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Research Article

Fixed Point Theorems for Manageable Contractions with Application to Integral Equations

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In this paper we utilize the concept of manageable functions to define multivalued $\alpha_* - \eta_*$ manageable contractions and prove fixed point theorems for such contractions. As applications we deduce certain fixed point theorems which generalize and improve Nadler's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, and some other well-known results in the literature. Also, we give an illustrating example showing that our results are a proper generalization of Nadler's theorem and provide an application to integral equations.

1. Introduction and Preliminaries

The Banach contraction principle [1] is an elementary result in metric fixed point theory. This golden principle has been broadened in several directions by different authors (see [1–18]). An interesting generalization is the elongation of the Banach contraction principle to multivalued maps, known as Nadler's fixed point theorem [19] and Mizoguchi-Takahashi's fixed point theorem [20]. In 2012, Samet et al. [18] defined α-ψ-contractive and α-admissible mappings and then Salimi et al. [17] generalized this idea by introducing function η and established fixed point theorems. Further Hasanzade Asl et al. [13] extended these notions to multivalued functions by introducing the concepts of α_*-ψ-contractive and α_*-admissible for multivalued mappings and proved some fixed point results.

Hussain et al. [14] modified the notions of α_*-admissible as follows.

Definition 1 (see [14]). Let $\mathscr{T} : \mathscr{X} \to 2^{\mathscr{X}}$ be a multifunction on a metric space $(\mathscr{X}, d)$ and $\alpha, \eta : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_+$ be two functions, where η is bounded; then $\mathscr{T}$ is an α_*-admissible mapping with respect to η if

$$\alpha(y,z) \geq \eta(y,z)$$
$$\text{implies that } \alpha_*(\mathscr{T}y, \mathscr{T}z) \geq \eta_*(\mathscr{T}y, \mathscr{T}z), \quad y, z \in \mathscr{X}, \tag{1}$$

where

$$\alpha_*(\mathscr{A}, \mathscr{B}) = \inf_{y\in\mathscr{A}, z\in\mathscr{B}} \alpha(y,z),$$
$$\eta_*(\mathscr{A}, \mathscr{B}) = \sup_{y\in\mathscr{A}, z\in\mathscr{B}} \eta(y,z). \tag{2}$$

Further, Ali et al. [3] generalized the results of Hussain et al. and introduced the following definition.

Definition 2 (see [3]). Let $T : X \to 2^X$ be a closed valued mapping on a metric space (X, d) and $\alpha, \eta : X \times X \to \mathbb{R}_+$

be two functions. We say that T is generalized α_*-admissible mapping with respect to η if

$$\alpha(y,z) \geq \eta(y,z) \text{ implies that } \alpha(u,v) \geq \eta(u,v), \quad \forall u \in Ty,\ v \in Tz. \tag{3}$$

Very recently, Ali et al. [2] modified Definition 2 for the sequence of multivalued functions as follows.

Definition 3 (see [2]). Let $\{\mathcal{T}_i : \mathcal{X} \to 2^{\mathcal{X}}\}_{i=1}^{\infty}$ be a sequence of closed valued maps on a metric space $(\mathcal{X}, d)$ and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions; then the sequence $\mathcal{T}_i$ is α_*-admissible mapping with respect to η if

$$\alpha(y,z) \geq \eta(y,z) \text{ implies that } \alpha(u,v) \geq \eta(u,v), \quad \forall u \in \mathcal{T}_i y,\ v \in \mathcal{T}_j z, \tag{4}$$

for each $i, j \in \mathbb{N}$. If, for all $y, z \in \mathcal{X}$ $\alpha(y,z) = 1$, the sequence $\{\mathcal{T}_i\}$ is called α_*-subadmissible and, for $\eta(y,z) = 1$, the sequence $\{\mathcal{T}_i\}$ is called α_*-admissible.

Recently, Du and Khojasteh [10] initiated the concept of manageable functions and proved some fixed point theorems. In this paper, we introduce multivalued $\alpha_* - \eta_*$ manageable contraction and prove certain fixed point results. We also prove common fixed point theorem for multivalued contraction. The investigated results of this paper conclude several existing fixed point results including Nadler's theorem.

Throughout this paper, $\mathrm{CL}(\mathcal{X})$ denotes the family of all nonempty closed subsets of a metric space $(\mathcal{X}, d)$. The Hausdorff metric H is defined on $\mathrm{CL}(\mathcal{X})$ by

$$H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{x\in\mathcal{A}} D(x,\mathcal{B}), \sup_{y\in\mathcal{B}} D(y,\mathcal{A})\right\}, \tag{5}$$

where $D(x,\mathcal{B}) = \inf\{d(x,y) : y \in \mathcal{B}\}$. In the sequel, $\mathrm{Fix}\{\mathcal{T}\}$ denotes the set of all fixed points of mapping $\mathcal{T}$, $\mathrm{Fix}\{\mathcal{T}_i\}$ denotes the set of all common fixed points of mappings $\mathcal{T}_i$, Σ denotes the class of all functions $\sigma : [0,+\infty) \to [0,1)$ fulfilling $\limsup_{t\to r^+}\sigma(t) < 1$, for all $r \in [0,+\infty)$, Φ denotes the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\int_0^{\epsilon}\varphi(t)dt$ exists and $\int_0^{\epsilon}\varphi(t)dt > \epsilon$, for each $\epsilon > 0$, Ψ denotes the class of all nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t) < +\infty$ for all $t > 0$, and Γ denotes the set of all L-functions. Recall that a function $\gamma : [0,+\infty) \to [0,+\infty)$ is said to be an L-function if $\gamma(0) = 0$ and $\gamma(t) > 0$ for all $t > 0$ and for every $t > 0$ there exists $s > t$ such that $\gamma(u) \leq t$ for $u \in [t,s]$ [21].

2. Fixed Point and Common Fixed Point Results for Multivalued Contractions via Manageable Function

Consistent with Du and Khojasteh [10], we denote by $\widehat{\mathrm{Man}(\mathbb{R})}$ the set of all manageable functions $\vartheta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ fulfilling the following conditions:

(ϑ_1) $\vartheta(t,s) < s - t$ for all $s, t > 0$;

(ϑ_2) for any bounded sequence $\{t_n\} \subset (0,+\infty)$ and any nondecreasing sequence $\{s_n\} \subset (0,+\infty)$, it holds that

$$\lim_{n\to\infty}\sup\frac{t_n + \vartheta(t_n,s_n)}{s_n} < 1. \tag{6}$$

Example 4 (see [10]). Let $r \in [0,1)$. Then $\vartheta_r : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\vartheta_r(t,s) = rs - t \tag{7}$$

is a manageable function.

Example 5. Let $\vartheta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\vartheta(t,s) = \begin{cases}\psi(s) - t & \text{if } (t,s) \in [0,+\infty)\times[0,+\infty), \\ f(t,s) & \text{otherwise,}\end{cases} \tag{8}$$

where $\psi \in \Psi$ and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is any function. Then $\vartheta(t,s) \in \widehat{\mathrm{Man}(\mathbb{R})}$. Indeed, by using Lemma 1 of [12], we have, for any $s, t > 0$,

$$\vartheta(t,s) = \psi(s) - t < s - t, \tag{9}$$

so, (ϑ_1) holds. Let $\{t_n\} \subset (0,+\infty)$ be a bounded sequence and let $\{s_n\} \subset (0,+\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty}s_n = \inf_{n\in\mathbb{N}}s_n = a$ for some $a \in [0,+\infty)$; we get

$$\lim_{n\to\infty}\sup\frac{t_n + \vartheta(t_n,s_n)}{s_n} = \lim_{n\to\infty}\sup\frac{\psi(s_n)}{(s_n)} < \lim_{n\to\infty}\frac{(s_n)}{(s_n)} = 1, \tag{10}$$

so, (ϑ_2) is also satisfied.

Definition 6. Let $(\mathcal{X}, d)$ be a metric space and $T : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping. Let $\alpha, \eta : \mathcal{X}\times\mathcal{X} \to \mathbb{R}_+$ be two functions and $\vartheta \in \widehat{\mathrm{Man}(\mathbb{R})}$. Then $\mathcal{T}$ is called a multivalued $\alpha_* - \eta_*$-manageable contraction with respect to ϑ if for all $y, z \in \mathcal{X}$

$$\alpha_*(\mathcal{T}y, \mathcal{T}z) \geq \eta_*(\mathcal{T}y, \mathcal{T}z) \text{ implies } \vartheta(H(\mathcal{T}y,\mathcal{T}z), d(y,z)) \geq 0. \tag{11}$$

Now we state and prove the main result of this section.

Theorem 7. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued map satisfying the following conditions:*

(1) *$\mathcal{T}$ is α_*-admissible map with respect to η;*

(2) *$\mathcal{T}$ is $\alpha_* - \eta_*$ manageable contraction with respect to ϑ;*

(3) *there exists $z_0 \in \mathcal{X}$ and $z_1 \in \mathcal{T}z_0$ such that $\alpha(z_0,z_1) \geq \eta(z_0,z_1)$;*

(4) *for a sequence $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = x$ and $\alpha(z_n,z_{n+1}) \geq \eta(z_n,z_{n+1})$, for all $n \in \mathbb{N}$, implies $\alpha(z_n,x) \geq \eta(z_n,x)$ for all $n \in \mathbb{N}$.*

Then $\mathrm{Fix}\{\mathcal{T}\} \neq \varnothing$.

Proof. Let $z_1 \in \mathscr{T}z_0$ be such that $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. Since $\mathscr{T}$ is α_*-admissible map with respect to η, then $\alpha_*(\mathscr{T}z_0, \mathscr{T}z_1) \geq \eta_*(\mathscr{T}z_0, \mathscr{T}z_1)$. Therefore, from (11), we have

$$\vartheta(H(\mathscr{T}z_0, \mathscr{T}z_1), d(z_0, z_1)) \geq 0. \quad (12)$$

If $z_1 = z_0$, then $z_0 \in \text{Fix}\{\mathscr{T}\}$; also if $z_1 \in \mathscr{T}z_1$, then $z_1 \in \text{Fix}\{\mathscr{T}\}$. So, we adopt that $z_0 \neq z_1$ and $z_1 \notin \mathscr{T}z_1$. Thus $0 < d(z_1, \mathscr{T}z_1) \leq H(\mathscr{T}z_0, \mathscr{T}z_1)$. Define $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\lambda(t, s) = \begin{cases} \dfrac{t + \vartheta(t, s)}{s} & \text{if } t, s > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

By (ϑ_1), we know that

$$0 < \lambda(t, s) < 1, \quad \forall t, s > 0. \quad (14)$$

Also note that if $\vartheta(t, s) \geq 0$, then

$$0 < t \leq s\lambda(t, s). \quad (15)$$

So, from (12) and (14), we get

$$0 < \lambda(H(\mathscr{T}z_0, \mathscr{T}z_1), d(z_0, z_1)) < 1. \quad (16)$$

Let

$$\varepsilon_1 = \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_0, \mathscr{T}z_1), d(z_0, z_1))}} - 1\right) d(z_1, \mathscr{T}z_1). \quad (17)$$

Since $d(z_1, \mathscr{T}z_1) > 0$. So, by using (16), we get $\varepsilon_1 > 0$ and

$$d(z_1, \mathscr{T}z_1) < d(z_1, \mathscr{T}z_1) + \varepsilon_1 = \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_0, \mathscr{T}z_1), d(z_0, z_1))}}\right) d(z_1, \mathscr{T}z_1). \quad (18)$$

This implies that there exists $z_2 \in \mathscr{T}z_1$ such that

$$d(z_1, z_2) < \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_0, \mathscr{T}z_1), d(z_0, z_1))}}\right) d(z_1, \mathscr{T}z_1). \quad (19)$$

Note that $z_1 \neq z_2$ (since $z_1 \notin \mathscr{T}z_1$). Now if $z_2 \in \mathscr{T}z_2$, then z_2 is a fixed point of $\mathscr{T}$. Otherwise, $0 < d(z_2, \mathscr{T}z_2) < H(\mathscr{T}z_1, \mathscr{T}z_2)$. Also, since $\alpha_*(\mathscr{T}z_0, \mathscr{T}z_1) \geq \eta_*(\mathscr{T}z_0, \mathscr{T}z_1)$, $z_1 \in \mathscr{T}z_0$, and $z_2 \in \mathscr{T}z_1$, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. So, $\alpha_*(\mathscr{T}z_1, \mathscr{T}z_2) \geq \eta_*(\mathscr{T}z_1, \mathscr{T}z_2)$, so, from (11), we get

$$\vartheta(H(\mathscr{T}z_1, \mathscr{T}z_2), d(z_1, z_2)) \geq 0. \quad (20)$$

By taking

$$\varepsilon_2 = \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_1, \mathscr{T}z_2), d(z_1, z_2))}} - 1\right) d(z_2, \mathscr{T}z_2), \quad (21)$$

there exists $z_3 \in \mathscr{T}z_2$ with $z_3 \neq z_2$ such that

$$d(z_2, z_3) < \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_1, \mathscr{T}z_2), d(z_1, z_2))}}\right) d(z_2, \mathscr{T}z_2). \quad (22)$$

Hence, by induction, we form a sequence $\{z_n\}$ in $\mathscr{X}$ satisfying for each $n \in \mathbb{N}$, $z_n \in \mathscr{T}z_{n-1}$, $z_n \neq z_{n-1}$, $z_n \notin \mathscr{T}z_n$, and $\alpha_*(z_{n-1}, z_n) \geq \eta_*(z_{n-1}, z_n)$

$$0 < d(x_n, \mathscr{T}x_n) \leq H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), \quad (23)$$

$$\vartheta(H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), d(z_{n-1}, z_n)) \geq 0, \quad (24)$$

$$d(z_n, z_{n+1}) < \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), d(z_{n-1}, z_n))}}\right) d(z_n, \mathscr{T}z_n), \quad (25)$$

by taking

$$\varepsilon_n = \left(\frac{1}{\sqrt{\lambda(H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), d(z_{n-1}, z_n))}} - 1\right) d(z_n, \mathscr{T}z_n). \quad (26)$$

By using (14), (15), (23), and (25), we get for each $n \in \mathbb{N}$

$$d(z_n, \mathscr{T}z_n) \leq d(z_{n-1}, z_n)\lambda(H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), d(z_{n-1}, z_n)) \leq d(z_{n-1}, z_n). \quad (27)$$

This implies that $\{d(z_n, \mathscr{T}z_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. By combining (25) and (27), for each $n \in \mathbb{N}$, we get

$$d(z_n, z_{n+1}) < \left(\sqrt{\lambda(H(\mathscr{T}z_{n-1}, \mathscr{T}z_n), d(z_{n-1}, z_n))}\right) d(z_{n-1}, z_n). \quad (28)$$

Which means that $\{d(z_{n-1}, z_n)\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. So, let

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = c \geq 0. \quad (29)$$

From (ϑ_2), we get

$$\lim_{n\to\infty}\sup \lambda\left(H\left(\mathcal{T}z_n, \mathcal{T}z_n\right), d\left(z_{n-1}, z_n\right)\right) < 1. \tag{30}$$

Now, if, in (29), $c > 0$, then, by taking $\lim_{n\to\infty}\sup$ in (28) and using (30), we have

$$c \leq \sqrt{\lim_{n\to\infty}\sup \lambda\left(H\left(\mathcal{T}z_{n-1}, \mathcal{T}z_n\right), d\left(z_{n-1}, z_n\right)\right)}c < c. \tag{31}$$

This contradiction shows that $c = 0$. Hence,

$$\lim_{n\to\infty} d\left(z_n, z_{n+1}\right) = 0. \tag{32}$$

Next, we prove that $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. Let, for each $n \in \mathbb{N}$,

$$\sigma_n = \sqrt{\lambda\left(H\left(\mathcal{T}z_{n-1}, \mathcal{T}z_n\right), d\left(z_{n-1}, z_n\right)\right)}. \tag{33}$$

Then, from (16), we have $\sigma_n \in (0,1)$. By (28), we obtain

$$d\left(z_n, z_{n+1}\right) < \sigma_n d\left(z_{n-1}, z_n\right). \tag{34}$$

Equation (30) implies that $\lim_{n\to\infty}\sigma_n < 1$, so there exists $\gamma \in [0,1)$ and $n_0 \in \mathbb{N}$, such that

$$\sigma_n \leq \gamma, \quad \forall n \in \mathbb{N},\ n \geq n_0. \tag{35}$$

For any $n \geq n_0$, since $\sigma_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\gamma \in [0,1)$, (34) and (35) imply that

$$\begin{aligned} d\left(z_n, z_{n+1}\right) &< \sigma_n d\left(z_{n-1}, z_n\right) < \sigma_n\sigma_{n-1} d\left(z_{n-2}, z_{n-1}\right) \\ &< \cdots \\ &< \sigma_n\sigma_{n-1}\sigma_{n-2}\cdots\sigma_{n_0} d\left(z_0, z_1\right) \\ &\leq \gamma^{n-n_0+1} d\left(z_0, z_1\right). \end{aligned} \tag{36}$$

Put $\beta_n = (\gamma^{n-n_0+1}/(1-\gamma))d(z_0, z_1)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we have from (36) that

$$d\left(z_n, z_m\right) \leq \sum_{j=n}^{m-1} d\left(z_j, z_{j+1}\right) < \beta_n. \tag{37}$$

Since $\gamma \in [0,1)$, $\lim_{n\to\infty}\beta_n = 0$. Hence

$$\lim_{n\to\infty}\sup\left\{d\left(z_n, z_m\right) : m > n\right\} = 0. \tag{38}$$

This shows that $\{z_n\}$ is a Cauchy sequence in $\mathcal{X}$. Completeness of $\mathcal{X}$ ensures the existence of $z \in \mathcal{X}$ such that $z_n \to z$ as $n \to \infty$. Now, since $\alpha(z_n, z) \geq \eta(z_n, z)$ for all $n \in \mathbb{N}$, $\alpha_*(\mathcal{T}z_n, \mathcal{T}z) \geq \eta_*(\mathcal{T}z_n, \mathcal{T}z)$, and so, from (11), we have

$$\vartheta\left(H\left(\mathcal{T}z_n, \mathcal{T}z\right), d\left(z_n, z\right)\right) \geq 0. \tag{39}$$

Then, from (14) and (15), we have

$$\begin{aligned} H\left(\mathcal{T}z_n, \mathcal{T}z\right) &\leq \lambda\left(H\left(\mathcal{T}z_n, \mathcal{T}z\right), d\left(z_n, z\right)\right) d\left(z_n, z\right) \\ &< d\left(z_n, z\right). \end{aligned} \tag{40}$$

Since $0 < d(z, \mathcal{T}z) \leq H(\mathcal{T}z_n, \mathcal{T}z) + d(z_n, z)$, so, by using (40), we get

$$0 < d\left(z, \mathcal{T}z\right) < 2d\left(z_n, z\right). \tag{41}$$

Letting limit $n \to \infty$ in the above inequality, we get

$$d\left(z, \mathcal{T}z\right) = 0. \tag{42}$$

Hence $z \in \operatorname{Fix}\{\mathcal{T}\}$. □

Example 8. Let $\mathcal{X} = \mathbb{R}$ with usual metric d. Then $(\mathcal{X}, d)$ is a complete metric space. Define $\mathcal{T} : \mathcal{X} \to CL(\mathcal{X})$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ and $\vartheta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}z = \begin{cases} \left\{1, \dfrac{1}{4z}\right\} & \text{if } z > 1; \\ \left\{0, \dfrac{z}{16}\right\} & \text{if } z \in [0,1]; \\ \{2,3\} & \text{otherwise,} \end{cases} \tag{43}$$

$$\alpha\left(y, z\right) = \begin{cases} 2 & \text{if } z, y \in [0,1]; \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

$\eta(y, z) = 1$, for all $z, y \in \mathcal{X}$ and $\vartheta(t, s) = as - t$, where $a \in [1/16, 1)$. Then ϑ is a manageable function. Indeed, for any $s, t > 0$, we have

$$\vartheta\left(t, s\right) = as - t < s - t, \tag{44}$$

so, (ϑ_1) holds. Let $\{t_n\} \subset (0, +\infty)$ be a bounded sequence and let $\{s_n\} \subset (0, +\infty)$ be a nonincreasing sequence. Then

$$\lim_{n\to\infty}\sup \frac{t_n + \vartheta\left(t_n, s_n\right)}{s_n} = a < 1, \tag{45}$$

which means that (ϑ_2) holds. Hence $\vartheta \in \widehat{\operatorname{Man}(\mathbb{R})}$.

Since $\alpha(y, z) \geq \eta(y, z)$ when $z, y \in [0,1]$, this implies that

$$\begin{aligned} \alpha_*\left(\mathcal{T}y, \mathcal{T}z\right) &= \inf_{u\in\mathcal{T}y, v\in\mathcal{T}z} \alpha\left(u, v\right) = 2 > 1 \\ &= \sup_{u\in\mathcal{T}y, v\in\mathcal{T}z} \eta\left(u, v\right) = \eta_*\left(\mathcal{T}y, \mathcal{T}z\right). \end{aligned} \tag{46}$$

Hence $\mathcal{T}$ is α_*-admissible mapping with respect to η.

Let $\alpha_*(\mathcal{T}y, \mathcal{T}z) \geq \eta_*(\mathcal{T}y, \mathcal{T}z)$; then $y, z \in [0,1]$. This implies that

$$\begin{aligned} \vartheta\left(H\left(\mathcal{T}y, \mathcal{T}z\right), d\left(y, z\right)\right) &= \vartheta\left(\left|\frac{y}{16} - \frac{z}{16}\right|, \left|y - z\right|\right) \\ &= a\left|y - z\right| - \left|\frac{y}{16} - \frac{z}{16}\right| \\ &= \left|y - z\right|\left(a - \frac{1}{16}\right) \geq 0. \end{aligned} \tag{47}$$

Thus, all conditions of Theorem 7 are satisfied and 0 is a fixed point of $\mathcal{T}$.

On the other hand, for $y = -1$ and $z = 0$, we have

$$H\left(\mathcal{T}y, \mathcal{T}z\right) = 2 > 1 = d\left(y, z\right). \tag{48}$$

This implies that $\mathcal{T}$ is not a multivalued contraction, so we cannot apply Nadler's theorem [19] with this example.

On bearing $\eta(y,z)=1$ in Theorem 7, we get the following corollary.

Corollary 9. *Let $(\mathcal{X},d)$ be a complete metric space and let $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$ be an α_*-admissible and closed valued map enjoying the following:*

(1) *$\alpha_*(\mathcal{T}y,\mathcal{T}z)\geq 1$ implies $\vartheta(H(\mathcal{T}y,\mathcal{T}y),d(y,z))\geq 0$;*

(2) *there exists $z_0\in\mathcal{X}$ and $z_1\in\mathcal{T}z_0$ such that $\alpha(z_0,z_1)\geq 1$;*

(3) *for a sequence $\{z_n\}\subset\mathcal{X}$, $\lim_{n\to\infty}\{z_n\}=x$, and $\alpha(z_n,z_{n+1})\geq 1$, for all $n\in\mathbb{N}$, one has $\alpha(z_n,x)\geq 1$ for all $n\in\mathbb{N}$,*

for all $y,z\in\mathcal{X}$ and $\vartheta\in\widehat{Man(\mathbb{R})}$. Then $Fix\{\mathcal{T}\}\neq\varnothing$.

By taking $\alpha(y,z)=1$ in Theorem 7, we get the following corollary.

Corollary 10. *Let $(\mathcal{X},d)$ be a complete metric space and let $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$ be an η_*-admissible and closed valued map enjoying the following conditions:*

(1) *$\eta_*(\mathcal{T}y,\mathcal{T}z)\leq 1$ implies $\vartheta(H(\mathcal{T}y,\mathcal{T}y),d(y,z))\geq 0$;*

(2) *there exists $z_0\in\mathcal{X}$ and $z_1\in\mathcal{T}z_0$ such that $\eta(z_0,z_1)\leq 1$;*

(3) *for a sequence $\{z_n\}\subset\mathcal{X}$, $\lim_{n\to\infty}\{z_n\}=x$, and $\eta(z_n,z_{n+1})\leq 1$, for all $n\in\mathbb{N}$, one has $\eta(z_n,x)\leq 1$ for all $n\in\mathbb{N}$,*

for all $y,z\in\mathcal{X}$ and $\vartheta\in\widehat{Man(\mathbb{R})}$. Then $Fix\{\mathcal{T}\}\neq\varnothing$.

Theorem 11. *Let $(\mathcal{X},d)$ be a complete metric space and let the sequence $\{\mathcal{T}_i:\mathcal{X}\to 2^{\mathcal{X}}\}$ of closed valued mappings enjoy the following with condition (4) of Theorem 7:*

(1) *$\{\mathcal{T}_i\}$ is α_*-admissible with respect to η;*

(2) *$\alpha(y,z)\geq\eta(y,z)$ implies $\vartheta(H(\mathcal{T}_iy,\mathcal{T}_jz),d(x,y))\geq 0$;*

(3) *there exists $z_0\in\mathcal{X}$ and $y_i\in\mathcal{T}_iz_0$ for each $i\in\mathbb{N}$ such that $\alpha(z_0,y_i)\geq\eta(z_0,y_i)$;*

for all $y,z\in\mathcal{X}$, $i,j\in\mathbb{N}$, and $\vartheta\in\widehat{Man(\mathbb{R})}$. Then $Fix\{\mathcal{T}_i\}\neq\varnothing$ for each $i\in\mathbb{N}$.

Proof. Let $z_1\in\mathcal{T}_1z_0$ be such that $\alpha(z_0,z_1)\geq\eta(z_0,z_1)$; then from (11) we have

$$\vartheta(H(\mathcal{T}_1z_0,\mathcal{T}_2z_1),d(z_0,z_1))\geq 0. \tag{49}$$

If $z_1\in\mathcal{T}_iz_1$, for each $i\in\mathbb{N}$, then $z_1\in\mathrm{Fix}\{\mathcal{T}_i\}$. Adopt that $z_1\notin\mathcal{T}_2z_1$. Thus

$$0<d(z_1,\mathcal{T}_2z_1)\leq H(\mathcal{T}_1z_0,\mathcal{T}_2z_1). \tag{50}$$

From (50), we get

$$0<\lambda(H(T_1z_0,T_2z_1),d(z_0,z_1))<1, \tag{51}$$

where λ is defined in (13). Let

$$\varepsilon_1=\left(\frac{\alpha(z_0,z_1)}{\eta(z_0,z_1)\sqrt{\lambda(H(\mathcal{T}_1z_0,\mathcal{T}_2z_1),d(z_0,z_1))}}-1\right)d(z_1,\mathcal{T}_2z_1). \tag{52}$$

Since $\alpha(z_0,z_1)\geq\eta(z_0,z_1)$, $d(z_1,\mathcal{T}_2z_1)>0$. So, by using (51), we get $\varepsilon_1>0$ and

$$d(z_1,\mathcal{T}_2z_1)<d(z_1,\mathcal{T}_2z_1)+\varepsilon_1=\left(\frac{\alpha(z_0,z_1)}{\eta(z_0,z_1)\sqrt{\lambda(H(\mathcal{T}_1z_0,\mathcal{T}_2z_1),d(z_0,z_1))}}\right)d(z_1,\mathcal{T}_2z_1). \tag{53}$$

This implies that there exists $z_2\in\mathcal{T}_2z_1$ such that

$$d(z_1,z_2)<\left(\frac{\alpha(z_0,z_1)}{\eta(z_0,z_1)\sqrt{\lambda(H(\mathcal{T}_1z_0,\mathcal{T}_2z_1),d(z_0,z_1))}}\right)d(z_1,\mathcal{T}_2z_1). \tag{54}$$

Note that $z_1\neq z_2$ (since $z_1\notin\mathcal{T}_2z_1$). Now if $z_2\in\mathcal{T}_iz_2$, for each $i\in\mathbb{N}$, then $z_2\in\mathrm{Fix}\{\mathcal{T}_i\}$. Let $z_2\notin\mathcal{T}_3z_2$; then

$$0<d(z_2,\mathcal{T}_3z_2)<H(\mathcal{T}_2z_1,\mathcal{T}_3z_2). \tag{55}$$

Since the sequence $\{\mathcal{T}_i\}$ is α_*-admissible with respect to η, so we have $\alpha(z_1,z_2)\geq\eta(z_1,z_2)$ and, from condition (11), we get

$$\vartheta(H(\mathcal{T}_2z_1,\mathcal{T}_3z_2),d(z_1,z_2))\geq 0. \tag{56}$$

By taking

$$\varepsilon_2=\left(\frac{\alpha(z_1,z_2)}{\eta(z_1,z_2)\sqrt{\lambda(H(\mathcal{T}_2z_1,\mathcal{T}_3z_2),d(z_1,z_2))}}-1\right)d(z_2,\mathcal{T}_3z_2), \tag{57}$$

there exists $z_3\in\mathcal{T}_3z_2$ with $z_3\neq z_2$ such that

$$d(z_2,z_3)<\left(\frac{\alpha(z_1,z_2)}{\eta(z_1,z_2)\sqrt{\lambda(H(\mathcal{T}_2z_1,\mathcal{T}_3z_2),d(z_1,z_2))}}\right)d(z_2,\mathcal{T}_3z_2). \tag{58}$$

Hence, by induction, we can establish a sequence $\{z_n\}$ in X satisfying for each $n \in \mathbb{N}$, $z_n \in \mathcal{T}_n z_{n-1}$, $z_n \neq z_{n-1}$, $z_n \notin \mathcal{T}_i z_n$ for each $n \in \mathbb{N}$, and $\alpha(z_{n-1}, z_n) \geq \eta(z_{n-1}, z_n)$

$$0 < d(z_n, \mathcal{T}_{n+1} z_n) \leq H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), \tag{59}$$

$$\vartheta(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n)) \geq 0, \tag{60}$$

$$d(z_n, z_{n+1}) < \left(\frac{\alpha(z_{n-1}, z_n)}{\eta(z_{n-1}, z_n)\sqrt{\lambda(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n))}} \right) \times d(z_n, \mathcal{T}_{n+1} z_n), \tag{61}$$

by taking

$$\varepsilon_n = \left(\frac{\alpha(z_{n-1}, z_n)}{\eta(z_{n-1}, z_n)\sqrt{\lambda(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n))}} - 1 \right) d(z_n, \mathcal{T}_{n+1} z_n). \tag{62}$$

By using (14), (15), (59), and (61), we get for each $n \in \mathbb{N}$

$$d(z_n, \mathcal{T}_{n+1} z_n) \leq d(z_{n-1}, z_n)\lambda(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n)) \leq d(z_{n-1}, z_n). \tag{63}$$

Which means that $\{d(z_n, \mathcal{T}_{n+1} z_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. By combining (61) and (63), for each $n \in \mathbb{N}$, we get

$$d(z_n, z_{n+1}) < \left(\sqrt{\lambda(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n))} \right) \times d(z_{n-1}, z_n). \tag{64}$$

Which means that $\{d(z_{n-1}, z_n)\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. So, let

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = c \geq 0. \tag{65}$$

From (ϑ_2), we get

$$\lim_{n\to\infty} \sup \lambda(d(z_n, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n)) < 1. \tag{66}$$

Now, if, in (65), $c > 0$, then, by taking $\lim_{n\to\infty}\sup$ in (64) and using (66), we have

$$c \leq \sqrt{\lim_{n\to\infty} \sup \lambda(H(\mathcal{T}_n z_{n-1}, \mathcal{T}_{n+1} z_n), d(z_{n-1}, z_n))} c < c. \tag{67}$$

This contradiction shows that $c = 0$. Hence,

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0. \tag{68}$$

Next, by following the same procedure as in proof of Theorem 7, we show that sequence $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Completeness of $\mathcal{X}$ guarantees that there is $z \in \mathcal{X}$ such that $z_n \to z$ as $n \to \infty$. Now, since $\alpha(z_n, z) \geq \eta(z_n, z)$ for all $n \in \mathbb{N}$, and from condition (11), we have

$$\vartheta(H(\mathcal{T}_{n+1} z_n, \mathcal{T}_i z), d(z_n, z)) \geq 0. \tag{69}$$

Then, from (14) and (15), we have

$$H(\mathcal{T}_{n+1} z_n, \mathcal{T}_i z) \leq \lambda(H(\mathcal{T}_{n+1} z_n, \mathcal{T}_i z), d(z_n, z)) d(z_n, z) < d(z_n, z). \tag{70}$$

Since $0 < d(z_{n+1}, \mathcal{T}_i z) < H(\mathcal{T}_{n+1} x_n, \mathcal{T}_i z)$, by using (70), we get

$$0 < d(z_{n+1}, \mathcal{T}_i z) \leq d(z_n, z). \tag{71}$$

Letting limit $n \to \infty$ in the above inequality, we get

$$d(z, \mathcal{T}_i z) = 0. \tag{72}$$

Hence $z \in \mathrm{Fix}\{\mathcal{T}_i z\}$. □

Let us take $\mathcal{T}_i = \mathcal{T}$ for each $i \in \mathbb{N}$; then Theorem 11 reduces to the following.

Theorem 12. *Let $(\mathcal{X}, d)$ be a complete metric space and let the sequence $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping enjoying the following with conditions (3) and (4) of Theorem 7:*

(1) *$\mathcal{T}$ is generalized α_*-admissible with respect to η;*

(2) *$\alpha(y, z) \geq \eta(y, z)$ implies $\vartheta(H(\mathcal{T} y, \mathcal{T} z), d(y, z)) \geq 0$;*

for all $y, z \in \mathcal{X}$ and $\vartheta \in \widehat{Man(\mathbb{R})}$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

As an application of Theorems 7 and 11, we can deduce the following result.

Theorem 13. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping satisfying for all $y, z \in \mathcal{X}$*

$$\vartheta(H(\mathcal{T} y, \mathcal{T} z), d(y, z)) \geq 0, \tag{73}$$

where $\vartheta \in \widehat{Man(\mathbb{R})}$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(y, z) = d(y, z), \quad \eta(y, z) = d(y, z), \tag{74}$$

for all $y, z \in \mathcal{X}$. So that $\alpha(y, z) = \eta(y, z)$. This implies that $\alpha_*(\mathcal{T} y, \mathcal{T} z) = \eta_*(\mathcal{T} y, \mathcal{T} z)$ for all $y, z \in \mathcal{X}$. That is, all the conditions of Theorem 7 hold true. Hence $\mathcal{T}$ has a fixed point. □

In the following corollaries, we obtain some known and some new results in literature via manageable functions.

Corollary 14 (see [19]). *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping satisfying for all $y, z \in \mathcal{X}$*

$$H(\mathcal{T}y, \mathcal{T}z) \le kd(y, z), \tag{75}$$

where $k \in (0, 1)$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

Proof. Define $\vartheta_N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_N(t, s) = ks - t \quad \forall (t, s) \in \mathbb{R} \times \mathbb{R},\ k \in [0, 1). \tag{76}$$

Then $\vartheta_N \in \widehat{\mathrm{Man}(\mathbb{R})}$, by Example 4. Therefore the result follows by taking $\vartheta = \vartheta_N$ in Theorem 13. □

Corollary 15 (see [20]). *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping satisfying for all $y, z \in \mathcal{X}$*

$$H(\mathcal{T}y, \mathcal{T}z) \le \sigma(d(y, z))\, d(y, z), \tag{77}$$

where $\sigma \in \Sigma$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

Proof. Define $\vartheta_M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_M(t, s) = \begin{cases} s\sigma(s) - t & \text{if } (t, s) \in [0, +\infty) \times [0, +\infty), \\ f(t, s) & \text{otherwise}, \end{cases} \tag{78}$$

where $\sigma \in \Sigma$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any function. Then $\vartheta_M \in \widehat{\mathrm{Man}(\mathbb{R})}$ (see Example C in [10]). Therefore the result follows by taking $\vartheta = \vartheta_M$ in Theorem 13. □

Corollary 16. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping satisfying for all $y, z \in \mathcal{X}$*

$$\int_0^{H(\mathcal{T}y, \mathcal{T}z)} \varphi(t)\, dt \le \alpha d(y, z), \tag{79}$$

where $\varphi \in \Phi$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

Proof. Define $\vartheta_I : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_I(t, s) = \begin{cases} s - \int_0^t \varphi(u)\, du & \text{if } (t, s) \in [0, +\infty) \times [0, +\infty), \\ f(t, s) & \text{otherwise}, \end{cases} \tag{80}$$

where $\varphi \in \Phi$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any function. Then for $s, t > 0$ we have

$$\vartheta_I(t, s) = s - \int_0^t \varphi(u)\, du < s - t. \tag{81}$$

Now let $\{t_n\} \subset (0, +\infty)$ be a bounded sequence and let $\{s_n\} \subset (0, +\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$ for some $a \in [0, +\infty)$ and

$$\begin{aligned} \lim_{n\to\infty} \sup \frac{t_n + \vartheta(t_n, s_n)}{s_n} &= \lim_{n\to\infty} \sup \frac{t_n + s_n - \int_0^t \varphi(u)\, du}{s_n} \\ &< \lim_{n\to\infty} \sup \frac{t_n + s_n - t_n}{s_n} = 1. \end{aligned} \tag{82}$$

Hence $\vartheta_I \in \widehat{\mathrm{Man}(\mathbb{R})}$. Therefore the result follows by taking $\vartheta = \vartheta_I$ in Theorem 13. □

Corollary 17. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping satisfying for all $y, z \in \mathcal{X}$*

$$H(\mathcal{T}y, \mathcal{T}z) < \gamma(d(y, z)), \tag{83}$$

where $\gamma \in \Gamma$. Then $Fix\{\mathcal{T}\} \neq \varnothing$.

Proof. Define $\vartheta_L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_L(t, s) = \begin{cases} \gamma(s) - t & \text{if } (t, s) \in [0, +\infty) \times [0, +\infty), \\ f(t, s) & \text{otherwise}, \end{cases} \tag{84}$$

where $\gamma \in \Gamma$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any function. Then $\vartheta_L \in \widehat{\mathrm{Man}(\mathbb{R})}$. Indeed, by using Lemma 10 of [22], we have for $s, t > 0$

$$\vartheta_L(t, s) = \gamma(s) - t < s - t, \tag{85}$$

so, (ϑ_1) holds. Let $\{t_n\} \subset (0, +\infty)$ be a bounded sequence and let $\{s_n\} \subset (0, +\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$ for some $a \in [0, +\infty)$ and

$$\begin{aligned} \lim_{n\to\infty} \sup \frac{t_n + \vartheta(t_n, s_n)}{s_n} &= \lim_{n\to\infty} \sup \frac{\gamma(s_n)}{s_n} < \lim_{n\to\infty} \frac{s_n}{s_n} \\ &= 1. \end{aligned} \tag{86}$$

So, (ϑ_2) is also satisfied. Therefore the result follows by taking $\vartheta = \vartheta_L$ in Theorem 13. □

Remark 18. Since $\mathcal{T}$ is a Meir-Keeler multivalued mapping of a metric space $(\mathcal{X}, d)$ if and only if there exists (nondecreasing, right continuous) mapping $\gamma \in \Gamma$ such that $H(\mathcal{T}y, \mathcal{T}z) < \gamma(d(y, z))$ (see [21], Theorem 2), therefore, from Corollary 17, we get the main result of [23].

By taking $\vartheta = \vartheta_M$ in Theorem 12, as defined in Corollary 15, we get the following.

Corollary 19 (see [3]). *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a closed valued mapping enjoying the following with conditions (3) and (4) of Theorem 7.*

(1) $\mathcal{T}$ *is generalized* α_**-admissible with respect to* η;

(2) $\alpha(y,z) \geq \eta(y,z)$ *implies* $H(\mathcal{T}y,\mathcal{T}z) \leq \sigma(d(y,z))d(y,z)$,

for all $y,z \in X$ *and* $\sigma \in \Sigma$. *Then* $Fix\{\mathcal{T}\} \neq \emptyset$.

On considering

$$\vartheta(t,s) = \begin{cases} \psi(s) - t & \text{if } (t,s) \in [0,+\infty) \times [0,+\infty); \\ f(t,s) & \text{otherwise,} \end{cases} \quad (87)$$

where $\psi \in \Psi$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is any function, in Theorems 7, 11, and 12, we get the following existing theorems, respectively.

Corollary 20 (see [14]). *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ *be a closed valued mapping fulfilling conditions (1), (3), and (4) with*

$$\alpha_*(\mathcal{T}y,\mathcal{T}z) \geq \eta_*(\mathcal{T}y,\mathcal{T}z) \\ \text{implies } H(\mathcal{T}y,\mathcal{T}z) \leq \psi(d(y,z)), \quad (88)$$

for all $y,z \in \mathcal{X}$ *and* $\psi \in \Psi$. *Then* $Fix\{\mathcal{T}\} \neq \emptyset$.

Corollary 21 (see [2]). *Let* $(\mathcal{X}, d)$ *be a complete metric space and let the sequence* $\{\mathcal{T}_i : \mathcal{X} \to 2^{\mathcal{X}}\}_{i=1}^{\infty}$ *be* α_**-admissible with respect to* η *fulfilling condition (4) of Theorem 7 and (3) of Theorem 11 with*

$$\alpha(y,z) \geq \eta(y,z) \\ \text{implies } H(\mathcal{T}_i y,\mathcal{T}_j z) \leq \psi(d(y,z)), \quad (89)$$

for all $y,z \in X$, $i,j \in \mathbb{N}$ *and* $\psi \in \Psi$. *Then* $Fix\{\mathcal{T}_i\} \neq \emptyset$.

Corollary 22 (see [2]). *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ *be a closed valued mapping and generalized* α_**-admissible with respect to* η *fulfilling conditions (3) and (4) of Theorem 7 with*

$$\alpha(y,z) \geq \eta(y,z) \\ \text{implies } H(\mathcal{T}y,\mathcal{T}z) \leq \psi(d(y,z)), \quad (90)$$

for all $y,z \in X$ *and* $\psi \in \Psi$. *Then* $Fix\{\mathcal{T}\} \neq \emptyset$.

3. Fixed Point Results in Partially Ordered Metric Spaces and an Application to Integral Equations

Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space. Recall that $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ is monotone increasing if $\mathcal{T}y \preceq \mathcal{T}z$ for all $y,z \in \mathcal{X}$, for which $y \preceq z$ (see [8]). There are many applications in differential and integral equations of monotone mappings in ordered metric spaces (see [15, 24–26] and references therein). In this section, from Theorems 7–13, we derive the following new results in partially ordered metric spaces and give an example to integral equations.

Theorem 23. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space and let* $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ *be a closed valued mapping satisfying the following assertions for all* $y,z \in \mathcal{X}$ *with* $y \preceq z$:

(1) $\mathcal{T}$ *is monotone increasing;*

(2) $\vartheta(H(\mathcal{T}y,\mathcal{T}z), d(y,z)) \geq 0$;

(3) *there exists* $z_0 \in X$ *and* $z_1 \in \mathcal{T}z_0$ *such that* $z_0 \preceq z_1$;

(4) *for a sequence* $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = z$, *and* $z_n \preceq z_{n+1}$ *for all* $n \in \mathbb{N}$, *one has* $z_n \preceq z$ *for all* $n \in \mathbb{N}$.

Then $Fix\{\mathcal{T}\} \neq \emptyset$.

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by

$$\alpha(y,z) = \begin{cases} 1 & y \preceq z \\ 0 & \text{otherwise} \end{cases} \\ \eta(y,z) = \begin{cases} \frac{1}{2} & y \preceq z \\ 0 & \text{otherwise.} \end{cases} \quad (91)$$

Then for $y,z \in \mathcal{X}$ with $y \preceq z$, $\alpha(y,z) \geq \eta(y,z)$ implies $\alpha_*(\mathcal{T}y,\mathcal{T}z) = 1 > 1/2 = \eta_*(\mathcal{T}y,\mathcal{T}z)$ and $\alpha_*(\mathcal{T}y,\mathcal{T}z) = \eta_*(\mathcal{T}y,\mathcal{T}z) = 0$ otherwise. Thus, all the conditions of Theorem 7 are satisfied and hence $\mathcal{T}$ has a fixed point. □

Theorem 24. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space and let the sequence* $\{\mathcal{T}_i : \mathcal{X} \to 2^{\mathcal{X}}\}$ *of closed valued mappings enjoy the following assertions, for all* $y,z \in \mathcal{X}$ *with* $x \preceq y$ *and for each* $i,j \in \mathbb{N}$:

(1) $\vartheta(H(\mathcal{T}_i y,\mathcal{T}_j z), d(y,z)) \geq 0$;

(2) *if* $y \preceq z$, *then* $\mathcal{T}_i y \preceq z$ *for each* $i \in \mathbb{N}$;

(3) *there exists* $z_0 \in X$ *and* $y_i \in \mathcal{T}_i z_0$ *for each* $i \in \mathbb{N}$ *such that* $z_0 \preceq y_i$;

(4) *for a sequence* $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = z$, *and* $z_n \preceq z_{n+1}$, *for all* $n \in \mathbb{N}$, *we have* $z_n \preceq z$ *for all* $n \in \mathbb{N}$.

Then $Fix\{\mathcal{T}_i\} \neq \emptyset$ *for all* $i \in \mathbb{N}$.

Proof. By defining $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ as in Theorem 23 and by using Theorem 11, we get the required result. □

Similar to the arguments of Theorems 23 and 24, we conclude the following result and omit its proof.

Theorem 25. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partial ordered metric space and let* $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ *be a closed valued mapping satisfying (73) of Theorem 13 for all* $x,y \in \mathcal{X}$ *with* $x \preceq y$. *Then* $Fix\{\mathcal{T}\} \neq \emptyset$.

In case of single valued mapping, Theorems 23–25 reduced to the following.

Theorem 26. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a self-map fulfilling the following assertions:*

(1) $\mathcal{T}$ *is monotone increasing;*

(2) $\vartheta(d(\mathcal{T}y, \mathcal{T}z), d(y, z)) \geq 0$;

(3) *there exists* $z_0 \in \mathcal{X}$ *and* $z_1 = \mathcal{T}z_0$ *such that* $z_0 \preceq z_1$;

(4) *for a sequence* $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = z$, *and* $z_n \preceq z_{n+1}$, *for all* $n \in \mathbb{N}$, *we have* $z_n \preceq z$ *for all* $n \in \mathbb{N}$,

for all $y, z \in \mathcal{X}$ *with* $y \preceq z$ *and* $\vartheta \in \widehat{Man(\mathbb{R})}$. *Then* $Fix\{\mathcal{T}\} \neq \varnothing$.

Theorem 27. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space and let the sequence* $\{\mathcal{T}_i : X \to X\}$ *of self-mappings fulfill the following assertions:*

(1) $\vartheta(d(\mathcal{T}_i y, \mathcal{T}_j z), d(y, z)) \geq 0$;

(2) $\mathcal{T}_i$ *is nondecreasing for each* i;

(3) *there exists* $z_0 \in X$ *and* $y_i = \mathcal{T}_i z_0$ *for each* $i \in \mathbb{N}$ *such that* $z_0 \preceq y_i$;

(4) *for a sequence* $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = z$ *and* $z_n \preceq z_{n+1}$ *for all* $n \in \mathbb{N}$, *we have* $z_n \preceq z$ *for all* $n \in \mathbb{N}$,

for all $y, z \in \mathcal{X}$ *with* $y \preceq z$, $\vartheta \in \widehat{Man(\mathbb{R})}$ *and* $i \in \mathbb{N}$. *Then* $Fix\{\mathcal{T}_i\} \neq \varnothing$.

Theorem 28. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partial ordered metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a self-map satisfying for all* $y, z \in \mathcal{X}$ *with* $y \preceq z$

$$\vartheta(d(\mathcal{T}y, \mathcal{T}z), d(y, z)) \geq 0, \tag{92}$$

where $\vartheta \in \widehat{Man(\mathbb{R})}$. *Then* $Fix\{\mathcal{T}_i\} \neq \varnothing$.

Now we give an application of our results and establish the existence of solution of the integral equation.

$$z(r) = \int_b^c \mathcal{B}(r, s, z(s))\,ds + g(r), \quad t \in [b, c]. \tag{93}$$

Let $\ll$ be a partial order relation on $\mathbb{R}^n$. Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}z(r) = \int_b^c \mathcal{B}(r, s, z(s))\,ds + g(r), \quad r \in [a, b]. \tag{94}$$

Theorem 29. *Let* $\mathcal{X} = C([b, c], \mathbb{R}^n)$ *with the usual supremum norm. Suppose that*

(1) $\mathcal{B} : [b, c] \times [b, c] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *are continuous;*

(2) *there exists a continuous function* $p : [b, c] \times [b, c] \to [b, c]$ *such that*

$$|\mathcal{B}(r, s, u) - \mathcal{B}(r, s, v)| \leq p(r, s)|u - v|, \tag{95}$$

for each $r, s \in [b, c]$ *and* $u, v \in \mathbb{R}^n$ *with* $u \ll v$;

(3) $\sup_{r\in[b,c]} \int_b^c p(r, s)ds = q \leq 1/4$;

(4) *there exists* $z_0 \in \mathcal{X}$ *and* $z_1 \in \mathcal{T}z_0$ *such that* $z_0 \preceq z_1$;

(5) *for a sequence* $\{z_n\} \subset \mathcal{X}$, $\lim_{n\to\infty}\{z_n\} = z$, *and* $\preceq z_{n+1}$, *for all* $n \in \mathbb{N}$, *one has* $z_n \preceq z$ *for all* $n \in \mathbb{N}$.

Then the integral equation (93) has a solution in $\mathcal{X}$.

Proof. Let $\mathcal{X} = C([b, c], \mathbb{R}^n)$ and $\|z\| = \max_{r\in[b,c]}|z(r)|$, for $z \in C([a, b])$. Consider a partial order defined on $\mathcal{X}$ by

$$\begin{gathered} y, z \in C([b, c], \mathbb{R}^n), \quad y \preceq z \\ \text{iff } y(r) \ll z(r), \quad \text{for } r \in [b, c]. \end{gathered} \tag{96}$$

Then $(\mathcal{X}, \|\cdot\|, \preceq)$ is a complete partial ordered metric space and for any increasing sequence $\{z_n\}$ in $\mathcal{X}$ converging to $z \in \mathcal{X}$, we have $z_n(r) \ll z(r)$ for any $r \in [b, c]$ (see [27]). By using (94) and conditions (2) and (3) and taking $\vartheta(r, s) = (1/2)s - r$ for all $y, z \in \mathcal{X}$ with $y \preceq z$, we obtain

$$\begin{aligned} &|\mathcal{T}y(r) - \mathcal{T}z(r)| \\ &\quad = \left|\int_b^c \mathcal{B}(r, s, y(s))\,ds - \int_b^c \mathcal{B}(r, s, z(s))\,ds\right| \\ &\quad \leq \int_b^c |\mathcal{B}(r, s, y(s)) - \mathcal{B}(r, s, z(s))|\,ds \\ &\quad \leq \int_b^c p(r, s)|y(s) - z(s)|\,ds \leq \frac{1}{4}\|y - z\|. \end{aligned} \tag{97}$$

This implies that

$$\begin{aligned} &\frac{1}{2}\|y - z\| - \|\mathcal{T}y - \mathcal{T}z\| \geq \frac{1}{2}\|y - z\| - \frac{1}{4}\|y - z\| \\ &\quad = \frac{1}{4}\|y - z\|. \end{aligned} \tag{98}$$

So $\vartheta(d(\mathcal{T}y, \mathcal{T}z), d(y, z)) \geq 0$ for all $y, z \in \mathcal{X}$ with $y \preceq z$. Hence all the conditions of Theorem 26 are satisfied. Therefore $\mathcal{T}$ has a fixed point; consequently, integral equation (93) has a solution in $\mathcal{X}$. □

Competing Interests

The authors declare that they have no competing interests.

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