t)$, $0 < \theta < 1$, v-slowly varying, the K- and J-methods are equivalent, but this is not true in the limiting cases $\theta = 0$ or $\theta = 1$. The problem of the relation between both real interpolation methods for Banach couples is treated in [13], where general theorems are proven and certain applications are given. More results about the relation between K- and J-methods in the limiting cases are obtained in [15, 16].

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$. Recall that the weight w is called slowly varying if for every $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to a nondecreasing one and the function $t^{-\varepsilon}w(t)$ is equivalent to a nonincreasing one.

The main results are announced in [17].

2. Formulae for the K-Functional

Using the Holmstedt argument, we can prove general formulae for the K-functional. Let $K(t,a) := K(t,a;A_0,A_1)$ and let

$$\begin{aligned} I(t,a) &:= \left\|\chi_{(0,t)}(u)\,K(u,a)\right\|_{\Phi_0}, \\ g_0(t) &:= t\left\|\chi_{(t,\infty)}(u)\right\|_{\Phi_0}, \\ h_0(t) &:= \left\|u\chi_{(0,t)}(u)\right\|_{\Phi_0}. \end{aligned} \tag{4}$$

Theorem 1 (case $((A_0,A_1)_{\Phi_0},A_1)$). *If $a \in (A_0,A_1)_{\Phi_0} + A_1$, then*

$$K\left(\rho(t),a;(A_0,A_1)_{\Phi_0},A_1\right) \approx I(t,a) + \rho(t)\frac{K(t,a)}{t}, \tag{5}$$

where $\rho(t) \approx g_0(t) + h_0(t)$.

Proof. Let $w := g_0 + h_0$, $B_0 := (A_0,A_1)_{\Phi_0}$, and $B_1 := A_1$. Using monotonicity of K, we get

$$I(t,a) \geq h_0(t)\frac{K(t,a)}{t}, \tag{6}$$

$$\|a\|_{B_0} \gtrsim \left(g_0(t) + h_0(t)\right)\frac{K(t,a)}{t}. \tag{7}$$

Let $a = a_0 + a_1$, $a_j \in A_j$, and $j = 0,1$. Then

$$\begin{aligned} K(t,a) &\lesssim K(t,a_0) + K(t,a_1) \\ &\lesssim \frac{t}{w(t)}\|a_0\|_{B_0} + t\|a_1\|_{A_1}, \end{aligned} \tag{8}$$

whence

$$w(t)\frac{K(t,a)}{t} \lesssim K\left(w(t),a;B_0,B_1\right). \tag{9}$$

Also,

$$\begin{aligned} I(t,a) &\lesssim \left\|\chi_{(0,t)}(u)\,K(u,a_0)\right\|_{\Phi_0} \\ &\quad + \left\|\chi_{(0,t)}(u)\,K(u,a_1)\right\|_{\Phi_0} \\ &\lesssim \|a_0\|_{B_0} + h_0(t)\|a_1\|_{B_1}. \end{aligned} \tag{10}$$

Hence

$$I(t,a) \lesssim K\left(w(t),a;B_0,B_1\right). \tag{11}$$

Now we estimate $K(w(t),a;B_0,B_1)$ from above. To this end, following Holmstedt, we choose decomposition $a = a_0 + a_1$ such that $K(t,a) \approx \|a_0\|_{A_0} + t\|a_1\|_{A_1}$. Then

$$\begin{aligned} K(u,a_0) &\leq \|a_0\|_{A_0} \lesssim K(t,a), \\ K(u,a_1) &\leq u\|a_1\|_{A_1} \lesssim u\frac{K(t,a)}{t}. \end{aligned} \tag{12}$$

Further,

$$\begin{aligned} \|a_0\|_{B_0} &= \|K(u,a_0)\|_{\Phi_0} \\ &\lesssim \left\|\chi_{(0,t)}(u)\,K(u,a_0)\right\|_{\Phi_0} \\ &\quad + \left\|\chi_{(t,\infty)}(u)\,K(u,a_0)\right\|_{\Phi_0}. \end{aligned} \tag{13}$$

Denote by C the first term on the right-hand side and by D the second one. We have

$$\begin{aligned} C &\lesssim I(t,a) + \left\|\chi_{(0,t)}(u)\,K(u,a_1)\right\|_{\Phi_0} \\ &\lesssim I(t,a) + h_0(t)\frac{K(t,a)}{t}, \\ D &\lesssim g_0(t)\frac{K(t,a)}{t}, \end{aligned} \tag{14}$$

whence

$$\|a_0\|_{B_0} \lesssim I(t,a) + w(t)\frac{K(t,a)}{t}. \tag{15}$$

Also,

$$w(t)\|a_1\|_{B_1} = w(t)\|a_1\|_{A_1} \lesssim w(t)\frac{K(t,a)}{t}. \tag{16}$$

Thus

$$K\left(w(t),a;B_0,B_1\right) \lesssim I(t,a) + w(t)\frac{K(t,a)}{t}. \tag{17}$$

□

Remark 2. If the spaces $(A_0,A_1)_{\Phi_0}$ and A_1 are too close, then formula (5) might be useless. For example, if $\Phi_0 = L^\infty(1/t)$, then $g_0 = 1 = h_0$. In applications we require ρ to be increasing (strictly).

Let

$$\begin{aligned} J(t,a) &:= \left\|\chi_{(t,\infty)}(u)\,K(u,a)\right\|_{\Phi_1}, \\ g_1(t) &:= t\left\|\chi_{(t,\infty)}(u)\right\|_{\Phi_1}, \\ h_1(t) &:= \left\|u\chi_{(0,t)}(u)\right\|_{\Phi_1}. \end{aligned} \tag{18}$$

Theorem 3 (case $(A_0,(A_0,A_1)_{\Phi_1})$). *If $a \in A_0 + (A_0,A_1)_{\Phi_1}$, then*

$$K\left(\rho(t),a;A_0,(A_0,A_1)_{\Phi_1}\right) \approx \rho(t)J(t,a) + K(t,a), \tag{19}$$

where $\rho(t) \approx t/(g_1(t) + h_1(t))$.

Proof. Let $w(t) := t/(g_1(t) + h_1(t))$, $B_0 := A_0$, and $B_1 := (A_0,A_1)_{\Phi_1}$. Using monotonicity of K, we get

$$J(t,a) \geq g_1(t)\frac{K(t,a)}{t}, \tag{20}$$

$$\|a\|_{B_1} \gtrsim \left(g_1(t) + h_1(t)\right)\frac{K(t,a)}{t}. \tag{21}$$

Let $a = a_0 + a_1$, $a_j \in A_j$, and $j = 0,1$. Then

$$\begin{aligned} K(t,a) &\lesssim K(t,a_0) + K(t,a_1) \\ &\lesssim \|a_0\|_{B_0} + w(t)\|a_1\|_{B_1}, \end{aligned} \tag{22}$$

whence

$$K(t,a) \lesssim K\left(w(t),a;B_0,B_1\right). \tag{23}$$

Also,

$$\begin{aligned} J(t,a) &\lesssim \left\|\chi_{(t,\infty)}(u)\,K(u,a_0)\right\|_{\Phi_1} \\ &\quad + \left\|\chi_{(t,\infty)}(u)\,K(u,a_1)\right\|_{\Phi_1} \\ &\lesssim \frac{g_1(t)}{t}\left\|a_0\right\|_{B_0} + \left\|a_1\right\|_{B_1}. \end{aligned} \tag{24}$$

Hence

$$w(t)\,J(t,a) \lesssim K\left(w(t),a;B_0,B_1\right). \tag{25}$$

Now we estimate $K(w(t),a;B_0,B_1)$ from above using decomposition $a = a_0 + a_1$ such that (12) are satisfied. Then

$$\left\|a_0\right\|_{B_0} = \left\|a_0\right\|_{A_0} \lesssim K(t,a). \tag{26}$$

We have

$$\begin{aligned} \left\|a_1\right\|_{B_1} &\lesssim \left\|\chi_{(0,t)}(u)\,K(u,a_1)\right\|_{\Phi_1} \\ &\quad + \left\|\chi_{(t,\infty)}(u)\,K(u,a_1)\right\|_{\Phi_1} := C + D, \\ &\qquad C \lesssim h_1(t)\frac{K(t,a)}{t}, \\ D &\lesssim J(t,a) + g_1(t)\frac{K(t,a)}{t}, \end{aligned} \tag{27}$$

whence

$$w(t)\left\|a_1\right\|_{B_1} \lesssim w(t)\,J(t,a) + K(t,a). \tag{28}$$

Hence

$$K\left(w(t),a;B_0,B_1\right) \lesssim w(t)\,J(t,a) + K(t,a). \tag{29}$$

□

Formulae for the couple $((A_0,A_1)_{\Phi_0},(A_0,A_1)_{\Phi_1})$ are more complicated. We need the following condition that in some sense require these two spaces to be not too close. Let

$$\begin{aligned} \frac{1}{g(t)} &:= \left\|\chi_{(t,\infty)}(u)\frac{u}{g_0(u)+h_0(u)}\right\|_{\Phi_1} < \infty, \\ h(t) &:= \left\|\chi_{(0,t)}(u)\frac{u}{g_1(u)+h_1(u)}\right\|_{\Phi_0} < \infty. \end{aligned} \tag{30}$$

Theorem 4 (case $((A_0,A_1)_{\Phi_0},(A_0,A_1)_{\Phi_1})$). *Let* $B_j := (A_0,A_1)_{\Phi_j}$, $j = 0,1$ *and let* $a \in B_0 + B_1$.

Case 1. If

$$\frac{g_0}{g_1+h_1} \lesssim \rho \lesssim \frac{g_0+h_0}{h_1}, \quad h \lesssim \rho \lesssim g, \tag{31}$$

then

$$\begin{aligned} K\left(\rho(t),a;B_0,B_1\right) &\approx I(t,a) + \rho(t)\,J(t,a) \\ &\quad + \left(g_0(t) + \rho(t)\,h_1(t)\right)\frac{K(t,a)}{t}. \end{aligned} \tag{32}$$

Case 2. If

$$\frac{g_0}{g_1+h_1} \lesssim \rho \lesssim \frac{h_0}{h_1}, \quad h \lesssim \rho \lesssim g, \tag{33}$$

then

$$\begin{aligned} K\left(\rho(t),a;B_0,B_1\right) &\approx I(t,a) + \rho(t)\,J(t,a) \\ &\quad + g_0(t)\frac{K(t,a)}{t}. \end{aligned} \tag{34}$$

Case 3. If

$$\frac{g_0}{g_1} \lesssim \rho \lesssim \frac{g_0+h_0}{h_1}, \quad h \lesssim \rho \lesssim g, \tag{35}$$

then

$$\begin{aligned} K\left(\rho(t),a;B_0,B_1\right) &\approx I(t,a) + \rho(t)\,J(t,a) \\ &\quad + \rho(t)\,h_1(t)\frac{K(t,a)}{t}. \end{aligned} \tag{36}$$

Case 4. If at least one of the conditions

$$\begin{aligned} &g_0/g_1 \lesssim \rho,\ h_1 \lesssim g_1,\ h \lesssim \rho \lesssim g, \\ &g_0 \lesssim h_0,\ \rho \lesssim h_0/h_1,\ h \lesssim \rho \lesssim g, \\ &g_0 \lesssim h_0,\ h_1 \lesssim g_1,\ h \lesssim \rho \lesssim g \end{aligned}$$

is satisfied, then

$$K\left(\rho(t),a;B_0,B_1\right) \approx I(t,a) + \rho(t)\,J(t,a). \tag{37}$$

Proof. From (6) and (20) it follows that

$$\begin{aligned} g_0(t)\frac{K(t,a)}{t} &\lesssim I(t,a) \quad \text{if } g_0 \lesssim h_0, \\ g_0(t)\frac{K(t,a)}{t} &\lesssim \rho(t)\,J(t,a) \quad \text{if } \frac{g_0}{g_1} \lesssim \rho, \\ \rho(t)\,h_1(t)\frac{K(t,a)}{t} &\lesssim I(t,a) \quad \text{if } \rho \lesssim \frac{h_0}{h_1}, \\ \rho(t)\,h_1(t)\frac{K(t,a)}{t} &\lesssim \rho(t)\,J(t,a) \quad \text{if } h_1 \lesssim g_1. \end{aligned} \tag{38}$$

Let $a = a_0 + a_1$, $a_j \in A_j$, and $j = 0,1$. Then, using estimates (7) and (21), we have

$$\begin{aligned} K(t,a) &\lesssim K(t,a_0) + K(t,a_1) \\ &\lesssim \frac{t}{g_0(t)+h_0(t)}\left\|a_0\right\|_{B_0} \\ &\quad + \frac{t}{g_1(t)+h_1(t)}\left\|a_1\right\|_{B_1}, \end{aligned} \tag{39}$$

whence

$$g_0(t)\frac{K(t,a)}{t} \lesssim K(\rho(t),a;B_0,B_1) \quad \text{if } \frac{g_0}{g_1+h_1} \lesssim \rho,$$
$$\rho(t)h_1(t)\frac{K(t,a)}{t} \lesssim K(\rho(t),a;B_0,B_1) \quad \text{if } \rho \lesssim \frac{g_0+h_0}{h_1}. \tag{40}$$

Also,

$$\begin{aligned} I(t,a) &\lesssim \|\chi_{(0,t)}(u)K(u,a_0)\|_{\Phi_0} + \|\chi_{(0,t)}(u)K(u,a_1)\|_{\Phi_0} \\ &\lesssim \|a_0\|_{B_0} + h(t)\|a_1\|_{B_1}, \\ J(t,a) &\lesssim \|\chi_{(t,\infty)}(u)K(u,a_0)\|_{\Phi_1} + \|\chi_{(t,\infty)}(u)K(u,a_1)\|_{\Phi_1} \\ &\lesssim \frac{1}{g(t)}\|a_0\|_{B_0} + \|a_1\|_{B_1}. \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} I(t,a) &\lesssim K(\rho(t),a;B_0,B_1) \quad \text{if } h \lesssim \rho, \\ \rho(t)J(t,a) &\lesssim K(\rho(t),a;B_0,B_1) \quad \text{if } \rho \lesssim g. \end{aligned} \tag{42}$$

To estimate $K(\rho(t),a;B_0,B_1)$ from above we use the same decomposition as before with properties (12). Then

$$\begin{aligned} \|a_0\|_{B_0} &\lesssim \|\chi_{(0,t)}(u)K(u,a_0)\|_{\Phi_0} + \|\chi_{(0,t)}(u)K(u,a_1)\|_{\Phi_0} + \|\chi_{(t,\infty)}(u)K(u,a_0)\|_{\Phi_0}, \end{aligned} \tag{43}$$

whence

$$\begin{aligned} \|a_0\|_{B_0} &\lesssim I(t,a) + (g_0(t)+h_0(t))\frac{K(t,a)}{t} \\ &\lesssim I(t,a) + g_0(t)\frac{K(t,a)}{t}. \end{aligned} \tag{44}$$

Also,

$$\begin{aligned} \|a_1\|_{B_1} &\lesssim \|\chi_{(0,t)}(u)K(u,a_1)\|_{\Phi_1} + \|\chi_{(t,\infty)}(u)K(u,a_1)\|_{\Phi_1} := C + D, \\ C &\lesssim h_1(t)\frac{K(t,a)}{t}, \\ D &\lesssim J(t,a) + g_1(t)\frac{K(t,a)}{t} \lesssim J(t,a), \end{aligned} \tag{45}$$

whence

$$\rho(t)\|a_1\|_{B_1} \lesssim \rho(t)J(t,a) + \rho(t)h_1(t)\frac{K(t,a)}{t}. \tag{46}$$

Therefore for all weights ρ we have

$$\begin{aligned} K(\rho(t),a;B_0,B_1) &\lesssim I(t,a) + \rho(t)J(t,a) \\ &\quad + (g_0(t)+\rho(t)h_1(t))\frac{K(t,a)}{t}. \end{aligned} \tag{47}$$

□

We give examples that are not entirely covered by the previous papers.

Example 5 (distant spaces). Let

$$\|g\|_{\Phi_j} := \left(\int_0^1 \left[t^{-\theta_j}v_j(t)g(t)\right]^{p_j}\frac{dt}{t}\right)^{1/p_j} + \left(\int_1^\infty \left[t^{-\eta_j}w_j(t)g(t)\right]^{q_j}\frac{dt}{t}\right)^{1/q_j}, \tag{48}$$

where $0 \le \theta_0 < \theta_1 \le 1$, $0 \le \eta_0 < \eta_1 \le 1$, $0 < p_j, q_j \le \infty$, and v_j, w_j are slowly varying weights and $j = 0,1$. We call these spaces distant if $\theta_0 < \theta_1$ or $\eta_0 < \eta_1$, in opposition to the case $\theta_0 = \theta_1$ and $\eta_0 = \eta_1$. We have

$$\begin{aligned} g_j(t) &= \begin{cases} t\left(1 + \left(\int_t^1 \left[s^{-\theta_j}v_j(s)\right]^{p_j}\frac{ds}{s}\right)^{1/p_j}\right), & 0 < t < 1, \\ t\left(\int_t^\infty \left[s^{-\eta_j}w_j(s)\right]^{q_j}\frac{ds}{s}\right)^{1/q_j}, & t \ge 1, \end{cases} \\ h_j(t) &= \begin{cases} \left(\int_0^t \left[s^{1-\theta_j}v_j(s)\right]^{p_j}\frac{ds}{s}\right)^{1/p_j}, & 0 < t < 1, \\ 1 + \left(\int_1^t \left[s^{1-\eta_j}w_j(s)\right]^{q_j}\frac{ds}{s}\right)^{1/q_j}, & t \ge 1. \end{cases} \end{aligned} \tag{49}$$

These integrals are convergent due to the property $\min(1,t) \in \Phi_j$, $j = 0,1$. Moreover,

$$\begin{aligned} g_1(t) &\approx \begin{cases} t^{1-\theta_1}v_1(t), & 0 < t < 1, \\ t^{1-\eta_1}w_1(t), & t \ge 1, \end{cases} \\ h_0(t) &\approx \begin{cases} t^{1-\theta_0}v_0(t), & 0 < t < 1, \\ t^{1-\eta_0}w_0(t), & t \ge 1. \end{cases} \end{aligned} \tag{50}$$

Also,

$$g(t) \approx \begin{cases} \dfrac{t^{\theta_1}}{v_1(t)} \left(1 + \left(\displaystyle\int_t^1 \left[s^{-\theta_0} v_0(s)\right]^{p_0} \dfrac{ds}{s}\right)^{1/p_0}\right), & 0 < t < 1, \\ \dfrac{t^{\eta_1}}{w_1(t)} \left(\displaystyle\int_t^\infty \left[s^{-\eta_0} w_0(s)\right]^{q_0} \dfrac{ds}{s}\right)^{1/q_0}, & t \geq 1, \end{cases}$$
$$h(t) \approx \begin{cases} t^{1-\theta_0} v_0(t) \left(\displaystyle\int_0^t \left[s^{1-\theta_1} v_1(s)\right]^{p_1} \dfrac{ds}{s}\right)^{-1/p_1}, & 0 < t < 1, \\ t^{1-\eta_0} w_0(t) \left(1 + \left(\displaystyle\int_1^t \left[s^{1-\eta_1} w_1(s)\right]^{q_1} \dfrac{ds}{s}\right)^{1/q_1}\right)^{-1}. & t \geq 1. \end{cases} \tag{51}$$

In particular,

$$\begin{aligned} g_0 &\gtrsim h_0, \\ g_1 &\lesssim h_1, \\ g &\gtrsim h. \end{aligned} \tag{52}$$

On the other hand,

$$\frac{g_0(t)}{h_1(t)} \approx t \left(1 + \left(\int_t^1 \left[s^{-\theta_0} v_0(s)\right]^{p_0} \frac{ds}{s}\right)^{1/p_0}\right) \cdot \left(\int_0^t \left[s^{1-\theta_1} v_1(s)\right]^{p_1} \frac{ds}{s}\right)^{-1/p_1} \tag{53}$$

for $0 < t < 1$ and

$$\frac{g_0(t)}{h_1(t)} \approx t \left(\int_t^\infty \left[s^{-\eta_0} w_0(s)\right]^{q_0} \frac{ds}{s}\right)^{1/q_0} \cdot \left(1 + \left(\int_1^t \left[s^{1-\eta_1} w_1(s)\right]^{q_1} \frac{ds}{s}\right)^{1/q_1}\right)^{-1} \tag{54}$$

for $t \geq 1$. Hence,

$$h \lesssim \frac{g_0}{h_1} \lesssim g \tag{55}$$

and therefore

$$K\left(\rho(t), a; B_0, B_1\right) \approx I(t,a) + \rho(t) J(t,a) + g_0(t) \frac{K(t,a)}{t}, \tag{56}$$

where $B_0 = (A_0, A_1)_{\Phi_0}$, $B_1 = (A_0, A_1)_{\Phi_1}$, $\rho(t) \approx g_0(t)/h_1(t)$.

Also,

$$K\left(\rho(t), a; (A_0, A_1)_{\Phi_0}, A_1\right) \approx I(t,a) + g_0(t) \frac{K(t,a)}{t}, \tag{57}$$

where $\rho(t) \approx g_0(t)$.

$$K\left(\rho(t), a; A_0, (A_0, A_1)_{\Phi_1}\right) \approx \rho(t) J(t,a) + K(t,a), \tag{58}$$

where $\rho(t) \approx t/h_1(t)$.

Moreover, if $0 < \theta_0 < \theta_1 < 1$ and $0 < \eta_0 < \eta_1 < 1$, then $g_j \approx h_j$ and $j = 0, 1$; hence, in (56), (57), and (58), we can drop the last term (this is the case treated in [3]).

Example 6 (nondistant spaces). Let Φ_j be as in Example 5 but with $\theta_0 = \theta_1 = 0$, $\eta_0 = \eta_1 = 1$, and $v_j(t) = (1 - \log t)^{a_j}$, and $w_j(t) = (1 + \log t)^{b_j}$, $j = 0, 1$.

If

$$\begin{aligned} a_0 - a_1 &< \frac{1}{p_1} - \frac{1}{p_0}, \\ a_j p_j + 1 &> 0, \\ b_1 - b_0 &< \frac{1}{q_0} - \frac{1}{q_1}, \\ b_j q_j + 1 &> 0, \\ p_1 &\leq p_0, \end{aligned} \tag{59}$$

then

$$\begin{aligned} g_j(t) &\approx \begin{cases} t(1 - \log t)^{a_j + 1/p_j}, & 0 < t < 1, \\ (1 + \log t)^{b_j}, & t \geq 1, \end{cases} \\ h_j(t) &\approx \begin{cases} t(1 - \log t)^{a_j}, & 0 < t < 1, \\ (1 + \log t)^{b_j + 1/q_j}, & t \geq 1. \end{cases} \end{aligned} \tag{60}$$

Also,

$$\begin{aligned} g(t) &\approx h(t) \\ &\approx \begin{cases} (1 - \log t)^{a_0 - a_1 + 1/p_0 - 1/p_1}, & 0 < t < 1, \\ (1 + \log t)^{b_0 - b_1 + 1/q_0 - 1/q_1}, & t \geq 1. \end{cases} \end{aligned} \tag{61}$$

In particular,

$$\begin{aligned} g_j(t) &\gtrsim h_j(t), \quad 0 < t < 1, \\ g_j(t) &\lesssim h_j(t), \quad t \geq 1. \end{aligned} \tag{62}$$

On the other hand, since $1/p_0 \le 1/p_1$, we have

$$\frac{g_0}{g_1+h_1} \lesssim h \lesssim \frac{h_0}{h_1} \tag{63}$$

and therefore

$$\begin{aligned} &K\left(\rho(t), a; (A_0, A_1)_{\Phi_0}, (A_0, A_1)_{\Phi_1}\right) \\ &\quad \approx I(t,a) + \rho(t) J(t,a) + g_0(t) \frac{K(t,a)}{t}, \end{aligned} \tag{64}$$

where

$$\rho(t) = \begin{cases} (1-\log t)^{a_0-a_1+1/p_0-1/p_1}, & 0 < t < 1, \\ (1+\log t)^{b_0-b_1+1/q_0-1/q_1}, & t \ge 1. \end{cases} \tag{65}$$

3. Reiteration

The formulae for the K-functional imply immediately theorems of reiteration or stability of the K-method. In particular, we recover many classical results. For another type of general reiteration theorems see [13]. For example, Theorems 1 and 3 imply the following results.

Theorem 7. *Let (A_0, A_1) be a compatible couple of quasi-normed spaces. Then*

$$\left((A_0, A_1)_{\Phi_0}, A_1\right)_{\Phi} = (A_0, A_1)_{\Psi}, \tag{66}$$

where

$$\|g\|_{\Psi} := \left\| \left\| \chi_{(0,\rho^{-1}(t))}(u)\, g(u) \right\|_{\Phi_0} \right\|_{\Phi} + \left\| \frac{t g\left(\rho^{-1}(t)\right)}{\rho^{-1}(t)} \right\|_{\Phi} \tag{67}$$

and ρ is the same as in Theorem 1 and $\rho(0) = 0$, $\rho(\infty) = \infty$, and $\rho(1) = 1$; ρ is increasing.

Proof. We only need to check that $\min(1,t) \in \Psi$. Since

$$\begin{aligned} &\left\| \chi_{(0,1)}(t) \left\| \chi_{(0,\rho^{-1}(t))}(u)\, u \right\|_{\Phi_0} \right\|_{\Phi} \\ &\quad = \left\| \chi_{(0,1)}(t)\, h_0\left(\rho^{-1}(t)\right) \right\|_{\Phi} \end{aligned} \tag{68}$$

and $h_0 \lesssim \rho$, therefore $h_0(\rho^{-1}(t)) \lesssim t$; we see that the above quantity is finite. Also

$$\begin{aligned} &\left\| \chi_{(1,\infty)}(t) \left\| \chi_{(0,1)}(u)\, u \right\|_{\Phi_0} \right\|_{\Phi} = \|\chi_{(1,\infty)}\|_{\Phi}\, h_0(1) < \infty, \\ &\left\| \chi_{(1,\infty)}(t) \left\| \chi_{(1,\rho^{-1}(t))} \right\|_{\Phi_0} \right\|_{\Phi} \le \|\chi_{(1,\infty)}\|_{\Phi} \|\chi_{(1,\infty)}\|_{\Phi_0} \\ &\quad < \infty. \end{aligned} \tag{69}$$

Further, for $t > 1$, we have $g_0(t) \le t$ and $h_0(t) \lesssim 1 + \|u\chi_{(1,t)}(u)\|_{\Phi_0} \lesssim t$; hence $t \lesssim \rho^{-1}(t)$. Therefore

$$\left\| \frac{\chi_{(1,\infty)}(t)\, t}{\rho^{-1}(t)} \right\|_{\Phi} \lesssim \|\chi_{(1,\infty)}\|_{\Phi} < \infty. \tag{70}$$

□

Theorem 8. *Let (A_0, A_1) be a compatible couple of quasi-normed spaces. Then*

$$\left(A_0, (A_0, A_1)_{\Phi_1}\right)_{\Phi} = (A_0, A_1)_{\Psi}, \tag{71}$$

where

$$\|g\|_{\Psi} := \left\| t \left\| \chi_{(\rho^{-1}(t),\infty)} g \right\|_{\Phi_1} \right\|_{\Phi} + \left\| g\left(\rho^{-1}(t)\right) \right\|_{\Phi} \tag{72}$$

and ρ is the same as in Theorem 3 and $\rho(0) = 0$, $\rho(\infty) = \infty$, and $\rho(1) = 1$; ρ is increasing.

Proof. We only need to check that $\min(1,t) \in \Psi$. We have

$$\begin{aligned} &\left\| \chi_{(0,1)}(t)\, t \left\| \chi_{(\rho^{-1}(t),\infty)} \min(1,u) \right\|_{\Phi_1} \right\|_{\Phi} \\ &\quad \lesssim \left\| \chi_{(0,1)}(t)\, t h_1(1) \right\|_{\Phi} < \infty \end{aligned} \tag{73}$$

and, using also $\rho(u) \lesssim u/g_1(u)$,

$$\begin{aligned} &\left\| \chi_{(1,\infty)}(t)\, t \left\| \chi_{(\rho^{-1}(t),\infty)}(u) \right\|_{\Phi_1} \right\|_{\Phi} \\ &\quad \lesssim \left\| \frac{\chi_{(1,\infty)} g_1\left(\rho^{-1}(t)\right)}{\rho^{-1}(t)} \right\|_{\Phi} < \infty. \end{aligned} \tag{74}$$

As above, we check that $g_1(t) \lesssim 1$ and $h_1(t) \lesssim 1$ for $0 < t < 1$. Hence $\rho(t) \gtrsim t$ for $0 < t < 1$. Then

$$\left\| \min\left(1, \rho^{-1}\right) \right\|_{\Phi} \lesssim 1 + \left\| \chi_{(0,1)}(t)\, t \right\|_{\Phi} < \infty. \tag{75}$$

□

In some cases the quasi norm of Ψ can be simplified.

Example 9 (distant spaces). Let

$$\begin{aligned} \|g\|_{\Phi_0} &:= \left(\int_0^1 \left[v_0(t)\, g(t) \right]^{p_0} \frac{dt}{t} \right)^{1/p_0} \\ &\quad + \left(\int_1^{\infty} \left[w_0(t)\, g(t) \right]^{q_0} \frac{dt}{t} \right)^{1/q_0}, \end{aligned} \tag{76}$$

where $0 < p_0, q_0 < \infty$ and v_0, w_0 are slowly varying weights and let

$$\begin{aligned} \|g\|_{\Phi} &:= \left(\int_0^1 \left[t^{-1} v(t)\, g(t) \right]^{p} \frac{dt}{t} \right)^{1/p} \\ &\quad + \left(\int_1^{\infty} \left[t^{-1} w(t)\, g(t) \right]^{q} \frac{dt}{t} \right)^{1/q}, \end{aligned} \tag{77}$$

where $0 < p, q < \infty$ and v, w are slowly varying weights. Then

$$\begin{aligned} &\left((A_0, A_1)_{\Phi_0}, A_1\right)_{\Phi} = (A_0, A_1)_{\Psi}, \\ &\|g\|_{\Psi} \\ &\quad = \left(\int_0^1 \left[t^{-1} v(\rho(t))\, g(t) \right]^{p} \frac{dt}{t} \right)^{1/p} \\ &\quad + \left(\int_1^{\infty} \left[t^{-1} w(\rho(t))\, g(t) \right]^{q} \frac{dt}{t} \right)^{1/q}, \end{aligned} \tag{78}$$

where

$$\rho(t)=\begin{cases} t\left(1+\int_0^t v_0^{p_0}(s)\frac{ds}{s}\right)^{1/p_0}, & 0<t\le 1,\\ t\left(\int_t^\infty w_0^{q_0}(s)\frac{ds}{s}\right)^{1/q_0}, & t\ge 1.\end{cases} \tag{79}$$

Indeed, we have

$$\begin{aligned} &K\left(\rho(t),a;(A_0,A_1)_{\Phi_0},A_1\right)\\ &\quad\approx I(t,a)+\frac{\rho(t)}{t}K(t,a;A_0,A_1)\end{aligned} \tag{80}$$

for $0<t<1$ and

$$\begin{aligned} &K\left(\rho(t),a;(A_0,A_1)_{\Phi_0},A_1\right)\\ &\quad\approx I_1(t,a)\\ &\qquad+\left(\int_0^1\left[v_0(s)K(s,a;A_0,A_1)\right]^{p_0}(s)\frac{ds}{s}\right)^{1/p_0}\\ &\qquad+\frac{\rho(t)}{t}K(t,a;A_0,A_1)\end{aligned} \tag{81}$$

for $t\ge 1$, where

$$\begin{aligned} I(t,a)&=\left(\int_0^t\left(v_0(u)K(u,a)\right)^{p_0}\frac{du}{u}\right)^{1/p_0},\\ I_1(t,a)&=\left(\int_1^t\left(w_0(u)K(u,a)\right)^{q_0}\frac{du}{u}\right)^{1/q_0}.\end{aligned} \tag{82}$$

Let $\rho(1)=1$. Then

$$\|a\|_{((A_0,A_1)_{\Phi_0},A_1)_\Phi}\approx B_0^{1/p}+B_1^{1/p}+C_0^{1/q}+C_1^{1/q}+C_2^{1/q}, \tag{83}$$

where

$$\begin{aligned} B_0&=\int_0^1\left(\frac{v(\rho(t))}{\rho(t)}\right)^p(I(t,a))^p\frac{d\rho}{\rho},\\ B_1&=\int_0^1\left(\frac{v(\rho(t))}{\rho(t)}\right)^p\left(\frac{\rho(t)K(t,a)}{t}\right)^p\frac{d\rho}{\rho},\\ C_0&=\left(\int_0^1\left[v_0(s)K(s,a)\right]^{p_0}(s)\frac{ds}{s}\right)^{1/p_0}\\ &\quad\cdot\int_1^\infty\left(\frac{w(\rho(t))}{\rho(t)}\right)^q\frac{d\rho}{\rho},\\ C_1&=\int_1^\infty\left(\frac{w(\rho(t))}{\rho(t)}\right)^q(I_1(t,a))^q\frac{d\rho}{\rho},\\ C_2&=\int_1^\infty\left(\frac{w(\rho(t))}{\rho(t)}\right)^q\left(\frac{\rho(t)K(t,a)}{t}\right)^q\frac{d\rho}{\rho}.\end{aligned} \tag{84}$$

We can choose equivalent ρ so that $d\rho/\rho\approx dt/t$ (see [6]); hence

$$\begin{aligned} B_1&\approx\int_0^1\left(u^{-1}v(\rho(u))K(u,a)\right)^p\frac{du}{u},\\ C_2&\approx\int_1^\infty\left(u^{-1}w(\rho(u))K(u,a)\right)^q\frac{du}{u}.\end{aligned} \tag{85}$$

It is sufficient to prove that

$$B_0+C_0\lesssim\int_0^1\left(u^{-1}v(\rho(u))K(u,a)\right)^p\frac{du}{u}, \tag{86}$$

$$C_1\lesssim\int_1^\infty\left(u^{-1}w(\rho(u))K(u,a)\right)^q\frac{du}{u}. \tag{87}$$

To estimate C_0 for $p\le p_0$, we use monotonicity of the interpolation scale, while for $p>p_0$ we apply Hölder's inequality. Namely, if $p\le p_0$,

$$\begin{aligned} &\int_0^1\left[v_0(t)K(t,a)\right]^{p_0}\frac{dt}{t}\\ &\quad\lesssim\left(\int_0^1\left[v_0(t)K(t,a)\right]^p\frac{dt}{t}\right)^{p_0/p}\\ &\quad\lesssim\left(\int_0^1\left[t^{-1}v(\rho(t))K(t,a)\right]^{p_0}\frac{dt}{t}\right)^{p_0/p}\end{aligned} \tag{88}$$

and if $p>p_0$, then

$$\begin{aligned} &\int_0^1\left[v_0(t)K(t,a)\right]^{p_0}\frac{dt}{t}\\ &\quad=\int_0^1\left[\frac{tv_0(t)}{v(\rho(t))}\right]^{p_0}\left[t^{-1}v(\rho(t))K(t,a)\right]^{p_0}\frac{dt}{t}\\ &\quad\lesssim\left(\int_0^1\left[t^{-1}v(\rho(t))K(t,a)\right]^p\frac{dt}{t}\right)^{p_0/p}.\end{aligned} \tag{89}$$

Next we check only the inequality for B_0 and the inequality for C_1 being similar. For $p=p_0$, it follows from Fubini's theorem. B_0 can be estimated from above if $p>p_0$ by the Muckenhoupt result [18].

Let $v_1=w_1^{1-s}(\varphi\int_t^1 w_1(u)du)^s$, where $1<s<\infty$, $\varphi>0$, and $w_1>0$. Then

$$\int_0^1\left(\int_0^t\varphi(u)h(u)\,du\right)^s w_1(t)\,dt\lesssim\int_0^1 h^s(t)v_1(t)\,dt. \tag{90}$$

We apply this for $\varphi(t)=v_0(t)^{p_0}/t$, $h(t)=K^{p_0}(t,a)$,

$$\begin{aligned} w_1(t)&=\frac{\left(v(\rho(t))/\rho(t)\right)^p\rho'(t)}{\rho(t)},\\ s&=\frac{p}{p_0}.\end{aligned} \tag{91}$$

Then $v_1(t)\approx[v_0(t)v(\rho(t))/\rho(t)]^p t^{-1}$ and hence

$$B_0\lesssim\int_0^1\left(v_0(t)K(t,a)\right)^p\left(\frac{v(\rho(t))}{\rho(t)}\right)^p\frac{dt}{t}. \tag{92}$$

Further, if $p < p_0$ we estimate B_0 from above using integration by parts. We have

$$B_0 = -\int_0^1 (I(t,a))^{p/p_0}\, d\left(\int_t^1 \left(\frac{v(\rho(t))}{\rho(t)}\right)^p \frac{d\rho}{\rho}\right). \quad (93)$$

Indeed, let $a \in (A_0, A_1)_\Psi$. Using that $v(\rho)/\rho$ is equivalent to a decreasing function, we have

$$\begin{aligned} S &:= (I(t,a))^{1/p_0} \frac{v(\rho(t))}{\rho(t)} \\ &\lesssim \int_0^t \left(\frac{v(\rho(u))\, K(u,a)\, v_0(u)}{\rho(u)}\right)^{p_0} \frac{du}{u}, \end{aligned} \quad (94)$$

and since $tv_0(t) \lesssim \rho(t)$, $p < p_0$,

$$\begin{aligned} S &\lesssim \int_0^t \left(\frac{v(\rho(u))\, K(u,a)}{u}\right)^{p_0} \frac{du}{u} \\ &\lesssim \int_0^t \left(\frac{v(\rho(u))\, K(u,a)}{u}\right)^{p} \frac{du}{u}. \end{aligned} \quad (95)$$

The integral above has a limit zero when $t \to 0$. Thus (93) is true. Further,

$$\begin{aligned} B_0 &\lesssim \int_0^1 (I(t,a))^{p/p_0-1} \left(v_0(t)\, K(t,a)\right) \\ &\quad \cdot \int_t^1 \left(\frac{v(\rho(t))}{\rho(t)}\right)^p \frac{d\rho}{\rho}\frac{dt}{t}, \end{aligned} \quad (96)$$

or

$$B_0 \lesssim \int_0^1 \left(v_0(t)\, K(t,a)\right)^p \left(\frac{v(\rho(t))}{\rho(t)}\right)^p \frac{dt}{t}. \quad (97)$$

Since $\rho(t) = g_0(t) \gtrsim tv_0(t)$, we get (86).

Now we consider examples of nondistant spaces: $B := (A_0, (A_0, A_1)_{w_1(t),p})_{h(t),q}$, w_1-slowly varying, $0 < p < \infty$, $0 < q < \infty$.

In order to handle this case, we need the following result.

Lemma 10. *One has*

$$\begin{aligned} &\int_0^\infty \left(\int_t^\infty \varphi(u)\, g(u)\, du\right)^s w_0(t)\, dt \\ &\quad \approx \int_0^\infty g^s(t)\, v_0(t)\, dt; \end{aligned} \quad (98)$$

$\varphi \geq 0, g \geq 0, w_0 \geq 0, v_0 \geq 0$, and $0 < s < \infty$, if the following conditions are satisfied:

$$\int_t^\infty \varphi(u)\, g(u)\, du \gtrsim g(t) \int_t^\infty \varphi(u)\, du, \quad (99)$$

$$\frac{\varphi(t)}{\int_t^\infty \varphi(u)\, du} \lesssim \frac{w_0(t)}{\int_0^t w_0(u)\, du}, \quad (100)$$

where

$$\begin{aligned} v_0(t) &= \varphi(t) \left(\int_t^\infty \varphi(u)\, du\right)^{s-1} \int_0^t w_0(u)\, du \\ &\qquad \text{for } 0 < s \leq 1, \\ v_0(t) &= w_0^{1-s}(t) \left(\varphi(t) \int_0^t w_0(u)\, du\right)^s \\ &\qquad \text{for } 1 \leq s < \infty. \end{aligned} \quad (101)$$

Proof. The estimate (98) from above is a consequence of Lemmas 3.2, 3.3 in [6]. Using (99), we get an estimate from below with $\int_0^\infty g^s(t) w_0(t) (\int_t^\infty \varphi(u)du)^s dt$. Now condition (100) ensures the equivalence in (98). □

Theorem 11. *Let w_1 be a slowly varying weight and let $\rho^p(t) := 1/\int_t^\infty w_1^p(u)du/u$, $0 < p < \infty$, satisfying $\rho(0) = 0$, $\rho(\infty) = \infty$. If*

$$\int_0^t h^q(s)\, s^q \frac{ds}{s} \approx h^q(t)\, t^q, \quad 0 < q < \infty, \quad (102)$$

then

$$B := \left(A_0, (A_0, A_1)_{w_1(t),p}\right)_{h(t),q} = (A_0, A_1)_{w(t),q}, \quad (103)$$

where

$$w^q(t) = h^q(\rho(t))\, \rho^p(t)\, w_1^p(t). \quad (104)$$

Proof. We have

$$\begin{aligned} &K\left(\rho(t), a; A_0, (A_0, A_1)_{w_1(t),p}\right) \\ &\quad \approx \rho(t) \left(\int_t^\infty w_1^p(s)\, K^p(s,a) \frac{ds}{s}\right)^{1/p}; \end{aligned} \quad (105)$$

hence

$$\begin{aligned} \|a\|_B^q &\approx \int_0^\infty h^q(\rho(t))\, \rho^q(t) \\ &\quad \cdot \left(\int_t^\infty w_1^p(s)\, K^p(s,a) \frac{ds}{s}\right)^{q/p} \frac{d\rho(t)}{\rho(t)}. \end{aligned} \quad (106)$$

We apply Lemma 10 with $\varphi(t) = w_1^p(t)/t$, $g(t) = K^p(t,a)$, $w_0(t) = h^q(\rho(t))\rho^{q-1}(t)\rho'(t)$, and $s = q/p$. Condition (99) follows from monotonicity of K. We have to check (100). To this end first we simplify

$$\begin{aligned} \int_0^t w_0(u)\, du &= \int_0^t h^q(\rho(u))\, \rho^q(u) \frac{d\rho(u)}{\rho(u)} \\ &= \int_0^{\rho(t)} h^q(u)\, u^q \frac{du}{u}. \end{aligned} \quad (107)$$

Using also (102), we get $\int_0^t w_0(u)du \lesssim h^q(\rho(t))\rho^q(t)$. Then (100) means that

$$\frac{w_1^p(t)}{\rho^p(t)} \lesssim \frac{t\rho'(t)}{\rho(t)}. \quad (108)$$

But this follows from the definition of ρ. □

For example, condition (102) is true if $h(t) = t^{-\theta}a(t)$, $0 \leq \theta < 1$, a -slowly varying.

Competing Interests

The authors declare that they have no competing interests.

References

[1] W. D. Evans and B. Opic, "Real interpolation with logarithmic functors and reiteration," *Canadian Journal of Mathematics*, vol. 52, no. 5, pp. 920–960, 2000.

[2] W. D. Evans, B. Opic, and L. Pick, "Real interpolation with logarithmic functors," *Journal of Inequalities and Applications*, vol. 7, no. 2, pp. 187–269, 2002.

[3] A. Gogatishvili, B. Opic, and W. Trebels, "Limiting reiteration for real interpolation with slowly varying functions," *Mathematische Nachrichten*, vol. 278, no. 1-2, pp. 86–107, 2005.

[4] P. Fernández-Martínez and T. Signes, "Real interpolation with symmetric spaces and slowly varying functions," *The Quarterly Journal of Mathematics*, vol. 63, no. 1, pp. 133–164, 2012.

[5] F. Cobos, L. M. Fernández-Cabrera, T. Kühn, and T. Ullrich, "On an extreme class of real interpolation spaces," *Journal of Functional Analysis*, vol. 256, no. 7, pp. 2321–2366, 2009.

[6] I. Ahmed, D. E. Edmunds, W. D. Evans, and G. E. Karadzhov, "Reiteration theorems for the K-interpolation method in limiting cases," *Mathematische Nachrichten*, vol. 284, no. 4, pp. 421–442, 2011.

[7] P. Fernández-Martínez and T. Signes, "Reiteration theorems with extreme values of parameters," *Arkiv för Matematik*, vol. 52, no. 2, pp. 227–256, 2014.

[8] F. Cobos and A. Segurado, "Limiting real interpolation methods for arbitrary Banach couples," *Studia Mathematica*, vol. 213, no. 3, pp. 243–273, 2012.

[9] F. Cobos, L. M. Fernández-Cabrera, and P. Silvestre, "New limiting real interpolation methods and their connection with the methods associated to the unit square," *Mathematische Nachrichten*, vol. 286, no. 5-6, pp. 569–578, 2013.

[10] F. Cobos and A. Segurado, "Some reiteration formulae for limiting real methods," *Journal of Mathematical Analysis and Applications*, vol. 411, no. 1, pp. 405–421, 2014.

[11] P. Fernández-Martínez and T. Signes, "Limit cases of reiteration theorems," *Mathematische Nachrichten*, vol. 288, no. 1, pp. 25–47, 2015.

[12] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, Germany, 1976.

[13] Ju. A. Brudnyi and N. Ja. Krugliak, *Interpolation Spaces and Interpolation Functors*, North-Holland, Amsterdam, The Netherlands, 1991.

[14] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, NY, USA, 1988.

[15] F. Cobos, L. M. Fernández-Cabrera, and P. Silvestre, "Limiting J-spaces for general couples," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 32, no. 1, pp. 83–101, 2013.

[16] F. Cobos and T. Kün, "Equivalence of $K-$ and $J-$methods for limiting real interpolation spaces," *Journal of Functional Analysis*, vol. 261, no. 12, pp. 3696–3722, 2011.

[17] I. Ahmed, G. E. Karadzhov, and A. Raza, "Reiteration in K-interpolation method for quasi-normed spaces," *Comptes Rendus de l'Academie Bulgare des Sciences*, vol. 69, no. 4, pp. 405–410, 2016.

[18] B. Muckenhoupt, "Hardy's inequality with weights," *Studia Mathematica*, vol. 44, pp. 31–38, 1972.