Hindawi
Journal of Function Spaces
Volume 2017, Article ID 6095293, 7 pages
https://doi.org/10.1155/2017/6095293

Research Article

A New Subclass of k-Janowski Type Functions Associated with Ruscheweyh Derivative

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Received 25 February 2017; Accepted 13 April 2017; Published 3 May 2017

Academic Editor: John R. Akeroyd

We introduce and investigate a new subclass $\mathscr{V}\mathscr{D}_k(A,B,b,\delta)$ of analytic functions using Ruscheweyh derivative. We derive the coefficient inequalities and other interesting properties and characteristics for functions belonging to the general class, which we have introduced and studied in this article. We also observe that this class is preserved under the Bernardi integral transform.

1. Introduction

Let $\mathscr{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disc $\mathfrak{D} = \{z : |z| < 1\}$. Also let $\mathscr{S}^*(\beta)$ and $\mathscr{C}(\beta)$ denote the well-known classes of starlike and convex functions of order β, respectively. For details, see [1]. For any two analytic functions $f(z)$ and $g(z)$ with

$$\begin{aligned} f(z) &= z + \sum_{n=2}^{\infty} a_n z^n, \\ g(z) &= z + \sum_{n=2}^{\infty} b_n z^n, \end{aligned} \qquad \text{for } z \in \mathfrak{D}, \tag{2}$$

their convolution (Hadamard product) is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad \text{for } z \in \mathfrak{D}. \tag{3}$$

In 1975, using the concept of convolution, Ruscheweyh [2] introduced a linear operator $D^\delta : \mathscr{A} \to \mathscr{A}$ defined by

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n, \qquad (\delta > -1), \tag{4}$$

with

$$\varphi_n(\delta) = \frac{(\delta+1)_{n-1}}{(n-1)!}, \tag{5}$$

where $(\rho)_n$ is a Pochhammer symbol given as

$$(\rho)_n = \begin{cases} 1, & n = 0, \\ \rho(\rho+1)(\rho+2)\cdots(\rho+n-1), & n \in \mathbb{N}. \end{cases} \tag{6}$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = z f'(z)$ and

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \qquad \forall \delta = n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{7}$$

The following identity can easily be established:

$$(\delta+1)\,D^{\delta+1}f(z)=\delta D^{\delta}f(z)+z\left(D^{\delta}f(z)\right)'. \tag{8}$$

The operator $D^{\delta}f(z)$ is called the Ruscheweyh derivative of $f(z)$; see [2].

Suppose also that, for $k\geq 0$, the classes $k-\mathscr{CV}$ and $k-\mathscr{ST}$ denote the well-known classes of k-uniformly convex and k-starlike functions, respectively. These classes were introduced by Kanas and Wisniowska [3, 4]. For some details see [3–5].

Consider the domain

$$\Omega_k=\left\{u+iv;\ u>k\sqrt{(u-1)^2+v^2}\right\}. \tag{9}$$

For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k=0$), the right branch of a hyperbola ($0<k<1$), a parabola ($k=1$), and an ellipse ($k>1$). This domain was studied by Kanas [3–5]. The function p_k, with $p_k(0)=1$, $p_k'(0)>0$, plays the role of extremal and is given by

$$p_k(z)=\begin{cases}\dfrac{1+z}{1-z}, & k=0,\\[2mm] 1+\dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1,\\[2mm] 1+\dfrac{2}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\operatorname{arctanh}\sqrt{z}\right], & 0<k<1,\\[2mm] 1+\dfrac{1}{k^2-1}\sin\left[\dfrac{\pi}{2R(t)}\displaystyle\int_0^{u(z)/\sqrt{t}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right]+\dfrac{1}{k^2-1}, & k>1,\end{cases} \tag{10}$$

where $u(z)=(z-\sqrt{t})/(1-\sqrt{tz})$, $t\in(0,1)$, $z\in\mathfrak{D}$, and t is chosen such that $k=\cosh(\pi R'(t)/4R(t))$, with $R(t)$ being Legendre's complete elliptic integral of the first kind and $R'(t)$ being complementary integral of $R(t)$ (see [3, 4, 6–8]). Let $\mathscr{P}_k$ denote the class of all those functions $p(z)$ which are analytic in $\mathfrak{D}$ with $p(0)=1$ and $p(z)\prec p_k(z)$ for $z\in\mathfrak{D}$. Clearly it can be seen that $\mathscr{P}_k\subset\mathscr{P}$ where $\mathscr{P}$ is the class of functions with positive real part (see [1, 9]). More precisely

$$\mathscr{P}_k\subset\mathscr{P}\left(\frac{k}{1+k}\right)\subset\mathscr{P}, \tag{11}$$

and, for $f(z)\in\mathscr{P}_k$, we have

$$\left|\arg p(z)\right|\leq\frac{\lambda\pi}{2}, \tag{12}$$

where

$$\lambda=\frac{2}{\pi}\arctan\left(\frac{1}{k}\right). \tag{13}$$

So we can write $p(z)=h^{\lambda}(z)$, $h(z)\in\mathscr{P}$.

Definition 1 (see [10]). A function $p(z)$ is said to be in the class $\mathscr{P}_k[A,B]$, if and only if

$$p(z)\prec p_k[A,B;z]=\frac{(A+1)\,p_k(z)-(A-1)}{(B+1)\,p_k(z)-(B-1)},\qquad k\geq 0, \tag{14}$$

where $p_k(z)$ is defined by (10) and $-1\leq B<A\leq 1$. Geometrically, the function $p(z)\in\mathscr{P}_k[A,B]$ takes all values from the domain $\Omega_k[A,B]$, $-1\leq B<A\leq 1$, $k\geq 0$ which is defined as

$$\begin{aligned}\Omega_k[A,B]=\Bigg\{w:\mathfrak{Re}\left(\frac{(B-1)\,w(z)-(A-1)}{(B+1)\,w(z)-(A+1)}\right)\\ >k\left|\frac{(B-1)\,w(z)-(A-1)}{(B+1)\,w(z)-(A+1)}-1\right|\Bigg\}\end{aligned} \tag{15}$$

or equivalently

$$\begin{aligned}\Omega_k[A,B]=\Big\{u+iv:\left[\left(B^2-1\right)\left(u^2+v^2\right)-2\,(AB-1)\right.\\ \left.\cdot\,u+\left(A^2-1\right)\right]^2>k^2\left[\left(-2\,(B+1)\left(u^2+v^2\right)\right.\right.\\ \left.\left.+\,2\,(A+B+2)\,u-2\,(A+1)\right)^2+4\,(A-B)^2\,v^2\right]\Big\}.\end{aligned} \tag{16}$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega_k[A,B]$. The impact of $\Omega_k[A,B]$ on the conic domain Ω_k changes the original shape of the conic regions. The ends of hyperbola and parabola get closer to each other but never meet anywhere and the ellipse gets the shape of oval. When $A\to 1$, $B\to -1$, the radius of the circular disk defined by $\Omega[A,B]$ tends to infinity, consequently the arms of hyperbola and parabola expand, and the oval turns into ellipse.

With the help of the above Ruscheweyh derivative, we now define the following class.

Definition 2. A function $f(z)\in\mathscr{A}$ of the form (1) is in the class $\mathscr{VD}_k(A,B,b,\delta)$ if and only if

$$\mathfrak{Re}\left(\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)}\right)$$
$$>k\left|\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)}-1\right|, \tag{17}$$

or equivalently,

$$\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right)\prec p_k[A,B;z],\quad z\in\mathfrak{D}, \tag{18}$$

with $p_k[A,B;z]$ being given by (14), $k\geq 0$, $-1\leq B<A\leq 1$, $\delta>-1$, and $b\in\mathbb{C}-\{0\}$.

For different permissible choices of parameters, we obtain several known as well as new subclasses of the class $\mathscr{A}$ of analytic functions as special cases; for example,

(i) For $\delta=0$ and $b=2$, we obtain $\mathscr{VD}_k(A,B,2,0)=k-\mathscr{ST}[A,B]$, and for $b=1$, we have the class $k-\mathscr{UCV}[A,B]$. These classes are recently introduced and studied by [10].

(ii) $\mathscr{VD}_k(1,-1,2,0)=k-ST$, $\mathscr{VD}_k(1,-1,1,0)=k-\mathscr{UCV}$, the well-known classes of k-uniformly convex and k-starlike functions, respectively, introduced by Kanas and Wisniowska [3, 4].

(iii) $\mathscr{VD}_k(1-2\alpha,-1,2,0)=\mathscr{SD}(k,\alpha)$, $\mathscr{VD}_k(1-2\alpha,-1,1,0)=\mathscr{KD}(k,\alpha)$, the classes, introduced by Shams et al. in [11].

(iv) $\mathscr{VD}_0(A,B,2,0)=\mathscr{S}^*[A,B]$, $\mathscr{VD}_0(A,B,1,0)=\mathscr{C}[A,B]$, the well-known classes of Janowski starlike and Janowski convex functions, respectively, introduced by Janowski [12].

Throughout this paper, we assume that $k\geq 0$, $\delta>-1$, $-1\leq B<A\leq 1$, and $b\in\mathbb{C}-\{0\}$ unless otherwise stated.

2. Preliminary Results

We need the following lemmas to obtain our results.

Lemma 3 (see [5]). *Let σ,λ with any complex numbers with $\lambda\neq 0$ and $0\leq\gamma\leq\mathfrak{Re}(\lambda k/(k+1)+\sigma)$. If $\phi(z)$ is analytic in $\mathfrak{D}$ with $\phi(0)=1$ and satisfies*

$$\left(\phi(z)+\frac{z\phi'(z)}{\lambda\phi(z)+\sigma}\right)\prec p_{k,\gamma}(z) \tag{19}$$

and $\varphi_{k,\gamma}(z)$ is an analytic solution of

$$\varphi_{k,\gamma}(z)+\frac{z\varphi'_{k,\gamma}(z)}{\lambda\varphi_{k,\gamma}(z)+\sigma}=p_{k,\gamma}(z), \tag{20}$$

then $\varphi_{k,\gamma}(z)$ is univalent,

$$\phi(z)\prec\varphi_{k,\gamma}(z)\prec p_{k,\gamma}(z), \tag{21}$$

and $\varphi_{k,\gamma}(z)$ is the best dominant of (19), where $\varphi_{k,\gamma}(z)$ is given as

$$\varphi_{k,\gamma}(z)=\left\{\left[\left(\int_0^1\exp\int_t^{tz}\frac{p_{k,\gamma}(u)-1}{u}\mathrm{d}u\right)\mathrm{d}t\right]^{-1}+\frac{\sigma}{\lambda}\right\}. \tag{22}$$

Lemma 4 (see [10]). *Let $p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in k-\mathscr{P}[A,B]$. Then*

$$|c_n|\leq|Q_{AB}|, \tag{23}$$

where

$$|Q_{AB}|=\frac{(A-B)|Q_k|}{2}, \tag{24}$$

$$Q_k=\begin{cases}\dfrac{8\left(\cos^{-1}k\right)^2}{\pi^2\left(1-k^2\right)}, & 0\leq k<1,\\[2mm] \dfrac{8}{\pi^2}, & k=1,\\[2mm] \dfrac{\pi^2}{4\sqrt{t}\left(k^2-1\right)R^2(t)(1+t)}, & k>1.\end{cases} \tag{25}$$

Lemma 5 (see [13]). *Let $p(z)=1+\sum_{n=1}^{\infty}p_nz^n\prec F(z)=1+\sum_{n=1}^{\infty}d_nz^n$ in $\mathfrak{D}$. If $F(z)$ is univalent in $\mathfrak{D}$ and $F(\mathfrak{D})$ is convex, then*

$$|p_n|\leq|d_1|,\quad n\geq 1. \tag{26}$$

Lemma 6 (see [14]). *Let $\beta_0>0$, $\beta_0+\gamma_0>0$ and $\alpha\in[\alpha_0,1)$, where*

$$\alpha_0=\max\left\{\frac{\beta_0-\gamma_0-1}{2\beta_0},\frac{-\gamma_0}{\beta_0}\right\}. \tag{27}$$

Let $h(z)\in\mathscr{A}$ in $\mathfrak{D}$ with $h(0)=1$ and let

$$\left\{h(z)+\frac{zh'(z)}{\beta_0h(z)+\gamma_0}\right\}\in\mathscr{P}(\alpha),\quad 0\leq\alpha<1. \tag{28}$$

Then

$$\mathfrak{Re}\,h(z)>\left[\frac{\beta_0+\gamma_0}{\beta_0\left[{}_2F_1\left[2\beta_0(1-\alpha),1,\beta_0+\gamma_0+1,1/2\right]\right]}-\frac{\gamma_0}{\beta_0}\right], \tag{29}$$

and the bound in (29) is sharp, the extremal functions being

$$h_0 = \frac{1}{\beta_0 g_0(z)} - \frac{\gamma_0}{\beta_0}, \tag{30}$$

with

$$g_0(z) = \int_0^1 \left[\frac{1-z}{1-tz}\right]^{2\beta_0(1-\alpha)} t^{\beta_0+\gamma_0-1} dt. \tag{31}$$

3. Main Results

Theorem 7. *A function $f(z) \in \mathscr{A}$ and of the form (1) is in the class $\mathscr{V}\mathscr{D}_k(A,B,b,\delta)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{\frac{4(n-1)(k+1)}{\delta+1} + \left|\frac{2(B+1)(\delta+n)}{\delta+1} + b(B-A) - 2(B+1)\right|\right\} \cdot \varphi_n(\delta)|a_n| < |b||B-A|. \tag{32}$$

Proof. Let we note that

$$\begin{aligned}
&\left|\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)} - 1\right| \\
&\quad = 4\left|\frac{D^{\delta+1}f(z) - D^{\delta}f(z)}{2(B+1)D^{\delta+1}f(z) + (b(B-A)-2(B+1))D^{\delta}f(z)}\right| \\
&\quad = \left|\frac{\sum_{n=2}^{\infty}\{4(n-1)/(\delta+1)\}\varphi_n(\delta)a_n z^n}{b(B-A)z + \sum_{n=2}^{\infty}\{2(B+1)(\delta+n)/(\delta+1) + b(B-A) - 2(B+1)\}\varphi_n(\delta)a_n z^n}\right| \\
&\quad \le \frac{\sum_{n=2}^{\infty}\{4(n-1)/(\delta+1)\}\varphi_n(\delta)|a_n|}{|b||B-A| - \sum_{n=2}^{\infty}|2(B+1)(\delta+n)/(\delta+1) + b(B-A) - 2(B+1)|\varphi_n(\delta)|a_n|},
\end{aligned} \tag{33}$$

because from (32) it follows that

$$\left\{|b||B-A| - \sum_{n=2}^{\infty}\left|\frac{2(B+1)(\delta+n)}{\delta+1} + b(B-A) - 2(B+1)\right| \cdot \varphi_n(\delta)|a_n|\right\} > 0. \tag{34}$$

To show that $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta)$ it suffices that

$$k\left|\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)} - 1\right| - \mathfrak{Re}\left(\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)}\right) < 1. \tag{35}$$

From (33), we have

$$k\left|\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)} - 1\right| - \mathfrak{Re}\left(\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)}\right)$$

$$\leq (k+1)\left|\frac{(B-1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A-1)}{(B+1)\left(1-2/b+(2/b)\left(D^{\delta+1}f(z)/D^{\delta}f(z)\right)\right)-(A+1)}-1\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}\{4(n-1)(k+1)/(\delta+1)\}\varphi_n(\delta)|a_n|}{|b||B-A|-\sum_{n=2}^{\infty}|2(B+1)(\delta+n)/(\delta+1)+b(B-A)-2(B+1)|\varphi_n(\delta)|a_n|}. \tag{36}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left\{\frac{4(n-1)(k+1)}{\delta+1}+\left|\frac{2(B+1)(\delta+n)}{\delta+1}+b(B-A)-2(B+1)\right|\right\}\cdot\varphi_n(\delta)|a_n|<|b||B-A|, \tag{37}$$

and this completes the proof. □

When we put $\delta=0$ and $b=2$ in the above theorem, we obtain the following known result, proved by Noor and Malik in [10].

Corollary 8. *A function $f(z)\in\mathcal{A}$ and of the form (1) is in the class $k-\mathcal{ST}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(n-1)(k+1)+|(B+1)n-(1+A)|\}\varphi_n(\delta)|a_n|<|B-A|. \tag{38}$$

For $A=1-2\alpha$, $B=-1$ with $0\leq\alpha<1$, $\delta=0$, and $b=2$, Theorem 7 reduces to the following known result, proved by Shams et al. [11].

Corollary 9. *A function $f(z)\in\mathcal{A}$ and of the form (1) is in the class $\mathcal{SD}(k,\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\alpha)\}|a_n|<1-\alpha. \tag{39}$$

Theorem 10. *Let $f(z)\in\mathcal{VD}_k(A,B,b,\delta)$. Then*

$$|a_n|\leq\frac{(\sigma)_{n-1}}{(n-1)!\varphi_n(\delta)},\quad \text{for } n\geq 2, \tag{40}$$

where

$$\sigma=\frac{|b||Q_{AB}|(\delta+1)}{2}, \tag{41}$$

where $|Q_{AB}|$ and $\varphi_n(\delta)$ are given by (24) and (5).

Proof. Set

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}=p(z), \tag{42}$$

so that $p(z)\in\mathcal{P}_k[A,B]$. Let $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$. Then (42) can be written as

$$2\left(D^{\delta+1}f(z)-D^{\delta}f(z)\right)=bD^{\delta}f(z)\sum_{n=1}^{\infty}c_nz^n, \tag{43}$$

which implies that

$$\frac{2\varphi_n(\delta)(n-1)a_n}{(\delta+1)}=b\left(c_{n-1}+\varphi_2(\delta)a_2c_{n-2}+\cdots+\varphi_{n-1}(\delta)a_{n-1}c_1\right). \tag{44}$$

Using the coefficient estimates $|c_n|\leq|Q_{AB}|$ for the class $\mathcal{P}_k[A,B]$ (see [10]), we obtain

$$|a_n|\leq\frac{|b||Q_{AB}|(\delta+1)}{2(n-1)\varphi_n(\delta)}\left(1+\varphi_2(\delta)|a_2|+\cdots+\varphi_{n-1}(\delta)|a_{n-1}|\right). \tag{45}$$

For $n=2$,

$$|a_2|\leq\frac{|b||Q_{AB}|}{2}. \tag{46}$$

Therefore (40) holds for $n=2$. Assume that (40) is true for $n=m$ and consider

$$\begin{aligned}|a_{m+1}|&\leq\frac{|b||Q_{AB}|(\delta+1)}{2m\varphi_{m+1}(\delta)}\left(1+\varphi_2(\delta)|a_2|+\cdots+\varphi_{n-1}(\delta)|a_m|\right)\\&\leq\frac{|b||Q_{AB}|(\delta+1)}{2m\varphi_{m+1}(\delta)}\left\{+\frac{|b||Q_{AB}|(\delta+1)}{2!}\left(1+\frac{|b||Q_{AB}|(\delta+1)}{2}\right)+\cdots+\frac{|b||Q_{AB}|(\delta+1)}{(m-1)!}\cdot\prod_{j=1}^{m-2}\left(1+\frac{|b||Q_{AB}|(\delta+1)}{2j}\right)\right\}\\&=\frac{|b||Q_{AB}|(\delta+1)}{2m\varphi_{m+1}(\delta)}\prod_{j=1}^{m-1}\left(1+\frac{|b||Q_{AB}|(\delta+1)}{2j}\right)\\&=\frac{(\sigma)_m}{(m)!\varphi_{m+1}(\delta)}.\end{aligned} \tag{47}$$

Therefore, the result is true for $n=m+1$. Using mathematical induction, (40) holds true for all $n\geq 2$. □

Corollary 11 (see [10]). *If $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,2,0) = k - \mathscr{S}\mathscr{T}[A,B]$, then*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|Q_k(A-\beta) - 2jB|}{2(j+1)}, \quad \text{for } n \ge 2, \tag{48}$$

where Q_k is defined by (8).

When $A = 1$, $B = -1$, $b = 2$, and $\delta = 0$, we obtain the following coefficient inequality for the class $k - \mathscr{S}\mathscr{T}$, introduced by Kanas and Wisniowska [4].

Corollary 12. *If $f(z) \in k - \mathscr{S}\mathscr{T}$, then*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|Q_k + j|}{j+1}, \quad \text{for } n \ge 2. \tag{49}$$

This result is sharp.

By taking the values $A = 1 - 2\alpha$ with $0 \le \alpha < 1$, $B = -1$, $b = 2$, and $\delta = 0$, we obtain the coefficient inequality of the class $\mathscr{S}\mathscr{D}(k,\alpha)$, introduced by Shams et al. [11].

Corollary 13. *If $f(z) \in \mathscr{S}\mathscr{D}(k,\alpha)$, then*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|Q_k(1-\alpha) + j|}{j+1}, \quad \text{for } n \ge 2. \tag{50}$$

This result is sharp.

Theorem 14. *For real $b > 0$, let $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta+1)$. Then $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b+1,\delta)$ for $z \in \mathfrak{D}$.*

Proof. Suppose $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta+1)$ and set

$$p(z) = 1 - \frac{2}{b+1} + \frac{2}{b+1}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}, \tag{51}$$

where $p(z)$ is analytic in $\mathfrak{D}$ with $p(0) = 1$. Then simple computations, together with (51) and (8), yield

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} = (1-\lambda_1) + \lambda_1\left[p(z) + \frac{\lambda_2 z p'(z)}{p(z)+\lambda_3}\right], \tag{52}$$

with $\lambda_1 = (\delta+1)(b+1)/b(\delta+2)$, $\lambda_2 = 2/(\delta+1)(b+1)$, $\lambda_3 = 2/(b+1) - 1$. Since $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta+1)$, it follows that

$$\left[(1-\lambda_1) + \lambda_1\left(p(z) + \frac{\lambda_2 z p'(z)}{p(z)+\lambda_3}\right)\right] = h_0(z) \in \mathscr{P}_k[A,B], \tag{53}$$

or, equivalently,

$$p(z) + \frac{z p'(z)}{(1/\lambda_2)p(z) + \lambda_3/\lambda_2} = \frac{1}{\lambda_1}h_0(z) + \left(1 - \frac{1}{\lambda_1}\right)p_0(z), \quad (p_0(z) = 1). \tag{54}$$

Since $h_0(z), p_0(z) \in \mathscr{P}_k[A,B]$ and $\mathscr{P}_k[A,B]$ is a convex set (see [10]), it follows that $(1/\lambda_1)h_0(z) + (1 - 1/\lambda_1)p_0(z)$, with $p_0(z) = 1$, belong to $\mathscr{P}_k[A,B]$ in $\mathfrak{D}$ and hence $p(z) + zp'(z)/((1/\lambda_2)p(z) + \lambda_3/\lambda_2) \in \mathscr{P}_k[A,B]$. We now use the Lemma 3 with $\eta = 1/\lambda_2$, $\sigma = \lambda_3/\lambda_2$, and $\mathfrak{Re}(\eta k/(k+1)+\sigma) > 0$ to obtain $p(z) \in \mathscr{P}_k[A,B]$ and hence $f(z) \in \mathscr{V}\mathscr{D}_k(A,B,b+1,\delta)$. This complete the proof. □

For a function $f(z) \in \mathscr{A}$, we consider the integral operator

$$F(z) = I_c(f(z)) = \frac{(c+1)}{z^c}\int_0^z t^{c-1}f(t)\,dt, \quad c > -1. \tag{55}$$

The operator I_c was introduced by Bernardi [15] for $c \in \mathbb{N}$. In particular, the operator I_1 was studied earlier by Libera [16] and Livingston [17].

Theorem 15. *Let $f(z) \in V\mathscr{D}_k(A,B,b,\delta)$ and let $F(z)$ be defined by (55). Then $F(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta)$.*

Proof. Let

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}(F(z))}{D^{\delta}(F(z))} = p(z), \tag{56}$$

where $p(z)$ is analytic in $\mathfrak{D}$, $p(0) = 1$. Then using (8), we have

$$\frac{z\left(D^{\delta}(F(z))\right)'}{D^{\delta}(F(z))} = ap(z) - a + 1, \tag{57}$$

where $a = b(\delta+1)/2$. Simple computation and use of (8), (55), and (56), we have

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}(f(z))}{D^{\delta}(f(z))} = p(z) + \frac{zp'(z)}{ap(z) - a + 1 + c} \prec p_k[A,B;z]. \tag{58}$$

Apply Lemma 3 with $\eta = a$, $\sigma = 1 - a + c$ and $\mathfrak{Re}(\eta k/(k+1) + \sigma) > 0$ to obtain $p(z) \prec p_k[A,B;z]$ and consequently $F(z) \in \mathscr{V}\mathscr{D}_k(A,B,b,\delta)$. □

Corollary 16. *Let $b > 0$, $f(z) \in V\mathscr{D}_k(A,B,b,\delta) \subset \mathscr{V}\mathscr{D}_{\alpha_1}(b,\delta)$, and $\alpha_1 = (2k+1-A)/(2k+1-B)$ and let $F(z)$ be defined by (55). Then $F(z) \in \mathscr{V}\mathscr{D}_\beta(b,\delta)$, where $c > -1$, and*

$$\beta = \left\{\frac{c+1}{a\left[{}_2F_1\left(2a(1-\alpha_1), 1; c+2; 1/2\right)\right]} - \frac{1+c-a}{a}\right\}. \tag{59}$$

Proof. Proceeding as in Theorem 15, it follows from (58) that

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}(f(z))}{D^{\delta}(f(z))} = p(z) + \frac{zp'(z)}{ap(z) - a + 1 + c} \in \mathscr{P}(\alpha_1), \quad \alpha_1 = \frac{2k+1-A}{2k+1-B}. \tag{60}$$

Applying Lemma 6, we obtain $p(z) \in \mathscr{P}(\beta)$, where β is given by (59). This proves that $F(z) \in \mathscr{V}\mathscr{D}_\beta(b,\delta)$ in $\mathfrak{D}$. □

Theorem 17. *If $f(z)$ is of the form (1) belonging to $\mathscr{V}\mathscr{D}_k(A, B, b, \delta)$ and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$, where $F(z)$ is the integral operator defined by (55), then*

$$|b_n| \leq \frac{(c+1)}{(c+n)} \frac{(\sigma)_{n-1}}{(n-1)!\varphi_n(\delta)}, \quad \text{for } n \geq 2. \tag{61}$$

Proof. From (55), we obtain

$$(c+1) f(z) = cF(z) + zF'(z). \tag{62}$$

Using the series for the functions $f(z)$ and $F(z)$, we obtain

$$\begin{aligned} &(1+c) z + \sum_{n=2}^{\infty} (1+c) a_n z^n \\ &= cz + \sum_{n=2}^{\infty} c b_n z^n + z + \sum_{n=2}^{\infty} n b_n z^n, \end{aligned} \tag{63}$$

and thus

$$(n+c) b_n = (1+c) a_n, \quad n \geq 2. \tag{64}$$

From the above we have

$$|b_n| \leq \frac{(c+1)}{(c+n)} |a_n|, \quad n \geq 2. \tag{65}$$

Using the estimates from Theorem 10, we obtain the required result. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors would also like to acknowledge *Professor Dr. Salim ur Rehman, V.C. Sarhad University of Science & IT,* for providing excellent research and academic environment.

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