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Research Article

Existence and Multiplicity of Positive Solutions for Schrödinger-Kirchhoff-Poisson System with Singularity

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We study a singular Schrödinger-Kirchhoff-Poisson system by the variational methods and the Nehari manifold and we prove the existence, uniqueness, and multiplicity of positive solutions of the problem under different conditions.

1. Introduction

In the present paper, we consider the following singular Schrödinger-Kirchhoff-Poisson system:

$$\begin{gathered} -\left(a+b\int_{\Omega}|\nabla u|^{2}\,dx\right)\Delta u+\varepsilon\eta\phi u=\lambda f(x)\,u^{-\beta},\quad \text{in } \Omega,\\ -\Delta\phi=\eta u^{2},\quad \text{in } \Omega,\\ u>0,\quad \text{in } \Omega,\\ u=\phi=0,\quad \text{on } \partial\Omega, \end{gathered}\tag{1}$$

where $\Omega\subset R^{3}$ is a smooth bounded domain with boundary $\partial\Omega$, $a>0$, $b\geq 0$, $\eta\geq 0$, and $\lambda>0$ are four parameters and $\varepsilon=\pm 1$, $\beta\in(0,1)$ is a constant, and $f(x)\in C^{1}(\overline{\Omega})$ is a nontrivial nonnegative function.

When $a=1$, $b=0$, $\eta=1$, and $f(x)\equiv 1$, system (1) reduces to the following singular Schrödinger-Poisson system:

$$\begin{gathered} -\Delta u+\varepsilon\phi u=\mu u^{-\beta},\quad \text{in } \Omega,\\ -\Delta\phi=u^{2},\quad \text{in } \Omega,\\ u>0,\quad \text{in } \Omega,\\ u=\phi=0,\quad \text{on } \partial\Omega, \end{gathered}\tag{2}$$

which has been studied in [1]. By using the variational method and the Nehari manifold, the existence, uniqueness, and multiplicity of solutions for system (2) have been obtained.

When $a=1$, $b=0$, and $\eta=0$, system (1) reduces to the following singular semilinear elliptic problem:

$$\begin{gathered} \Delta u+\lambda f(x)\,u^{-\beta}=0,\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega; \end{gathered}\tag{3}$$

the existence and uniqueness of positive solution have been studied in [2, 3] for $\lambda=1$.

When $\eta=0$, the system (1) reduces to the following singular Kirchhoff type problem:

$$\begin{gathered} -\left(a+b\int_{\Omega}|\nabla u|^{2}\,dx\right)\Delta u=\lambda f(x)\,u^{-\beta},\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega. \end{gathered}\tag{4}$$

It is well known that the following Kichhoff type problem:

$$\begin{gathered} -\left(a+b\int_{\Omega}|\nabla u|^{2}\,dx\right)\Delta u=h(x,u),\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega, \end{gathered}\tag{5}$$

where $\Omega\subset R^{3}$ is a smooth bounded domain and $h:\overline{\Omega}\times R\to R$ is a continuous function, has been extensively studied;

see [4–7] and so forth. Problem (5) is called nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 dx$ which implies, when $b \neq 0$, that the equation in (5) is no longer a pointwise identity.

Recently, the singular Kirchhoff type problems have been considered (see [8–11]). In [8], by using the Nehari manifold methods, Liu and Sun proved that problem (5) with $h(x) = f(x)u^{-\beta} + \lambda g(x)(u^p/|x|^s)$ has at least two positive solutions for $\lambda > 0$ small enough. In [9], by using the variational methods, Lei et al. obtained that problem (5) with $h(x) = u^5 + \mu u^{-\beta}$ has at least two positive solutions for $\mu > 0$ small enough. The common characteristic of [8–10] is that the nonlinear terms contain both singular and superlinear. The superlinear term u^p $(p \geq 3)$ can overcome the difficulties which are caused by the Kirchhoff type perturbation $\|u\|^4_{H_0^1(\Omega)}$.

Motivated by the above references, especially by [8–10], we study the singular Schrödinger-Kirchhoff-Poisson system (1). Let $H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v, \quad \|u\| = (u, u)^{1/2}. \tag{6}$$

As usual, $|u|_s = (\int_\Omega |\nabla u|^s)^{1/s}$, $s \in [1, +\infty)$, denotes the norm of the Lebesgue space $L^s(\Omega)$. Let $S > 0$ be the usual Sobolev constant defined by

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega u^6 dx\right)^{1/3}}; \tag{7}$$

that is,

$$|u|_6^2 \leq S^{-1} \|u\|^2, \quad \forall u \in H_0^1(\Omega). \tag{8}$$

By the Hölder inequality and (8), we have

$$\begin{aligned} \int_\Omega f(x)|u|^{1-\beta} &\leq |f|_\infty |u|_4^{1-\beta} |\Omega|^{(3+\beta)/4} \\ &\leq |f|_\infty |u|_6^{1-\beta} |\Omega|^{(5+\beta)/6} \\ &\leq |f|_\infty S^{-(1-\beta)/2} \|u\|^{1-\beta} |\Omega|^{(5+\beta)/6}, \end{aligned} \tag{9}$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω.

Before we state the main results about system (1), we first recall the following well-known facts (see [1]).

Lemma 1. *Let $\eta \geq 0$; then for every $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in H_0^1(\Omega)$ solution of*

$$\begin{aligned} -\Delta\phi &= \eta u^2, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

Moreover

(i) $\|\phi_u\|^2 = \eta \int_\Omega \phi_u |u|^2$;

(ii) $\phi_u \geq 0$. *Moreover* $\phi_u > 0$ *when* $u \neq 0$ *and* $\eta > 0$;

(iii) *for each* $t \neq 0$, $\phi_{tu} = t^2 \phi_u$;

(iv)

$$\begin{aligned} \int_\Omega \phi_u u^2 &\leq \eta S^{-1} |u|_{12/5}^4 \leq \eta S^{-1} |u|_4^4 |\Omega|^{2/3} \\ &\leq \eta S^{-3} \|u\|^4 |\Omega|, \quad u \in H_0^1(\Omega); \end{aligned} \tag{11}$$

(v) *assume that* $u_n \rightharpoonup u$; *then* $\phi_{u_n} \to \phi_u$ *in* $H_0^1(\Omega)$ *and* $\int_\Omega \phi_{u_n} u_n v \to \int_\Omega \phi_u u v$ *for any* $v \in H_0^1(\Omega)$;

(vi) $\phi_u \in W^{2,3}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega})$;

(vii) *for* $u, v \in H_0^1(\Omega)$, $\eta \int_\Omega (\phi_u u - \phi_v v)(u - v) \geq (1/2)\|\phi_u - \phi_v\|^2$.

By Lemma 1, we easily see that system (1) can be converted into a binonlocal type problem of the singular Schrödinger-kirchhoff

$$\begin{aligned} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u + \varepsilon\eta\phi_u u &= \lambda f(x) u^{-\beta}, \\ & \text{in } \Omega, \\ u > 0, \quad & \text{in } \Omega, \\ u = 0, \quad & \text{on } \partial\Omega. \end{aligned} \tag{12}$$

We define the functional

$$\begin{aligned} J_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\varepsilon\eta}{4}\int_\Omega \phi_u u^2 \\ - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u|^{1-\beta}, \quad u \in H_0^1(\Omega). \end{aligned} \tag{13}$$

For any $u, v \in H_0^1(\Omega)$, by (9),

$$\begin{aligned} \left|\int_\Omega f(x)\left(|u|^{1-\beta} - |v|^{1-\beta}\right)\right| &\leq |f|_\infty \int_\Omega |u - v|^{1-\beta} \\ &\leq |f|_\infty |u - v|_4^{1-\beta} |\Omega|^{(3+\beta)/4} \\ &\leq |f|_\infty S^{-(1-\beta)/2} \|u - v\|^{1-\beta} |\Omega|^{(5+\beta)/6}. \end{aligned} \tag{14}$$

Then, by Lemma 1, $J_\lambda(u)$ is well defined and continuous on $H_0^1(\Omega)$.

In general, a function u is called a solution of (12), if $u \in H_0^1(\Omega)$, $u(x) > 0$, $\forall x \in \Omega$, and

$$\begin{aligned} \left(a + b\int_\Omega |\nabla u|^2\right)\int_\Omega \nabla u \cdot \nabla v + \varepsilon\eta\int_\Omega \phi_u u v - \lambda \\ \cdot \int_\Omega f(x) u^{-\beta} v = 0, \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{15}$$

Moreover, $(u, \phi_u) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of system (1) if and only if $u \in H_0^1(\Omega)$ is a solution of (12).

As far as we know, the singular Schrödinger-Kirchhoff-Poisson system has not been considered up to now, and the study on the existence, uniqueness, and multiplicity of solutions for system (1) is meaningful in mathematics. We emphasize that the combined effects of the two nonlocal

terms it contains cause some mathematical difficulties which make the study of such class of problems particularly interesting.

We consider system (1) in two cases: $\varepsilon = 1$ and $\varepsilon = -1$. By using the variational method, the Nehari manifold, and Ekeland's variational principle, we obtain the existence and uniqueness of positive solution for $\varepsilon = 1$ and the multiplicity of positive solutions for $\varepsilon = -1$. To prove the multiplicity of positive solutions, it is necessary that $N_\lambda^\pm \neq \emptyset$, where $N_\lambda^\pm$ are defined in Section 3.2. But the method of getting $N_\lambda^\pm \neq \emptyset$ in [1] cannot apply to our problem. In order to show that $N_\lambda^\pm \neq \emptyset$, we introduce a special nonempty set. Let

$$\begin{aligned} \eta_0 &= b^{1/2} S^{3/2} |\Omega|^{-1/2}, \\ \lambda_* &= \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \frac{|f|_\infty^{-1} S^{(1-\beta)/2} |\Omega|^{-(5+\beta)/6}}{\left(\eta^2 S^{-3}|\Omega| - b\right)^{(1+\beta)/2}}. \end{aligned} \tag{16}$$

Our main results can be described as follows.

Theorem 2. *Assume $a > 0$, $b \geq 0$, $\varepsilon = 1$, and $\beta \in (0,1)$, then system (1) has a unique positive solution for all $\eta \geq 0$ and $\lambda > 0$.*

Theorem 3. *Assume $a > 0$, $b \geq 0$, $\varepsilon = -1$, and $\beta \in (0,1)$, then*

(i) *When $0 \leq \eta \leq \eta_0$, system (1) has at least one positive solution for all $\lambda > 0$;*

(ii) *When $\eta > \eta_0$, system (1) has at least two positive solutions for all $0 < \lambda < \lambda_*$.*

Remark 4. When $a = 1$, $b = 0$, $\eta = 1$, and $f(x) \equiv 1$, we easily see that $\lambda_* = \mu^*$, where μ^* is as in Theorem 1.2 in [1]. Consequently, our results generalize and improve those of [1].

2. Proof of Theorem 2

Lemma 5. *For all $a, \lambda > 0$, $b, \eta \geq 0$, and $\varepsilon = 1$, the functional J_λ attains the global minimizer in $H_0^1(\Omega)$; that is, there exists $u_* \in H_0^1(\Omega)$ such that*

$$J_\lambda(u_*) = m_\lambda = \inf_{u \in H_0^1(\Omega)} J_\lambda(u) < 0. \tag{17}$$

Proof of Lemma 5. For $u \in H_0^1(\Omega)$, by Lemma 1(ii) and (9),

$$\begin{aligned} J_\lambda(u) &\geq \frac{a}{2}\|u\|^2 \\ &\quad - \frac{\lambda}{1-\beta} S^{-(1-\beta)/2} |f|_\infty |\Omega|^{(5+\beta)/6} \|u\|^{1-\beta}, \end{aligned} \tag{18}$$

so, $J_\lambda(u)$ is coercive and bounded from below on $H_0^1(\Omega)$ for any $\lambda > 0$. Thus $m_\lambda = \inf_{u \in H_0^1(\Omega)} J_u(u)$ is well defined. For $t > 0$ and given $u \in H_0^1(\Omega) \setminus \{0\}$,

$$\begin{aligned} J_\lambda(tu) &= \frac{at^2}{2}\|u\|^2 + \frac{b}{4}t^4\|u\|^4 + \frac{\eta t^4}{4}\int_\Omega \phi_u u^2 \\ &\quad - \frac{\lambda}{1-\beta} t^{1-\beta} \int_\Omega f(x)|u|^{1-\beta}; \end{aligned} \tag{19}$$

we can see that for $t > 0$ small enough, $J_\lambda(tu) < 0$. Therefore, we have $m_\lambda = \inf_{u \in H_0^1(\Omega)} J_u(u) < 0$.

According to the definition of m_λ, there exists a minimizing sequence $\{u_n\} \subset H$ such that $\lim_{n\to\infty} J_\lambda(u_n) = m_\lambda < 0$. Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we may assume that $u_n \geq 0$. From (18), it is easy to see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, up to a subsequence; there exists $u_* \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_*, \quad \text{in } H_0^1(\Omega) \\ u_n &\longrightarrow u_*, \quad \text{in } L^p(\Omega), \ p \in [1,6) \\ u_n(x) &\longrightarrow u_*(x), \quad \text{a.e. in } \Omega. \end{aligned} \tag{20}$$

Then by the weakly lower semicontinuity of the norm, Lemma 1(v) and (14), we have

$$\begin{aligned} m_\lambda &\leq J_\lambda(u_*) \\ &= \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\eta}{4}\int_\Omega \phi_{u_*} u_*^2 \\ &\quad - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u_*|^{1-\beta} \leq \liminf_{n\to\infty} J_\lambda(u_n) \\ &= m_\lambda. \end{aligned} \tag{21}$$

So, we have $J_\lambda(u_*) = m_\lambda < 0$. □

Proof of Theorem 2. We divide three steps to prove Theorem 2.

(1) We show $u_* > 0$ in Ω.

From Lemma 5, $u_* \geq 0$ and $u_* \neq 0$. Fix $\varphi \in H_0^1(\Omega)$, $\varphi > 0$, and $t \geq 0$; we have

$$\begin{aligned} 0 &\leq \liminf_{t\to 0} \frac{J_\lambda(u_* + t\varphi) - J_\lambda(u_*)}{t} \\ &= \int_\Omega \left[\left(a + b\|u_*\|^2\right)\nabla u_* \nabla\varphi + \eta\phi_{u_*} u_* \varphi\right] \\ &\quad - \frac{\lambda}{1-\beta} \limsup_{t\to 0} \int_\Omega f(x) \frac{(u_* + t\varphi)^{1-\beta} - u_*^{1-\beta}}{t}; \end{aligned} \tag{22}$$

that is

$$\begin{aligned} &\frac{\lambda}{1-\beta} \limsup_{t\to 0} \int_\Omega f(x) \frac{(u_* + t\varphi)^{1-\beta} - u_*^{1-\beta}}{t} \\ &\quad \leq \int_\Omega \left[\left(a + b\|u_*\|^2\right)\nabla u_* \nabla\varphi + \eta\phi_{u_*} u_* \varphi\right]. \end{aligned} \tag{23}$$

Notice that

$$\int_\Omega f(x)\frac{(u_*+t\varphi)^{1-\beta}-u_*^{1-\beta}}{t} = (1-\beta)\int_\Omega f(x)(u_*+t\varphi\xi)^{-\beta}\varphi, \tag{24}$$

where $\xi(x) \in (0,1)$ and $(u_*(x)+t\varphi(x)\xi(x))^{-\beta}\varphi(x) \to (u_*(x))^{-\beta}\varphi(x)$; a.e $x \in \Omega$, $t \to 0$. Since $(u_*(x)+t\varphi(x)\xi(x))^{-\beta}\varphi(x) \geq 0$, by using Fatou's lemma, we have

$$\lambda\int_\Omega f(x)u_*^{-\beta}\varphi \leq \int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi\right]. \tag{25}$$

By the idea of approximation, the above expression also holds for $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$; that is,

$$\int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi\right] - \lambda\int_\Omega f(x)u_*^{-\beta}\varphi \geq 0. \tag{26}$$

Therefore,

$$-\left(a+b\|u_*\|^2\right)\Delta u_*+\eta\phi_{u_*}u_* \geq 0 \quad \text{in the weak sense.} \tag{27}$$

Since $u_* \geq 0$ and $u_* \neq 0$, by Lemma 1(ii) and (vi), $\phi_{u_*} > 0$ and $\phi_{u_*} \in C^0(\overline{\Omega})$. Therefore, by the strong maximum principle for weak solutions, we obtain that $u_* > 0$ a.e. in Ω.

(2) We show that u_* is a solution of (12); that is, we prove u_* satisfying (15) for $\varepsilon = 1$.

For given $\delta > 0$, define $h : [-\delta,\delta] \to (-\infty,+\infty)$ by $h(t) = J_\lambda(u_*+tu_*)$; then h attains its minimum at $t = 0$ by Lemma 5. It implies that

$$h'(0) = a\|u_*\|^2+b\|u_*\|^4+\eta\int_\Omega\phi_{u_*}u_*^2 - \lambda\int_\Omega f(x)|u_*|^{1-\beta} = 0. \tag{28}$$

We take $\varphi \in H_0^1(\Omega)\setminus\{0\}$, $\rho > 0$ and define $\Psi = (u_*+\rho\varphi)^+$. Let

$$\Omega_1 = \{x\in\Omega : u_*(x)+\rho\varphi(x) > 0\}, \quad \Omega_2 = \{x\in\Omega : u_*(x)+\rho\varphi(x) \leq 0\}. \tag{29}$$

Then $\Psi|_{\Omega_1} = u_*+\rho\varphi$, $\Psi|_{\Omega_2} = 0$. Inserting Ψ into (26) and using (28), we can obtain that

$$\begin{aligned}
0 &\leq \int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\Psi+\eta\phi_{u_*}u_*\Psi - \lambda f(x)u_*^{-\beta}\Psi\right] \\
&= \int_{\Omega_1}\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla(u_*+\rho\varphi) + \eta\phi_{u_*}u_*(u_*+\rho\varphi)-\lambda f(x)u_*^{-\beta}(u_*+\rho\varphi)\right] \\
&= \int_{\Omega\setminus\Omega_2}\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla(u_*+\rho\varphi) + \eta\phi_{u_*}u_*(u_*+\rho\varphi)-\lambda f(x)u_*^{-\beta}(u_*+\rho\varphi)\right] \\
&= \rho\int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi - \lambda f(x)u_*^{-\beta}\varphi\right] - \rho\int_{\Omega_2}\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi \right. \\
&\quad \left. + \eta\phi_{u_*}u_*\varphi-\lambda f(x)u_*^{-\beta}\varphi\right] \\
&\leq \rho\int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi - \lambda f(x)u_*^{-\beta}\varphi\right] - \rho\int_{\Omega_2}\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi \right. \\
&\quad \left. + \eta\phi_{u_*}u_*\varphi\right].
\end{aligned} \tag{30}$$

Since $u_* > 0$ and the measure of the domain $\Omega_2 = \{x\in\Omega : u_*(x)+\rho\varphi(x) \leq 0\}$ tends to zero as $\rho \to 0$, it follows that

$$\int_{\Omega_2}\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi\right] \longrightarrow 0. \tag{31}$$

Then dividing by $\rho > 0$ and letting $\rho \to 0$ in (30), we see that

$$\int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi-\lambda f(x)u_*^{-\beta}\varphi\right] \geq 0, \quad \varphi\in H_0^1(\Omega). \tag{32}$$

This inequality also holds for $-\varphi$, so we get

$$\int_\Omega\left[\left(a+b\|u_*\|^2\right)\nabla u_*\nabla\varphi+\eta\phi_{u_*}u_*\varphi-\lambda f(x)u_*^{-\beta}\varphi\right] = 0, \quad \varphi\in H_0^1(\Omega). \tag{33}$$

Then $u_* \in H_0^1(\Omega)$ is a solution of (12) for $\varepsilon = 1$, $\lambda > 0$, and $\eta \geq 0$.

(3) We show that u_* is the unique solution of (12) for $\varepsilon = 1$, $\lambda > 0$, and $\eta \geq 0$.

Assume that $v_* \in H_0^1(\Omega)$ is also a solution of (12) for $\varepsilon = 1$, $\lambda > 0$, and $\eta \geq 0$. It follows from (15) that

$$\begin{aligned}&\left(a + b\int_\Omega |\nabla u_*|^2\right)\int_\Omega \nabla u_* \cdot \nabla (u_* - v_*)\\&\quad + \eta \int_\Omega \phi_{u_*} u_* (u_* - v_*)\\&\quad - \lambda \int_\Omega f(x) u_*^{-\beta} (u_* - v_*) = 0.\end{aligned} \tag{34}$$

$$\begin{aligned}&\left(a + b\int_\Omega |\nabla v_*|^2\right)\int_\Omega \nabla v_* \cdot \nabla (u_* - v_*)\\&\quad + \eta \int_\Omega \phi_{v_*} v_* (u_* - v_*)\\&\quad - \lambda \int_\Omega f(x) v_*^{-\beta} (u_* - v_*) = 0\end{aligned} \tag{35}$$

Subtracting (34) and (35), we obtain that

$$\begin{aligned}&a\|u_* - v_*\|^2\\&\quad + b\left[\|u_*\|^4 - \left(\|u_*\|^2 + \|v_*\|^2\right)\int_\Omega \nabla u_* \nabla v_* + \|v_*\|^4\right]\\&\quad + \eta \int_\Omega \left(\phi_{u_*} u_* - \phi_{v_*} v_*\right)(u_* - v_*)\\&= \lambda \int_\Omega f(x)\left(u_*^{-\beta} - v_*^{-\beta}\right)(u_* - v_*).\end{aligned} \tag{36}$$

Since $\beta \in (0,1)$, $u_*, v_* > 0$ in Ω, the following inequality holds:

$$\lambda \int_\Omega f(x)\left(u_*^{-\beta} - v_*^{-\beta}\right)(u_* - v_*) \leq 0. \tag{37}$$

Consequently, it follows from Lemma 1(vii) and (36) that

$$\|u_* - v_*\| \leq 0, \tag{38}$$

which implies that

$$\|u_* - v_*\| = 0; \tag{39}$$

that is $(u_*, \phi_{u_*}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is the unique solution of system (1). □

3. Proof of Theorem 3

3.1. The Case of $0 \leq \eta \leq \eta_0$. In this part, let $0 \leq \eta \leq \eta_0$, where $\eta_0 = b^{1/2} S^{3/2} |\Omega|^{-1/2}$. From Lemma 1(iv) and (9), we have

$$\begin{aligned}J_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\eta}{4}\int_\Omega \phi_u u^2\\&\quad - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u|^{1-\beta}\\&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\eta^2}{4} S^{-3}|\Omega|\|u\|^4\\&\quad - \frac{\lambda}{1-\beta}|f|_\infty S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u\|^{1-\beta}\\&\geq \frac{a}{2}\|u\|^2\\&\quad - \frac{\lambda}{1-\beta}|f|_\infty S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u\|^{1-\beta},\\&\qquad \forall u \in H_0^1(\Omega).\end{aligned} \tag{40}$$

Thus, $J_\lambda(u)$ is coercive and bounded from below on $H_0^1(\Omega)$ for all $0 \leq \eta \leq \eta_0$ and $\lambda > 0$. So

$$\alpha_\lambda = \inf_{u \in H_0^1(\Omega)} J_\lambda(u) \tag{41}$$

is well defined. Obviously, $\alpha_\lambda < 0$. Similar to the proof of Lemma 5, we easily obtain the following proposition.

Proposition 6. *For all $a > 0$, $b \geq 0$, $\varepsilon = -1$, and $\beta \in (0,1)$, the functional J_λ attains the global minimizer in $H_0^1(\Omega)$; that is, there exists $u^* \in H_0^1(\Omega)$ such that*

$$J_\lambda(u^*) = \alpha_\lambda = \inf_{u \in H_0^1(\Omega)} J_\lambda(u) < 0. \tag{42}$$

Proof of Theorem 3(i). The proof is similar to parts (1) and (2) of the proof in Theorem 2, so we omit it. □

3.2. The Case of $\eta > \eta_0$. In this part, let $\eta > \eta_0$. $J_\lambda(u)$ is not necessarily coercive and bounded from below on $H_0^1(\Omega)$ for $\eta > \eta_0$. We define a Nehari manifold N_λ by

$$\begin{aligned}N_\lambda = \Big\{&u \in H_0^1(\Omega) : a\|u\|^2 + b\|u\|^4 - \eta\int_\Omega \phi_u u^2\\&- \lambda\int_\Omega f(x)|u|^{1-\beta}\,dx = 0\Big\}.\end{aligned} \tag{43}$$

Then the solutions of (12) must lie in N_λ. Obviously, N_λ is a closed set in $H_0^1(\Omega)$. In order to obtain the multiplicity, we split N_λ into the following three parts:

$$N_\lambda^+ = \left\{u \in N_\lambda : 2a\|u\|^2 + 4b\|u\|^4 - 4\eta\int_\Omega \phi_u u^2 - \lambda(1-\beta)\int_\Omega f(x)|u|^{1-\beta}dx > 0\right\}, \tag{44}$$

$$N_\lambda^0 = \left\{u \in N_\lambda : 2a\|u\|^2 + 4b\|u\|^4 - 4\eta\int_\Omega \phi_u u^2 - \lambda(1-\beta)\int_\Omega f(x)|u|^{1-\beta}dx = 0\right\}, \tag{45}$$

$$N_\lambda^- = \left\{u \in N_\lambda : 2a\|u\|^2 + 4b\|u\|^4 - 4\eta\int_\Omega \phi_u u^2 - \lambda(1-\beta)\int_\Omega f(x)|u|^{1-\beta}dx < 0\right\}. \tag{46}$$

When $u \in N_\lambda$, we have

$$\begin{aligned} &2a\|u\|^2 + 4b\|u\|^4 - 4\eta\int_\Omega \phi_u u^2 \\ &\quad - \lambda(1-\beta)\int_\Omega f(x)|u|^{1-\beta} = -2a\|u\|^2 \\ &\quad + (3+\beta)\lambda\int_\Omega f(x)|u|^{1-\beta} = 2b\|u\|^4 - 2\eta\int_\Omega \phi_u u^2 \\ &\quad + (1+\beta)\lambda\int_\Omega f(x)|u|^{1-\beta} = (1+\beta)a\|u\|^2 \\ &\quad + (3+\beta)b\|u\|^4 - (3+\beta)\eta\int_\Omega \phi_u u^2. \end{aligned} \tag{47}$$

Lemma 7. *For any $\eta > \eta_0$ there exists $\lambda_* > 0$ such that $N_\lambda^\pm \neq \emptyset$ and $N_\lambda^0 = \{0\}$ for $\lambda \in (0, \lambda_*)$.*

Proof. For any given $u \in H_0^1(\Omega) \setminus \{0\}$, $t \geq 0$, by calculating, we can get that

$$\begin{aligned} t\frac{d}{dt}[J_\lambda(tu)] &= at^2\|u\|^2 + bt^4\|u\|^4 - \eta t^4\int_\Omega \phi_u u^2 \\ &\quad - \lambda t^{1-\beta}\int_\Omega f(x)|u|^{1-\beta} = t^{1-\beta}\Big(at^{1+\beta}\|u\|^2 \\ &\quad + bt^{3+\beta}\|u\|^4 - \eta t^{3+\beta}\int_\Omega \phi_u u^2 - \lambda\int_\Omega f(x)|u|^{1-\beta}\Big). \end{aligned} \tag{48}$$

Let

$$\Phi_u(t) = at^{1+\beta}\|u\|^2 + bt^{3+\beta}\|u\|^4 - \eta t^{3+\beta}\int_\Omega \phi_u u^2, \quad t \geq 0. \tag{49}$$

Then,

$$\begin{aligned} \Phi_u'(t) &= (1+\beta)at^\beta\|u\|^2 + (3+\beta)bt^{2+\beta}\|u\|^4 \\ &\quad - (3+\beta)\eta t^{2+\beta}\int_\Omega \phi_u u^2. \end{aligned} \tag{50}$$

Since $\eta > \eta_0$, it is easy to obtain that

$$G = \left\{u \in H_0^1(\Omega) \mid b\|u\|^4 - \eta\int_\Omega \phi_u u^2 < 0\right\} \neq \emptyset. \tag{51}$$

Therefore, for any $u \in G$, let $\Phi_u'(t) = 0$. Then

$$t_u = \left[\frac{a(1+\beta)\|u\|^2}{(3+\beta)\left(\eta\int_\Omega \phi_u u^2 - b\|u\|^4\right)}\right]^{1/2}. \tag{52}$$

Moreover, $\Phi_u'(t) > 0$ for all $0 < t < t_u$ and $\Phi_u'(t) < 0$ for all $t > t_u$. Thus Φ_u is increasing for all $0 < t < t_u$ and decreasing for all $t > t_u$. So

$$\max_{t\in[0,+\infty)} \Phi_u(t) = \Phi_u(t_u) = \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \cdot \frac{\|u\|^{3+\beta}}{\left(\eta\int_\Omega \phi_u u^2 - b\|u\|^4\right)^{(1+\beta)/2}} > 0, \tag{53}$$

$$\begin{aligned} &\Phi_u(0) = 0, \\ &\lim_{t\to\infty}\Phi_u(t) = -\infty. \end{aligned} \tag{54}$$

Thus from Lemma 1(iv), (9), and (53), it follows that

$$\begin{aligned} &\Phi_u(t_u) - \lambda\int_\Omega f(x)|u|^{1-\beta}dx \\ &\quad \geq \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \cdot \frac{\|u\|^{3+\beta}}{\left(\eta\int_\Omega \phi_u u^2 - b\|u\|^4\right)^{(1+\beta)/2}} - \lambda|f|_\infty \\ &\quad \cdot S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u\|^{1-\beta} \\ &\quad \geq \left\{\frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \cdot \frac{1}{(\eta^2 S^{-3}|\Omega| - b)^{(1+\beta)/2}} - \lambda|f|_\infty \cdot S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\right\}\|u\|^{1-\beta} > 0 \end{aligned} \tag{55}$$

for all $0 < \lambda < \lambda_*$.

From (54) and (55), we obtain that there exist unique positive numbers $t_u^+ < t_u < t_u^-$ such that

$$\begin{aligned} &\Phi_u(t_u^+) = \lambda\int_\Omega f(x)|u|^{1-\beta}dx = \Phi_u(t_u^-), \\ &\Phi_u'(t_u^+) > 0, \\ &\Phi_u'(t_u^-) < 0. \end{aligned} \tag{56}$$

From (48)–(50), it follows that $t_u^+ u \in N_\lambda^+$ and $t_u^- u \in N_\lambda^-$ for all $0 < \lambda < \lambda_*$.

Next, we prove that $N_\lambda^0 = \{0\}$ for all $0 < \lambda < \lambda_*$. Assume that there exists $u_0 \in N_\lambda^0$ and $u_0 \neq 0$. Then it follows that

$$\begin{aligned} &a(1+\beta)\|u_0\|^2 + b(3+\beta)\|u_0\|^4 \\ &\quad - (3+\beta)\eta\int_\Omega \phi_{u_0}(u_0)^2 = 0, \\ &0 = a\|u_0\|^2 + b\|u_0\|^4 - \eta\int_\Omega \phi_{u_0}u_0^2 \\ &\quad - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx = \frac{2a}{3+\beta}\|u_0\|^2 \\ &\quad - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx. \end{aligned} \tag{57}$$

From (57) and (55), one has $u_0 \in G$ and

$$\begin{aligned} 0 &< \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\ &\quad \cdot \frac{\|u_0\|^{3+\beta}}{\left(\eta\int_\Omega \phi_{u_0}u_0^2 - b\|u_0\|^4\right)^{(1+\beta)/2}} - \lambda|f|_\infty \\ &\quad \cdot S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u_0\|^{1-\beta} \\ &\le \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\ &\quad \cdot \frac{\|u_0\|^{3+\beta}}{\left(\eta\int_\Omega \phi_{u_0}u_0^2 - b\|u_0\|^4\right)^{(1+\beta)/2}} \\ &\quad - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx \\ &= \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\ &\quad \cdot \frac{\|u_0\|^{3+\beta}}{\left((a(1+\beta)/(3+\beta))\|u_0\|^2\right)^{(1+\beta)/2}} \\ &\quad - \frac{2a}{3+\beta}\|u_0\|^2 = 0 \end{aligned} \tag{58}$$

for all $0 < \lambda < \lambda_*$, which implies a contradiction. Thus $N_\lambda^0 = \{0\}$ for all $0 < \lambda < \lambda_*$. □

Lemma 8. *N_λ^- is a closed set in $H_0^1(\Omega)$ for $\lambda \in (0, \lambda_*)$.*

Proof. Suppose $u_n \subset N_\lambda^-$ such that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. From the definition of N_λ^-, one has

$$\begin{aligned} &a\|u_n\|^2 + b\|u_n\|^4 - \eta\int_\Omega \phi_{u_n}u_n^2 \\ &\quad - \lambda\int_\Omega f(x)|u_n|^{1-\beta}dx = 0, \end{aligned} \tag{59}$$

$$\begin{aligned} &a(1+\beta)\|u_n\|^2 + b(3+\beta)\|u_n\|^4 \\ &\quad - \eta(3+\beta)\int_\Omega \phi_{u_n}u_n^2 < 0. \end{aligned} \tag{60}$$

Thus it follows from $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$ that

$$\begin{aligned} &a\|u\|^2 + b\|u\|^4 - \eta\int_\Omega \phi_u u^2 - \lambda\int_\Omega f(x)|u|^{1-\beta}dx \\ &\quad = 0, \\ &a(1+\beta)\|u\|^2 + b(3+\beta)\|u\|^4 - \eta(3+\beta)\int_\Omega \phi_u u^2 \\ &\quad \le 0, \end{aligned} \tag{61}$$

so $u \in N_\lambda^0 \cup N_\lambda^-$. If $u \in N_\lambda^0$, then $u = 0$. However, from (60), we obtain

$$\|u\|^2 \ge \frac{a(1+\beta)}{(3+\beta)(\eta^2 S^{-3}|\Omega| - b)}, \quad \forall u_n \in N_\lambda^-, \tag{62}$$

which contracts $u = 0$. Thus $u \in N_\lambda^-$ for all $0 < \lambda < \lambda_*$. Therefore, N_λ^- is closed for all $0 < \lambda < \lambda_*$. □

Lemma 9. *Assume $\eta > \eta_0$ and $\lambda \in (0, \lambda_*)$, $J_\lambda(u)$ is coercive and bounded from below on N_λ and $\inf_{u \in N_\lambda^+ \cup N_\lambda^0} J_\lambda(u) < 0$.*

Proof. For all $u \in N_\lambda$, from (9) we have

$$\begin{aligned} &J_\lambda(u) \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\eta}{4}\int_\Omega \phi_u u^2 \\ &\quad - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u|^{1-\beta} \\ &= \frac{a}{2}\|u\|^2 + \frac{1}{4}\left(-a\|u\|^2 + \lambda\int_\Omega f(x)|u|^{1-\beta}dx\right) \\ &\quad - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u|^{1-\beta} \\ &= \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{1-\beta} - \frac{1}{4}\right)\int_\Omega f(x)|u|^{1-\beta} \\ &\ge \frac{a}{4}\|u\|^2 \\ &\quad - \lambda\left(\frac{1}{1-\beta} - \frac{1}{4}\right)|f|_\infty S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u\|^{1-\beta}. \end{aligned} \tag{63}$$

It follows from $0 < \beta < 1$ that $J_\lambda(u)$ is coercive and bounded from below on N_λ for any $\lambda \in (0, \lambda_*)$.

Since $N_\lambda^+ \cup N_\lambda^0$ and N_λ^- are two nonempty closed subsets of $H_0^1(\Omega)$ for all $\eta > \eta_0$ and $\lambda \in (0, \lambda_*)$, it follows that $\alpha_\lambda^+ = \inf_{u \in N_\lambda^+ \cup N_\lambda^0} J_\lambda(u)$ and $\alpha_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u)$ are well defined. For any given $u \in N_\lambda^+$, by (47), we have

$$a(1+\beta)\|u\|^2 + b(3+\beta)\|u\|^4 - \eta(3+\beta)\int_\Omega \phi_u u^2 > 0. \tag{64}$$

Consequently,

$$\begin{aligned} J_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\eta}{4}\int_\Omega \phi_u u^2 - \frac{\lambda}{1-\beta}\int_\Omega f(x)|u|^{1-\beta} \\ &= -\frac{a(1+\beta)}{2(1-\beta)}\|u\|^2 - \frac{b(3+\beta)}{4(1-\beta)}\|u\|^4 + \frac{\eta(3+\beta)}{4(1-\beta)}\int_\Omega \phi_u u^2 \\ &< \frac{-a(1+\beta)\|u\|^2 - b(3+\beta)\|u\|^4 + \eta(3+\beta)\int_\Omega \phi_u u^2}{4(1-\beta)} \\ &< 0. \end{aligned} \tag{65}$$

Thus, by Lemma 7, $\alpha_\lambda^+ = \inf_{u \in N_\lambda^+ \cup N_\lambda^0} J_\lambda(u) < 0$ for all $\eta > \eta_0$ and $0 < \lambda < \lambda_*$. □

Lemma 10. *Assume that $\eta > \eta_0$ and $\lambda \in (0, \lambda_*)$, given $u \in N_\lambda^+$ (resp., N_λ^-), there exists $\varepsilon > 0$ and a continuous function $f = f(w) > 0$, $w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$, satisfying that*

$$\begin{aligned} &f(0) = 1, \\ &f(w)(u+w) \in N_\lambda^+ \ (\textit{resp. } N_\lambda^-), \\ &\qquad \forall w \in H_0^1(\Omega),\ \|w\| < \varepsilon. \end{aligned} \tag{66}$$

Proof. Please see Lemma 3.5 of [1] for the similar proof. □

Proof of Theorem 3(ii). By Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset N_\lambda^+ \cup N_\lambda^0$ satisfying

(i) $J_\lambda(u_n) \le \inf_{u \in N_\lambda^+ \cup N_\lambda^0} J_\lambda(u) + 1/n^2$,

(ii) $J_\lambda(u) \ge J_\lambda(u_n) - (1/n)\|u - u_n\|$, $u \in N_\lambda^+ \cup N_\lambda^0$.

From $J_\lambda(|u|) = J_\lambda(u)$ and Lemma 9, we may assume that $u_n \in N_\lambda^+$ and $u_n \ge 0$. It is easy to obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$; we assume that $\|u_n\| \le C_0$. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u_0, \quad \text{in } H_0^1(\Omega) \\ &u_n \longrightarrow u_0, \quad \text{in } L^p(\Omega),\ p \in [1,6) \\ &u_n(x) \longrightarrow u_0(x), \quad \text{a.e. in } \Omega. \end{aligned} \tag{67}$$

Since $\{u_n\} \subset N_\lambda^+ \subset N_\lambda$, we get

$$a\|u_n\|^2 + b\|u_n\|^4 - \eta\int_\Omega \phi_{u_n} u_n^2 - \lambda\int_\Omega f(x)|u_n|^{1-\beta}dx = 0. \tag{68}$$

By the weakly lower semicontinuity of the norm, Lemma 1(v), (14), and (47), we have

$$\begin{aligned} &a\|u_0\|^2 + b\|u_0\|^4 - \eta\int_\Omega \phi_{u_0} u_0^2 - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx \le 0, \\ &-2a\|u_0\|^2 + (3+\beta)\lambda\int_\Omega f(x)|u_0|^{1-\beta}dx \ge 0, \\ &J_\lambda(u_0) \le \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in N_\lambda^+ \cup N_\lambda^0} J_\lambda(u) < 0. \end{aligned} \tag{69}$$

Therefore, $u_0(x) \ge 0$ and $u_0(x) \not\equiv 0$.

First, we prove that $u_0(x) > 0$ in Ω. Since $\{u_n\} \subset N_\lambda^+$, we can claim that there exists a constant $C_1 > 0$ such that up to a subsequence we have

$$a(1+\beta)\|u_n\|^2 + b(3+\beta)\|u_n\|^4 - \eta(3+\beta)\int_\Omega \phi_{u_n} u_n^2 \ge C_1. \tag{70}$$

In order to prove (70), it suffices to verify

$$a(1+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^2 + b(3+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^4 > \eta(3+\beta)\int_\Omega \phi_{u_0} u_0^2. \tag{71}$$

Since $u_n \in N_\lambda^+$,

$$a(1+\beta)\|u_n\|^2 + b(3+\beta)\|u_n\|^4 - \eta(3+\beta)\int_\Omega \phi_{u_n} u_n^2 > 0. \tag{72}$$

It follows that

$$a(1+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^2 + b(3+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^4 \ge \eta(3+\beta)\int_\Omega \phi_{u_0} u_0^2. \tag{73}$$

By contradiction, we assume that

$$a(1+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^2 + b(3+\beta)\overline{\lim_{n\to\infty}}\|u_n\|^4 = \eta(3+\beta)\int_\Omega \phi_{u_0} u_0^2. \tag{74}$$

From the boundedness of $\{u_n\}$ and (72), one has

$$a(1+\beta)\underline{\lim_{n\to\infty}}\|u_n\|^2 + b(3+\beta)\underline{\lim_{n\to\infty}}\|u_n\|^4 \ge \eta(3+\beta)\int_\Omega \phi_{u_0} u_0^2, \tag{75}$$

which combines with (74); it follows that

$$a(1+\beta)\lim_{n\to\infty}\|u_n\|^2 + b(3+\beta)\lim_{n\to\infty}\|u_n\|^4 = \eta(3+\beta)\int_\Omega \phi_{u_0}u_0^2. \tag{76}$$

Let $\lim_{n\to\infty}\|u_n\| = \ell$; then $\ell > 0$ and

$$a(1+\beta)\ell + b(3+\beta)\ell^2 = \eta(3+\beta)\int_\Omega \phi_{u_0}u_0^2. \tag{77}$$

Since $u_n \in N_\lambda$, one has

$$a\ell + b\ell^2 - \eta\int_\Omega \phi_{u_0}u_0^2 - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx = 0. \tag{78}$$

Consequently, we have

$$\frac{2a\ell}{3+\beta} = \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx. \tag{79}$$

For all $0 < \lambda < \lambda_*$, from (55), (77), and (79),

$$\begin{aligned}
0 &< \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\
&\quad\cdot \frac{\|u_0\|^{3+\beta}}{\left(\eta\int_\Omega \phi_{u_0}u_0^2 - b\|u_0\|^4\right)^{(1+\beta)/2}} - \lambda|f|_\infty \\
&\quad\cdot S^{-(1-\beta)/2}|\Omega|^{(5+\beta)/6}\|u_0\|^{1-\beta} \\
&< \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\
&\quad\cdot \frac{\|u_0\|^{3+\beta}}{\left(\eta\int_\Omega \phi_{u_0}u_0^2 - b\|u_0\|^4\right)^{(1+\beta)/2}} \\
&\quad - \lambda\int_\Omega f(x)|u_0|^{1-\beta}dx \\
&= \frac{2a}{3+\beta}\left[\frac{a(1+\beta)}{3+\beta}\right]^{(1+\beta)/2} \\
&\quad\cdot \frac{\ell^{(3+\beta)/2}}{\left((a(1+\beta)/(3+\beta))\ell\right)^{(1+\beta)/2}} - \frac{2a\ell}{3+\beta} = 0,
\end{aligned} \tag{80}$$

which is a contradiction. Thus our claim is true; that is, (70) and (71) must hold.

Taking $N = 2(1-\beta)C_0C_1/(1+\beta)$, for fixed $\varphi \in H_0^1(\Omega)$ and $\varphi > 0$ in Ω, we apply Lemma 10 with $u = u_n$, and $w = s\varphi$, $s > 0$, small enough; it is easy to see that $g_n(s) = f_n(s\varphi)$ such that $g_n(0) = 1$ and $g_n(u_n + s\varphi) \in N_\lambda^+$. It follows from u_n, $g_n(s)(u_n + s\varphi) \in N_\lambda^+ \subset N_\lambda$ that

$$\begin{aligned}
&a\|u_n\|^2 + b\|u_n\|^4 - \eta\int_\Omega \phi_{u_n}u_n^2 \\
&\quad - \lambda\int_\Omega f(x)|u_n|^{1-\beta}dx = 0, \\
&ag_n^2(s)\|u_n + s\varphi\|^2 + bg_n^4(s)\|u_n + s\varphi\|^4 \\
&\quad - \eta g_n^4(s)\int_\Omega \phi_{u_n+s\varphi}(u_n + s\varphi)^2 \\
&\quad - \lambda g_n^{1-\beta}(s)\int_\Omega f(x)|u_n + s\varphi|^{1-\beta}dx = 0.
\end{aligned} \tag{81}$$

By the above equalities, we have

$$\begin{aligned}
0 &= \left(g_n^2(s) - 1\right)a\|u_n + s\varphi\|^2 \\
&\quad + a\left(\|u_n + s\varphi\|^2 - \|u_n\|^2\right) \\
&\quad + \left(g_n^4(s) - 1\right)b\|u_n + s\varphi\|^4 \\
&\quad + b\left(\|u_n + s\varphi\|^4 - \|u_n\|^4\right) \\
&\quad - \left(g_n^4(s) - 1\right)\eta\int_\Omega \phi_{u_n+s\varphi}(u_n + s\varphi)^2 \\
&\quad - \eta\int_\Omega \phi_{u_n+s\varphi}^2(u_n + s\varphi)^2 - \phi_{u_n}u_n^2dx \\
&\quad - \left(g_n^{1-\beta}(s) - 1\right)\lambda\int_\Omega f(x)|u_n + s\varphi|^{1-\beta}dx \\
&\quad - \lambda\int_\Omega f(x)\left(|u_n + s\varphi|^{1-\beta} - |u_n|^{1-\beta}\right)dx \\
&\le \left(g_n^2(s) - 1\right)a\|u_n + s\varphi\|^2 \\
&\quad + a\left(\|u_n + s\varphi\|^2 - \|u_n\|^2\right) \\
&\quad + \left(g_n^4(s) - 1\right)b\|u_n + s\varphi\|^4 \\
&\quad + b\left(\|u_n + s\varphi\|^4 - \|u_n\|^4\right) \\
&\quad - \left(g_n^4(s) - 1\right)\eta\int_\Omega \phi_{u_n+s\varphi}(u_n + s\varphi)^2 \\
&\quad - \eta\int_\Omega \phi_{u_n+s\varphi}^2(u_n + s\varphi)^2 - \phi_{u_n}u_n^2dx \\
&\quad - \left(g_n^{1-\beta}(s) - 1\right)\lambda\int_\Omega f(x)|u_n + s\varphi|^{1-\beta}dx.
\end{aligned} \tag{82}$$

Denote $D^+g_n(0)$ the right upper Dini derivative of g_n at zero. Then, dividing by $s > 0$ and letting $s \to 0^+$, we deduce that

$$\begin{aligned}
0 &\le 2aD^+g_n(0)\|u_n\|^2 + 2a\int_\Omega \nabla u_n\nabla\phi\,dx \\
&\quad + 4bD^+g_n(0)\|u_n\|^4 + 4b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\,dx
\end{aligned}$$

$$
\begin{aligned}
&-4\eta D^{+}g_n(0)\int_\Omega \phi_{u_n}u_n^2-4\eta\int_\Omega \phi_{u_n}u_n^2-(1-\beta)\\
&\cdot\lambda D^{+}g_n(0)\int_\Omega f(x)\,u_n^{1-\beta}dx=D^{+}g_n(0)\Big[2a\|u_n\|^2\\
&+4b\|u_n\|^4-4\eta\int_\Omega \phi_{u_n}u_n^2\\
&-(1-\beta)\lambda\int_\Omega f(x)\,u_n^{1-\beta}dx\Big]+\Big[2a\int_\Omega \nabla u_n\nabla\varphi\,dx\\
&+4b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\,dx-4\eta\int_\Omega \phi_{u_n}u_n^2\Big]\\
&=D^{+}g_n(0)\Big[(1+\beta)a\|u_n\|^2+(3+\beta)b\|u_n\|^4\\
&-(3+\beta)\eta\int_\Omega \phi_{u_n}u_n^2\Big]+\Big[2a\int_\Omega \nabla u_n\nabla\varphi\,dx\\
&+4b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\,dx-4\eta\int_\Omega \phi_{u_n}u_n^2\Big].
\end{aligned}
\tag{83}
$$

By $\|u_n\|\le C_0$ and Lemma 1(iv), there exist $C_2,C_3>0$ independent of n such that

$$
\begin{aligned}
&\Big|2a\int_\Omega \nabla u_n\nabla\varphi\,dx+4b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\,dx\\
&\quad-4\eta\int_\Omega \phi_{u_n}u_n^2\Big|\le C_2,
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&\Big|\frac{1}{1-\beta}\Big[(1+\beta)a\int_\Omega \nabla u_n\nabla\varphi\\
&\quad+(3+\beta)b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\\
&\quad-(3+\beta)\eta\int_\Omega \phi_{u_n}u_n\varphi\Big]\Big|\le C_3.
\end{aligned}
\tag{85}
$$

It follows from (70), (83), and (84) that

$$
D^{+}g_n(0)\ge -C_1^{-1}C_2. \tag{86}
$$

In the following part, we prove

$$
D^{+}g_n(0)\le 2\left(\|\varphi\|+C_3\right)C_1^{-1},\quad \forall n\ge N. \tag{87}
$$

Fixing $n\ge N$, without loss of generality, we may assume $D^{+}g_n(0)\ge 0$. Thus, from condition (ii) and (65), we have

$$
\begin{aligned}
&\frac{|g_n(s)-1|\cdot\|u_n\|}{n}+\frac{|sg_n(s)|\cdot\|\varphi\|}{n}\\
&\ge\frac{1}{n}\|g_n(s)(u_n+s\varphi)-u_n\|\\
&\ge J_\lambda(u_n)-J_\lambda[g_n(s)(u_n+s\varphi)]\\
&=\frac{1+\beta}{2(1-\beta)}\left(a\|u_n+s\varphi\|^2-a\|u_n\|^2\right)\\
&\quad+\frac{1+\beta}{2(1-\beta)}a\left[g_n^2(s)-1\right]\|u_n+s\varphi\|^2\\
&\quad+\frac{3+\beta}{4(1-\beta)}\left(b\|u_n+s\varphi\|^4-b\|u_n\|^4\right)\\
&\quad+\frac{3+\beta}{4(1-\beta)}b\left[g_n^4(s)-1\right]\|u_n+s\varphi\|^4\\
&\quad-\eta\frac{3+\beta}{4(1-\beta)}g_n^4(s)\int_\Omega\left[\phi_{u_n+s\varphi}(u_n+s\varphi)^2-\phi_{u_n}u_n^2\right]\\
&\quad-\eta\frac{3+\beta}{4(1-\beta)}\left[g_n^4(s)-1\right]\int_\Omega \phi_{u_n}u_n^2dx.
\end{aligned}
\tag{88}
$$

Dividing by $s>0$ and letting $s\to 0^{+}$, we derive that

$$
\begin{aligned}
&D^{+}g_n(0)\frac{\|u_n\|}{n}+\frac{\|\varphi\|}{n}\\
&\ge\frac{a(1+\beta)}{1-\beta}\int_\Omega \nabla u_n\nabla\varphi+\frac{a(1+\beta)}{1-\beta}D^{+}g_n(0)\|u_n\|^2\\
&\quad+\frac{b(3+\beta)}{1-\beta}\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\\
&\quad+\frac{b(3+\beta)}{1-\beta}D^{+}g_n(0)\|u_n\|^4\\
&\quad-\frac{\eta(3+\beta)}{1-\beta}\int_\Omega \phi_{u_n}u_n\phi\\
&\quad-\frac{\eta(3+\beta)}{1-\beta}D^{+}g_n(0)\int_\Omega \phi_{u_n}u_n^2.
\end{aligned}
\tag{89}
$$

From (89) and (70), we have

$$
\begin{aligned}
\|\varphi\|\ge\frac{\|\varphi\|}{n}\ge\frac{D^{+}g_n(0)}{1-\beta}&\Big[(1+\beta)a\|u_n\|^2\\
&+(3+\beta)b\|u_n\|^4-(3+\beta)\eta\int_\Omega \phi_{u_n}u_n^2\\
&-\frac{1-\beta}{n}\|u_n\|\Big]+\frac{a(1+\beta)}{1-\beta}\int_\Omega \nabla u_n\nabla\varphi\,dx
\end{aligned}
$$

$$+\frac{b(3+\beta)}{1-\beta}\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi-\frac{\eta(3+\beta)}{1-\beta}\cdot\int_\Omega \phi_{u_n}u_n\varphi\geq\frac{D^+g_n(0)}{1-\beta}\left[\frac{1-\beta}{2}C_1\right]+\frac{a(1+\beta)}{1-\beta}\cdot\int_\Omega \nabla u_n\nabla\varphi\,dx+\frac{b(3+\beta)}{1-\beta}\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi-\frac{\eta(3+\beta)}{1-\beta}\int_\Omega \phi_{u_n}u_n\varphi,\quad \forall n\geq N. \tag{90}$$

From (85) and (90), it is easy to see that (87) holds. Thus, (86) and (87) imply that

$$\left|D^+g_n(0)\right|\leq C,\quad \forall n\geq N, \tag{91}$$

where $C>0$ is independent of n.

From condition (ii) and (13), we can obtain that

$$\begin{aligned}
&\frac{1}{n}\left[\left|g_n(s)-1\right|\cdot\|u_n\|+sg_n(s)\|\varphi\|\right]\\
&\geq\frac{1}{n}\|g_n(s)(u_n+s\varphi)-u_n\|\\
&\geq J_\lambda(u_n)-J_\lambda\left[g_n(s)(u_n+s\varphi)\right]\\
&=-\left\{\frac{g_n^2(s)-1}{2}\right\}a\|u_n\|^2-\left\{\frac{g_n^4(s)-1}{4}\right\}b\|u_n\|^4\\
&\quad+\left\{\frac{g_n^4(s)-1}{4}\right\}\eta\int_\Omega \phi_{u_n}^2u_n^2\\
&\quad+\frac{\lambda}{1-\beta}\left[g_n^{1-\beta}(s)-1\right]\int_\Omega f(x)\left|u_n\right|^{1-\beta}\\
&\quad+\frac{g_n^2(s)}{2}a\left(\|u_n\|^2-\|u_n+s\varphi\|^2\right)\\
&\quad+\frac{g_n^4(s)}{4}b\left(\|u_n\|^4-\|u_n+s\varphi\|^4\right)\\
&\quad+\frac{g_n^4(s)}{4}\eta\int_\Omega\left[\phi_{u_n+s\varphi}(u_n+s\varphi)^2-\phi_{u_n}u_n^2\right]\\
&\quad+\frac{\lambda}{1-\beta}g_n^{1-\beta}(s)\int_\Omega f(x)\left[(u_n+s\varphi)^{1-\beta}-u_n^{1-\beta}\right].
\end{aligned}\tag{92}$$

Dividing by $s>0$ and letting $s\to0^+$, we derive that

$$\begin{aligned}
&\frac{1}{n}\left(D^+g_n(0)\|u_n\|+\|\varphi\|\right)\\
&\geq-a\int_\Omega \nabla u_n\nabla\varphi-b\|u_n\|^2\int_\Omega \nabla u_n\nabla\varphi\\
&\quad+\eta\int_\Omega \phi_{u_n}u_n\varphi\,dx\\
&\quad+\varliminf_{s\to0^+}\frac{\lambda}{1-\beta}\int_\Omega\frac{f(x)\left[(u_n+s\varphi)^{1-\beta}-u_n^{1-\beta}\right]}{s}dx.
\end{aligned}\tag{93}$$

Since $f(x)[(u_n+s\varphi)^{1-\beta}-u_n^{1-\beta}]\geq0$, $\forall x\in\Omega$, $t>0$, then by Fatou's Lemma we obtain

$$\begin{aligned}
&\int_\Omega f(x)u_n^{-\beta}\varphi\,dx\\
&\leq\varliminf_{s\to0^+}\frac{1}{1-\beta}\int_\Omega\frac{f(x)\left[(u_n+s\varphi)^{1-\beta}-u_n^{1-\beta}\right]}{s}dx.
\end{aligned}\tag{94}$$

Combining (93) and (94) we deduce

$$\begin{aligned}
\lambda\int_\Omega f(x)u_n^{-\beta}\varphi\,dx&\leq\frac{1}{n}\left(\left|D^+g_n(0)\right|\|u_n\|+\|\varphi\|\right)\\
&\quad+\left(a+b\|u_n\|^2\right)\int_\Omega \nabla u_n\nabla\varphi\\
&\quad-\eta\int_\Omega \phi_{u_n}u_n\varphi\,dx\\
&\leq\frac{C_0C+\|\varphi\|}{n}\\
&\quad+\left(a+b\|u_n\|^2\right)\int_\Omega \nabla u_n\nabla\varphi\,dx\\
&\quad-\eta\int_\Omega \phi_{u_n}u_n\varphi\,dx
\end{aligned}\tag{95}$$

for all $n\geq N$, which implies that

$$\begin{aligned}
&\lambda\varliminf_{n\to\infty}\int_\Omega f(x)u_n^{-\beta}\varphi\,dx\\
&\leq\left(a+b\varliminf_{n\to\infty}\|u_n\|^2\right)\int_\Omega \nabla u_0\nabla\varphi-\eta\int_\Omega \phi_{u_0}u_0\varphi\,dx.
\end{aligned}\tag{96}$$

Using Fatou's Lemma, we infer that

$$\begin{aligned}
\lambda\int_\Omega f(x)u_0^{-\beta}\varphi\,dx&\leq\left(a+b\varliminf_{n\to\infty}\|u_n\|^2\right)\int_\Omega \nabla u_0\nabla\varphi\\
&\quad-\eta\int_\Omega \phi_{u_0}u_0\varphi\,dx.
\end{aligned}\tag{97}$$

On one hand, since $u_0(x)\geq0$ a.e. in Ω, choosing $\varphi=u_0$ in (97)

$$\begin{aligned}
&\left(a+b\varliminf_{n\to\infty}\|u_n\|^2\right)\|u_0\|^2\\
&\geq\eta\int_\Omega \phi_{u_0}u_0^2dx+\lambda\int_\Omega f(x)u_0^{1-\beta}dx.
\end{aligned}\tag{98}$$

On the other hand, according to (67) and (68), it follows that

$$\begin{aligned}
&\left(a+b\varliminf_{n\to\infty}\|u_n\|^2\right)\|u_0\|^2\\
&\leq\left(a+b\varlimsup_{n\to\infty}\|u_n\|^2\right)\varlimsup_{n\to\infty}\|u_n\|^2\\
&=\eta\int_\Omega \phi_{u_0}u_0^2dx+\lambda\int_\Omega f(x)u_0^{1-\beta}dx.
\end{aligned}\tag{99}$$

Combining (98) and (99) yields

$$\underline{\lim}_{n\to\infty} \|u_n\|^2 = \overline{\lim}_{n\to\infty} \|u_n\|^2 = \|u_0\|^2. \tag{100}$$

Thus, according to (97) and (100), we have

$$\begin{aligned}\left(a + b\|u_0\|^2\right) \int_\Omega \nabla u_0 \nabla \varphi \, dx - \eta \int_\Omega \phi_{u_0} u_0 \varphi \, dx \\ - \lambda \int_\Omega f(x) u_0^{-\beta} \varphi \geq 0\end{aligned} \tag{101}$$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi > 0$. Therefore, by the idea of approximation and (101), we get

$$\int_\Omega \nabla u_0 \nabla \varphi \, dx \geq 0, \quad \forall \varphi \in H_0^1(\Omega), \ \varphi \geq 0. \tag{102}$$

$u_0 \geq 0$ and $u_0 \neq 0$ in Ω. By the strong maximum principle for weak solutions, we obtain that $u_0 > 0$ a.e. in Ω.

Second, we prove that u_0 is a solution of problem (12). In fact, we can repeat the proof of Theorem 2 and obtain that

$$\begin{aligned}\left(a + b\|u_0\|^2\right) \int_\Omega \nabla u_0 \nabla \varphi \, dx - \eta \int_\Omega \phi_{u_0} u_0 \varphi \, dx \\ - \lambda \int_\Omega f(x) u_0^{-\beta} \varphi \, dx = 0, \quad \forall \varphi \in H_0^1(\Omega).\end{aligned} \tag{103}$$

Then $u_0 \in H_0^1(\Omega)$ is a positive solution of (12) for $\varepsilon = -1$, $0 < \lambda < \lambda_*$, and $\eta > \eta_0$.

Finally, we prove that $u_0 \in N_\lambda^+$. According to (71) and (100), we have

$$\begin{aligned}a(1+\beta)\|u_0\|^2 + b(3+\beta)\|u_0\|^4 \\ > \eta(3+\beta) \int_\Omega \phi_{u_0} u_0^2 dx,\end{aligned} \tag{104}$$

which implies that $u_0 \in N_\lambda^+$.

By Lemma 8, we know that N_λ^- is a closed set in $H_0^1(\Omega)$ for $\lambda \in (0, \lambda_*)$. Thus, applying Lemma 9 and Ekeland's variational principle, we can find a bounded and nonnegative sequence $\{v_n\} \subset N_\lambda^-$ and $v_0 \in H_0^1(\Omega)$ such that

(i) $J_\lambda(v_n) \leq \inf_{u \in N_\lambda^-} J_\lambda(u) + 1/n^2$,

(ii) $J_\lambda(v) \geq J_\lambda(v_n) - (1/n)\|v - v_n\|$, $v \in N_\lambda^-$,

(iii) $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$,

(iv) $v_n \to v_0$ in $L^p(\Omega)$, $p \in [1, 6)$,

(v) $v_n(x) \to v_0(x)$, a.e. in Ω.

At this point we can repeat the proof as above and conclude that $v_0 \in N_\lambda^-$ is another positive solution of (12) for $\varepsilon = -1$, $0 < \lambda < \lambda_*$, and $\eta > \eta_0$. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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