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Research Article

The Embedding Theorem of an L^0-Prebarreled Module into Its Random Biconjugate Space

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We first prove Mazur's lemma in a random locally convex module endowed with the locally L^0-convex topology. Then, we establish the embedding theorem of an L^0-prebarreled random locally convex module, which says that if $(S, \mathscr{P})$ is an L^0-prebarreled random locally convex module such that S has the countable concatenation property, then the canonical embedding mapping J of S onto $J(S) \subset (S_s^*)_s^*$ is an L^0-linear homeomorphism, where $(S_s^*)_s^*$ is the strong random biconjugate space of S under the locally L^0-convex topology.

1. Introduction

Mazur's lemma in a locally convex space is a very useful fact in convex analysis. The embedding theorem of a locally convex space into its biconjugate space has played a crucial role in the study of semireflexivity and reflexivity of a locally convex space. The purpose of this paper is to generalize the two basic results from a locally convex space to a random locally convex module.

Based on the idea of randomizing functional space theory, a new approach to random functional analysis was initiated by Guo in [1–3]; in particular, the study of random normed modules and random inner product modules together with their random conjugate spaces was already the central theme in random functional analysis in [2, 3]. Currently, random normed modules, random inner product modules, random locally convex modules, and the theory of random conjugate spaces still occupy a central place in random functional analysis. At the early stage, motivated by the theory of probabilistic metric spaces [4], random normed modules and random locally convex modules used to be endowed with the (ε, λ)-topology, which also leads to the theory of random conjugate spaces under the (ε, λ)-topology [5, 6]. In 2009, motivated by financial applications, Filipović et al. presented the notion of a locally L^0-convex module while the locally L^0-convex topology for random normed modules and random locally convex modules was also introduced in [7]. Subsequently, Guo established the relations between some basic results derived from the (ε, λ)-topology and the locally L^0-convex topology for a random locally convex module in [8]. The (ε, λ)-topology is too weak, whereas the locally L^0-convex topology is too strong, and the advantages and disadvantages of the two kinds of topologies often complement each other so that simultaneously considering the two kinds of topologies for a random locally convex module or a random normed module will make random functional analysis deeply developed, which also leads to a series of recent advances in random functional analysis and its applications [9–14].

In 2009, Guo et al. first proved Mazur's lemma in a random locally convex module endowed with the (ε, λ)-topology in [6]. Recently, Zapata [15] studied Mazur's lemma in a random normed module endowed with the locally L^0-convex topology. This paper will give Mazur's lemma in the sense of all kinds of random duality, in particular Mazur's lemma in a random locally convex module endowed with the locally L^0-convex module. The notion of an L^0-prebarreled module is a proper random generalization of that of a barreled space; in particular, a characterization for a random locally convex module to be L^0-prebarreled was established in [10]. Based on [10], this paper will prove an L^0-linear homeomorphically embedding theorem of an L^0-prebarreled random locally convex module into its strong random biconjugate space.

The remainder of this paper is organized as follows. Section 2 states and proves the main results of this paper.

2. Main Results and Their Proofs

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a given probability space and K the scalar field R of real numbers or C of complex numbers. Now, we can state the main results of this paper as follows.

Theorem 1. *Let $(S, \mathcal{P})$ be a random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ and G an L^0-convex subset of S such that G has the countable concatenation property. Then, $G_c^- = G_{\varepsilon,\lambda}^- = [G]^-_{\sigma_{\varepsilon,\lambda}(S, S^*_{\varepsilon,\lambda})} = [G]^-_{\sigma_{\varepsilon,\lambda}(S, S^*_c)} = [G]^-_{\sigma_c(S, S^*_c)}$.*

Theorem 2. *Let $(S, \mathcal{P})$ be an L^0-prebarreled random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ such that S has the countable concatenation property. Then, S is L^0-linearly homeomorphically embedded into $(S_s^*)_s^*$ by the canonical mapping $J : S \to J(S) \subset (S_s^*)_s^*$ defined by $J(x)(f) = f(x)$, $\forall x \in S$ and $f \in S_s^*$, where S_s^* denotes the random conjugate space S_c^* endowed with its strong locally L^0-convex topology.*

For the sake of readers' convenience and proofs of Theorems 1 and 2, let us first recapitulate some notations and known terminology.

In the sequel, $L^0(\mathcal{F}, K)$ denotes the algebra of equivalence classes of K-valued random variables on Ω and $\overline{L}^0(\mathcal{F})$ the set of equivalence classes of extended real-valued random variables on Ω, where two random variables are equivalent if they are equal almost everywhere (briefly, a.s.).

It is well known from [16] that $\overline{L}^0(\mathcal{F})$ is an order complete lattice under the partial order: $\xi \le \eta$ iff $\xi^0(\omega) \le \eta^0(\omega)$ for almost all ω in Ω, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively; further, $\vee A$ and $\wedge A$ stand for the supremum and infimum of a subset A of $\overline{L}^0(\mathcal{F})$, respectively. In addition, it is also well known that if A is directed upwards (downwards), then there exists a nondecreasing (nonincreasing) sequence $\{a_n, n \in N\}$ ($\{b_n, n \in N\}$) in A such that $a_n \uparrow \vee A$ ($b_n \downarrow \wedge A$). $\overline{L}^0(\mathcal{F})$ has the largest element and the smallest element, denoted by $+\infty$ and $-\infty$, respectively; namely, $+\infty$ and $-\infty$ stand for the equivalence classes of constant functions with values $+\infty$ and $-\infty$ on Ω, respectively. Particularly, $L^0(\mathcal{F}, R)$ is order complete as a sublattice of $\overline{L}^0(\mathcal{F})$.

Let $A \in \mathcal{F}$ and ξ and η be in $\overline{L}^0(\mathcal{F})$; we say that $\xi > \eta$ on A ($\xi \ge \eta$ on A) if $\xi^0(\omega) > \eta^0(\omega)$ (accordingly, $\xi^0(\omega) \ge \eta^0(\omega)$) for almost all $\omega \in A$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Similarly, one can understand $\xi \ne \eta$ on A and $\xi = \eta$ on A. In particular, $\tilde{I}_A$ stands for the equivalence class of I_A, where $I_A(\omega) = 1$ if $\omega \in A$ and 0 if $\omega \notin A$.

This paper always employs the following notation:

$L^0(\mathcal{F}) = L^0(\mathcal{F}, R)$.

$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \ge 0\}$.

$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$.

Similarly, one can understand $\overline{L}^0_+(\mathcal{F})$ and $\overline{L}^0_{++}(\mathcal{F})$.

Let E be a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module); the module multiplication $\xi \cdot x$ is simply denoted by ξx for any $\xi \in L^0(\mathcal{F}, K)$ and $x \in E$. A mapping $\|\cdot\| : E \to L^0_+(\mathcal{F})$ is called an L^0-seminorm on E if it satisfies the following:

(1) $\|\xi x\| = |\xi|\|x\|$, $\forall \xi \in L^0(\mathcal{F}, K)$ and $x \in E$.

(2) $\|x + y\| \le \|x\| + \|y\|$, $\forall x, y \in E$.

If, in addition, $\|x\| = 0$ implies $x = \theta$ (the null element of E), then $\|\cdot\|$ is called an L^0-norm on E; at this time, the ordered pair $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, \mu)$.

An ordered pair $(E, \mathcal{P})$ is called a random locally convex module (briefly, an RLC module) over K with base $(\Omega, \mathcal{F}, P)$ if E is an $L^0(\mathcal{F}, K)$-module and $\mathcal{P}$ is a family of L^0-seminorms on E such that $\vee\{\|x\| : \|\cdot\| \in \mathcal{P}\} = 0$ implies $x = \theta$. Clearly, when $\mathcal{P}$ is a singleton consisting of an L^0-norm $\|\cdot\|$, an RLC module $(E, \mathcal{P})$ becomes an RN module $(E, \|\cdot\|)$, so the notion of an RN module is a special case of that of an RLC module.

Motivated by Schweizer and Sklar's work on random metric spaces and random normed linear spaces [4], Guo introduced the notions of RN modules and random inner product modules (briefly, RIP modules) in [2, 3]. The importance of RN modules lies in their $L^0(\mathcal{F}, K)$-module structure which makes RN modules and their random conjugate spaces possess the same nice behaviors as normed spaces and their conjugate spaces. At almost the same time, Haydon et al. also independently introduced the notion of an RN module over the real number field R with base being a measure space (called randomly normed L^0-module in terms of [17]) as a tool for the study of ultrapowers of Lebesgue–Bochner function spaces. The notion of an RLC module was first introduced by Guo and deeply developed by Guo and others in [6].

Given an RLC module $(E, \mathcal{P})$ over K with base $(\Omega, \mathcal{F}, P)$, we always denote by $\mathcal{P}(F)$ the family of finite nonempty subsets of $\mathcal{P}$. For each $Q \in \mathcal{P}(F)$, $\|\cdot\|_Q : E \to L^0_+(\mathcal{F})$ is the L^0-seminorm defined by $\|x\|_Q = \vee\{\|x\| : \|\cdot\| \in Q\}$ for all $x \in E$. Now, we can speak of the (ε, λ)-topology as follows.

Proposition 3 (see [6]). *Let $(E, \mathcal{P})$ be an RLC module over K with base $(\Omega, \mathcal{F}, P)$. For any positive numbers ε and λ with $0 < \lambda < 1$ and for any $Q \in \mathcal{P}(F)$, let $N_\theta(Q, \varepsilon, \lambda) = \{x \in E : P\{\omega \in \Omega \mid \|x\|_Q(\omega) < \varepsilon\} > 1 - \lambda\}$. Then, $\{N_\theta(Q, \varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1, \text{ and } Q \in \mathcal{P}(F)\}$ forms the local base at θ of some Hausdorff linear topology for E, called the (ε, λ)-topology induced by $\mathcal{P}$.*

From now on, for any RLC module $(E,\mathscr{P})$, we always use $\mathcal{T}_{\varepsilon,\lambda}$ for the (ε,λ)-topology for E induced by $\mathscr{P}$. It is clear that the absolute value $|\cdot|$ is an L^0-norm on $L^0(\mathscr{F},K)$. $\mathcal{T}_{\varepsilon,\lambda}$ induced by $|\cdot|$ is exactly the topology of convergence in probability; namely, a sequence $\{\xi_n : n\in N\}$ converges in $\mathcal{T}_{\varepsilon,\lambda}$ to ξ in $L^0(\mathscr{F},K)$ if and only if it converges in probability to ξ. It is easy to check that $(L^0(\mathscr{F},K),\mathcal{T}_{\varepsilon,\lambda})$ is a metrizable topological algebra for an RLC module $(E,\mathscr{P})$ over K with base $(\Omega,\mathscr{F},P)$. $(E,\mathcal{T}_{\varepsilon,\lambda})$ is a topological module over the topological algebra $(L^0(\mathscr{F},K),\mathcal{T}_{\varepsilon,\lambda})$.

In 2009, Filipović et al. introduced another kind of topology for $L^0(\mathscr{F},K)$: let ε belong to $L^0_{++}(\mathscr{F})$ and $U(\varepsilon)=\{\xi\in L^0(\mathscr{F},K)\mid |\xi|\le\varepsilon\}$. A subset G of $L^0(\mathscr{F},K)$ is said to be $\mathcal{T}_c$-open if for each $g\in G$ there exists some $U(\varepsilon)$ such that $g+U(\varepsilon)\subset G$. Denote by $\mathcal{T}_c$ the family of $\mathcal{T}_c$-open subsets of $L^0(\mathscr{F},K)$; then, $(L^0(\mathscr{F},K),\mathcal{T}_c)$ is a topological ring; namely, the multiplication and addition operations on $L^0(\mathscr{F},K)$ are both jointly continuous. Let E be an $L^0(\mathscr{F},K)$-module and $\mathcal{T}$ a topology for E; then, the topological space $(E,\mathcal{T})$ is called a topological L^0-module in [7] if $(E,\mathcal{T})$ is a topological module over the topological ring $(L^0(\mathscr{F},K),\mathcal{T}_c)$, namely, the module operations: the module multiplication operation and addition operation are both jointly continuous. In [7], a topological L^0-module $(E,\mathcal{T})$ is called a locally L^0-convex module if $\mathcal{T}$ possesses a local base at θ whose each element is L^0-convex, L^0-absorbent, and L^0-balanced, at which time $\mathcal{T}$ is also called a locally L^0-convex topology. Here, a subset U of E is said to be L^0-convex if $\xi x+(1-\xi)y\in U$ for all $x,y\in U$ and $\xi\in L^0_+(\mathscr{F})$ such that $0\le\xi\le1$; L^0-absorbent if for each $x\in E$ there exists some $\eta\in L^0_{++}(\mathscr{F})$ such that $\xi x\in U$ for any $\xi\in L^0(\mathscr{F},K)$ such that $|\xi|\le\eta$; and L^0-balanced if $\xi x\in U$ for all $x\in U$ and all $\xi\in L^0(\mathscr{F},K)$ such that $|\xi|\le 1$. The work in [7] leads directly to the following.

Proposition 4 (see [7]). *Let $(E,\mathscr{P})$ be an RLC module over K with base $(\Omega,\mathscr{F},P)$. For any $\varepsilon\in L^0_{++}(\mathscr{F})$ and $Q\in\mathscr{P}(F)$, let $N_\theta(Q,\varepsilon)=\{x\in E\mid \|x\|_Q\le\varepsilon\}$. Then, $\{N_\theta(Q,\varepsilon)\mid Q\in\mathscr{P}(F),\ \varepsilon\in L^0_{++}(\mathscr{F})\}$ forms a local base at θ of some Hausdorff locally L^0-convex topology, which is called the locally L^0-convex topology induced by $\mathscr{P}$.*

From now on, for an RLC module $(E,\mathscr{P})$, we always use $\mathcal{T}_c$ for the locally L^0-convex topology induced by $\mathscr{P}$. Recently, it is proved independently in [18, 19] that the converse of Proposition 4 is no longer true; namely, not every locally L^0-convex topology is necessarily induced by a family of L^0-seminorms.

For the sake of convenience, this paper needs the following.

Definition 5 (see [8]). Let E be an $L^0(\mathscr{F},K)$-module and G a subset of E. G is said to have the countable concatenation property if for each sequence $\{g_n : n\in N\}$ in G and each countable partition $\{A_n : n\in N\}$ of Ω to $\mathscr{F}$ there always exists $g\in G$ such that $\tilde{I}_{A_n}g=\tilde{I}_{A_n}g_n$ for each $n\in N$. If E has the countable concatenation property, $H_{cc}(G)$ denotes the countable concatenation hull of G, namely, the smallest set containing G and having the countable concatenation property.

Remark 6. As pointed out in [8], when $(E,\mathscr{P})$ is an RLC module, g in Definition 5 must be unique, at which time we can write $g=\sum_{n=1}^{\infty}\tilde{I}_{A_n}g_n$.

In [7], a family $\mathscr{P}$ of L^0-seminorms on an $L^0(\mathscr{F},K)$-module is said to have the countable concatenation property if each L^0-seminorm $\|\cdot\|:=\sum_{n=1}^{\infty}\tilde{I}_{A_n}\|\cdot\|_{Q_n}$ still belongs to $\mathscr{P}$ for each countable partition $\{A_n : n\in N\}$ of Ω to $\mathscr{F}$ and each sequence $\{Q_n : n\in N\}$ in $\mathscr{P}(F)$. We always denote $\mathscr{P}_{cc}=\{\sum_{n=1}^{\infty}\tilde{I}_{A_n}\cdot\|\cdot\|_{Q_n} : \{A_n : n\in N\}$ as a countable partition of Ω to $\mathscr{F}$ and $\{Q_n : n\in N\}$ as a sequence of $\mathscr{P}(F)\}$, called the countable concatenation hull of $\mathscr{P}$. Clearly, $\mathscr{P}$ has the countable concatenation property iff $\mathscr{P}_{cc}=\mathscr{P}$.

In random functional analysis, the notion of random conjugate spaces is crucial, which is defined as follows.

Definition 7 (see [8]). Let $(E,\mathscr{P})$ be an RLC module over K with base $(\Omega,\mathscr{F},P)$. Denote by $(E,\mathscr{P})^*_{\varepsilon,\lambda}$ the $L^0(\mathscr{F},K)$-module of continuous module homomorphisms from $(E,\mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathscr{F},K),\mathcal{T}_{\varepsilon,\lambda})$, called the random conjugate space of $(E,\mathscr{P})$ under $\mathcal{T}_{\varepsilon,\lambda}$; denote by $(E,\mathscr{P})^*_c$ the $L^0(\mathscr{F},K)$-module of continuous module homomorphisms from $(E,\mathcal{T}_c)$ to $(L^0(\mathscr{F},K),\mathcal{T}_c)$, called the random conjugate space of $(E,\mathscr{P})$ under $\mathcal{T}_c$.

From now on, when $\mathscr{P}$ is understood, we often briefly write $E^*_{\varepsilon,\lambda}$ for $(E,\mathscr{P})^*_{\varepsilon,\lambda}$ and E^*_c for $(E,\mathscr{P})^*_c$. When $\mathscr{P}$ has the countable concatenation property, it is proved in [8] that $E^*_{\varepsilon,\lambda}=E^*_c$. In general, $E^*_c\subset E^*_{\varepsilon,\lambda}$ and $E^*_{\varepsilon,\lambda}$ has the countable concatenation property. Recently, in [12], Guo et al. established the following precise relation between $E^*_{\varepsilon,\lambda}$ and E^*_c.

Proposition 8 (see [12]). *Let $(E,\mathscr{P})$ be an RLC module. Then, $E^*_{\varepsilon,\lambda}=H_{cc}(E^*_c)$.*

Remark 9. For an RLC module $(E,\mathscr{P})$, since $\mathscr{P}$ and $\mathscr{P}_{cc}$ induce the same (ε,λ)-topology on E, then $(E,\mathscr{P})^*_{\varepsilon,\lambda}=(E,\mathscr{P}_{cc})^*_{\varepsilon,\lambda}$. Since $\mathscr{P}_{cc}$ has the countable concatenation property, $(E,\mathscr{P}_{cc})^*_{\varepsilon,\lambda}=(E,\mathscr{P}_{cc})^*_c$; in fact, Proposition 8 has shown that $E^*_{\varepsilon,\lambda}=(E,\mathscr{P}_{cc})^*_c=H_{cc}(E^*_c)$.

To state and prove the main result of this section, we still need Lemma 10.

Lemma 10 (see [8]). *Let $(E,\mathscr{P})$ be an RLC module with base $(\Omega,\mathscr{F},P)$ and $G\subset E$ such that G has the countable concatenation property. Then, $\overline{G}_{\varepsilon,\lambda}=\overline{G}_c$, where $\overline{G}_{\varepsilon,\lambda}$ and $\overline{G}_c$ stand for the closures of G under $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$, respectively.*

Guo et al. started the study of random duality under the (ε,λ)-topology in [10]; further, in [10], Guo et al. studied random duality under the locally L^0-convex topology. Let us recall some notions and results used in proofs of the main results in this paper.

Definition 11 (see [5, 10]). Let X and Y be two $L^0(\mathcal{F},K)$-modules and $\langle\cdot,\cdot\rangle : X\times Y \to L^0(\mathcal{F},K)$ an L^0-bilinear functional. Then, $\langle X,Y\rangle$ is called a random duality pair (briefly, a random duality) over K with base $(\Omega,\mathcal{F},P)$ if the following conditions are satisfied:

(1) $\langle x,y\rangle = 0$, $\forall y\in Y$ iff $x=\theta$ (the null in X).

(2) $\langle x,y\rangle = 0$, $\forall x\in X$ iff $y=\theta$ (the null in Y).

Let $\langle X,Y\rangle$ be a random duality over K with base $(\Omega,\mathcal{F},P)$. For any given $y\in Y$, $\|\cdot\|_y : X\to L^0_+(\mathcal{F})$ defined by $\|x\|_y = |\langle x,y\rangle|$ ($\forall x\in X$) is an L^0-seminorm on X; denote $\{\|\cdot\|_y \mid y\in Y\}$ by $\sigma(X,Y)$; then, $(X,\sigma(X,Y))$ is a random locally convex module over K with base $(\Omega,\mathcal{F},P)$; the (ε,λ)-topology and the locally L^0-convex topology induced by $\sigma(X,Y)$ are denoted by $\sigma_{\varepsilon,\lambda}(X,Y)$ and $\sigma_c(X,Y)$, respectively. In particular, it was proved in [10] that $(X,\sigma_c(X,Y))^* = Y$. A subset B of Y is said to be $\sigma_c(Y,X)$-bounded if B is L^0-absorbed by each $\sigma_c(Y,X)$-neighborhood U of the null of Y; namely, there exists $\eta\in L^0_{++}(\mathcal{F})$ such that $\lambda B\subset U$ whenever $\lambda\in L^0(\mathcal{F},K)$ and $|\lambda|\le\eta$, which is equivalent to saying that $\vee\{|\langle x,y\rangle| : y\in B\}\in L^0_+(\mathcal{F})$, $\forall x\in X$. Denote $\{B : B\subset Y \text{ and } B \text{ is } \sigma_c(Y,X)\text{-bounded}\}$ by $\mathcal{B}(Y,X)$; for each $B\in\mathcal{B}(Y,X)$, the L^0-seminorm $\|\cdot\|_B : X\to L^0_+(\mathcal{F})$ is defined by $\|x\|_B = \vee\{|\langle x,y\rangle| : y\in B\}$ ($\forall x\in X$); then, $(X,\{\|\cdot\|_B : B\in\mathcal{B}(Y,X)\})$ is a random locally convex module over K with base $(\Omega,\mathcal{F},P)$; the locally L^0-convex topology induced by $\{\|\cdot\|_B : B\in\mathcal{B}(Y,X)\}$ is denoted by $\beta(X,Y)$.

Let $(S,\mathcal{P})$ be a random locally convex module over K with base $(\Omega,\mathcal{F},P)$. An L^0-balanced, L^0-absorbent, and $\mathcal{T}_c$-closed L^0-convex set of S is an L^0-barrel. $(S,\mathcal{T}_c)$ is an L^0-barreled module if each L^0-barrel is a $\mathcal{T}_c$-neighborhood of $\theta\in S$, whereas $(S,\mathcal{T}_c)$ is an L^0-prebarreled module if each L^0-barrel with the countable concatenation property is a $\mathcal{T}_c$-neighborhood of $\theta\in S$. Clearly, both $\langle S,S^*_{\varepsilon,\lambda}\rangle$ and $\langle S,S^*_c\rangle$ are a random duality pair over K with base $(\Omega,\mathcal{F},P)$. Further, let S possess the countable concatenation property; then, it is proved in [10] that $(S,\mathcal{T}_c)$ is L^0-prebarreled iff $\mathcal{T}_c=\beta(S,S^*_c)$.

In 2009, Guo et al. proved Mazur's lemma in a random locally convex module under the (ε,λ)-topology, which is stated as follows.

Proposition 12 (see [11]). *Let $(S,\mathcal{P})$ be a random locally convex module over K with base $(\Omega,\mathcal{F},P)$ and G an L^0-convex subset of S. Then, $G^-_{\varepsilon,\lambda} = [G]^-_{\sigma_{\varepsilon,\lambda}(S,S^*_{\varepsilon,\lambda})}$.*

Now, we can prove Theorem 1.

Proof of Theorem 1. Since G has the countable concatenation property, $G^-_c = G^-_{\varepsilon,\lambda}$ by Lemma 10. Further, $G^-_{\varepsilon,\lambda} = [G]^-_{\sigma_{\varepsilon,\lambda}(S,S^*_{\varepsilon,\lambda})}$ by Proposition 12. Since $S^*_{\varepsilon,\lambda} = H_{cc}(S^*_c)$ by Proposition 8, it is easy to see that $\sigma(S,S^*_{\varepsilon,\lambda})$ and $\sigma(S,S^*_c)$ induce the same (ε,λ)-topology on S, so that $[G]^-_{\sigma_{\varepsilon,\lambda}(S,S^*_{\varepsilon,\lambda})} = [G]^-_{\sigma_{\varepsilon,\lambda}(S,S^*_c)}$. Applying Lemma 10 to the random locally convex module $(S,\sigma(S,S^*_c))$ leads to $[G]^-_{\sigma_{\varepsilon,\lambda}(S,S^*_c)} = [G]^-_{\sigma_c(S,S^*_c)}$.

This completes the proof. □

Remark 13. In [15], Zapata proved the following result: let $(S,\|\cdot\|)$ be a random normed module and G an L^0-convex subset of S such that G has the relative countable concatenation property; in addition, if S possesses the property (sum of any two subsets with the relative countable concatenation property still has the relative countable concatenation property), then $G^-_c = [G]^-_{\sigma_c(S,S^*_c)}$. The advantage of Theorem 1 only requires that G has the countable concatenation property and $(S,\mathcal{P})$ is arbitrary, which is convenient to applications. On the other hand, as far as $G^-_c = [G]^-_{\sigma_c(S,S^*_c)}$ in Theorem 1 is concerned, the conclusion is also directly derived from Guo et al.'s separation theorem between a point and a $\mathcal{T}_c$-closed L^0-convex subset in [12].

Let $(S,\mathcal{P})$ be a random locally convex module over K with base $(\Omega,\mathcal{F},P)$. A subset $B\subset S$ is $\mathcal{T}_c$-bounded if B is L^0-absorbed by each $\mathcal{T}_c$-neighborhood U of $\theta\in S$ (namely, there exists $\eta\in L^0_{++}(\mathcal{F})$ such that $\lambda B\subset U$ whenever $\lambda\in L^0(\mathcal{F},K)$ and $|\lambda|\le\eta$); this is equivalent to saying that $\vee\{\|b\| : b\in B\}\in L^0_+(\mathcal{F})$ for any $\|\cdot\|\in\mathcal{P}$. Denote $\{B\subset S \mid B \text{ is } \mathcal{T}_c\text{-bounded}\}$ by $\mathcal{B}(S)$; for any given $B\in\mathcal{B}(S)$, the L^0-seminorm $\|\cdot\|_B : S^*_c\to L^0_+(\mathcal{F})$ is defined by $\|f\|_B = \vee\{|f(x)| : x\in B\}$ ($\forall f\in S^*_c$); then, $(S^*_c,\{\|\cdot\|_B : B\in\mathcal{B}(S)\})$ is a random locally convex module; the locally L^0-convex topology induced by $\{\|\cdot\|_B : B\in\mathcal{B}(S)\}$ is called the strong locally L^0-convex topology for S^*_c; we use S^*_s for S^*_c endowed with this strong locally L^0-convex topology; similarly, $(S^*_s)^*_s$ stands for $(S^*_s)^*_c$ endowed with its strong locally L^0-convex topology. For any given $x\in S$, $J(x) : S^*_c\to L^0(\mathcal{F},K)$ is defined by $J(x)(f) = f(x)$ ($\forall f\in S^*_c$). Since $J(x)$ is a continuous module homomorphism from $(S^*_c,\sigma_c(S^*_c,S))$ to $(L^0(\mathcal{F},K),\mathcal{T}_c)$ for each fixed $x\in S$, $J(x)$ also belongs to $(S^*_s)^*_c$ by an obvious fact that the strong locally L^0-convex topology is stronger than $\sigma_c(S^*_c,S)$, which shows that the canonical embedding mapping $J : S\to(S^*_s)^*_c$ is well defined, and J is also injective by the Hahn–Banach theorem established in [8]. For a subset A of S, the random right polar A° is defined by $A^\circ = \{f\in S^*_c \mid |f(x)|\le 1\ \forall x\in A\}$; similarly, the random left polar ${}^\circ B$ of a subset B of S^*_c is defined by ${}^\circ B = \{x\in S \mid |f(x)|\le 1\ \forall f\in B\}$.

Now, we can prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{B}(S^*_s)$ denote the family of $\mathcal{T}_c$-bounded sets of S^*_s; then, $\{(B^*)^\circ : B^*\in\mathcal{B}(S^*_s)\}$ forms a local base of $(S^*_s)^*_s$. Let $\mathcal{B}(S^*_c,S)$ denote the family of $\sigma_c(S^*_c,S)$-bounded sets of S^*_c; then, $\{{}^\circ(B^*) : B^*\in\mathcal{B}(S^*_c,S)\}$ forms a local base of $\beta(S,S^*_c)$. Since $(S,\mathcal{P})$ is an L^0-prebarreled random locally convex module such that S has the countable concatenation property, $\mathcal{T}_c = \beta(S,S^*_c)$ by the characterization theorem established by Guo et al. in [10]. It remains to check that $\mathcal{B}(S^*_s) = \mathcal{B}(S^*_c,S)$.

It is obvious that $\mathcal{B}(S^*_s)\subset\mathcal{B}(S^*_c,S)$. As for the reverse inclusion, let B^* be any element in $\mathcal{B}(S^*_c,S)$; then, ${}^\circ(B^*)$ is a neighborhood of $\theta\in S$, so ${}^\circ(B^*)$ L^0-absorbs each

$\mathcal{T}_c$-bounded set of the random locally convex module $(S, \mathcal{P})$, which implies $B^* \in \mathcal{B}(S_s^*)$. To sum up, $\mathcal{B}(S_s^*) = \mathcal{B}(S_c^*, S)$.

Finally, it is easy to observe that $J(^\circ(B^*)) = (B^*)^\circ \cap J(S)$ for each $B^* \in \mathcal{B}(S_s^*)$, which shows that $J : S \to J(S) \subset (S_s^*)_s^*$ is an L^0-linear homeomorphism.

This completes the proof. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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