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Research Article

Iterative Schemes for Nonconvex Quasi-Variational Problems with V-Prox-Regular Data in Banach Spaces

M. Bounkhel[1] and Dj. Bounekhel[2]

[1] *Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Faculté des Sciences et Techniques, Université de Jijel, P.O. Box 98, Cité Ouled Aissa, Jijel, Algeria*

Correspondence should be addressed to M. Bounkhel; bounkhel@ksu.edu.sa

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In this paper, we propose an extension of quasi-equilibrium problems from the convex case to the nonconvex case and from Hilbert spaces to Banach spaces. The proposed problem is called quasi-variational problem. We study the convergence of some algorithms to solutions of the proposed nonconvex problems in Banach spaces.

1. Introduction

Let X be a Banach space and let X^* be the dual space of X. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing of X^* and X. Let $C : X \rightrightarrows X$ be a set-valued mapping with nonempty closed values and let $F : X \times X \to \mathbf{R}$ be a bifunction satisfying $F(x,x) = 0$ for all $x \in \text{Fix}(C) := \{x \in X : x \in C(x)\}$. We associate with a *closed convex* valued set-valued mapping C and a *convex* bifunction F the following well known quasi-equilibrium problem:

$$\begin{aligned} \text{Find} \quad & \overline{x} \in C(\overline{x}), \\ \text{such that} \quad & F(\overline{x}, x) \geq 0, \\ & \forall x \in C(\overline{x}). \end{aligned} \qquad (\text{QEP}[C,F])$$

In this paper we propose the following appropriate extensions of (QEP$[C,F]$) from the convex case to the nonconvex case in Banach spaces setting. We associate with C and F the following nonconvex quasi-variational problem equilibrium problems:

$$\begin{aligned} \text{Find} \quad & \overline{x} \in C(\overline{x}), \\ \text{s.t.} \quad & \left[-\partial^{\pi} F(\overline{x},\cdot)(\overline{x})\right] \cap N^{\pi}(C(\overline{x});\overline{x}) \\ & \neq \emptyset, \end{aligned} \qquad (\text{NQVP}[C,F])$$

where ∂^{π} (resp. N^{π}) is the V-proximal subdifferential (resp. V-proximal normal cone) introduced and studied in [1].

The proposed nonconvex quasi-variational problem extends many existing quasi-equilibrium problems and quasi-variational inequalities from the convex case to the nonconvex case and from Hilbert spaces setting to Banach spaces setting.

(1) If X is a Hilbert space, the proposed (NQVP$[C,F]$) becomes

$$\begin{aligned} \text{Find} \quad & \overline{x} \in C(\overline{x}), \\ \text{such that} \quad & \left[-\partial^{P} F(\overline{x},\cdot)(\overline{x})\right] \cap N^{P}(C(\overline{x});\overline{x}) \neq \emptyset, \end{aligned} \qquad (1)$$

where ∂^P and N^P are the usual proximal subdifferential and proximal normal cone in Hilbert spaces. This problem has been introduced and studied in Bounkhel et al. [2]. Since then it has been studied and extended in various ways in Hilbert spaces by the authors in [3] and in Noor [4] and many works (see for instances Noor et al. [5, 6]).

(2) If X is a Hilbert space, C is a convex closed set in X, F is a convex bifunction, and $\rho = 0$, then

(NQVP$[C, F]$) becomes the following well known convex equilibrium problem:

$$\begin{aligned} &\text{Find} \quad \overline{x} \in C, \\ &\text{such that} \quad F(\overline{x}, x) \geq 0, \\ &\forall x \in C, \end{aligned} \tag{2}$$

which has been studied in various works (see for instance Moudafi [7], M. A. Noor and K. I. Noor [5], and the references therein).

(3) If $F(x, y) = \langle T(x), y - x\rangle$, with $T : X \to X^*$, is a nonlinear operator then (NQVP$[C, F]$) reduces to

$$\begin{aligned} &\text{Find} \quad \overline{x} \in C(\overline{x}), \\ &\text{s.t.} \quad -T(\overline{x}) \in N^\pi (C(\overline{x}); \overline{x}) \end{aligned} \tag{3}$$

which will be shown in Section 4 to be equivalent in the uniform V-prox-regular case, for some $\rho \geq 0$, to the following quasi-variational inequality:

$$\begin{aligned} &\text{Find} \quad \overline{x} \in C(\overline{x}), \\ &\text{s.t.} \quad \langle T(\overline{x}), x - \overline{x}\rangle + \rho V(J(\overline{x}), x) \geq 0, \\ &\forall x \in C(\overline{x}). \end{aligned} \tag{4}$$

This inequality is new in Banach spaces. However, it has been studied, in Hilbert spaces, in Bounkhel et al. [2], when C is a uniformly V-prox-regular set (see also Bounkhel and Al-Sinan [8] and Noor et al. [5, 6]).

When $\rho = 0$ and $C(x) \equiv C$ the last inequality becomes

$$\begin{aligned} &\text{Find} \quad \overline{x} \in C, \\ &\text{such that} \quad \langle T(\overline{x}), x - \overline{x}\rangle \geq 0, \\ &\forall x \in C, \end{aligned} \tag{5}$$

which is known as the classical variational inequality introduced and studied in Stampacchia [9].

Our main objective of the present paper is to prove the convergence of some algorithms to solutions of the proposed nonconvex quasi-variational problem (NQVP$[C, F]$).

2. Preliminaries

In order to prepare the framework of our study we need to state some concepts and results. First we recall (see for instance [1, 10]) the definition of p-uniformly convex and q-uniformly smooth Banach spaces. The space X is said to be p-uniformly convex (resp. q-uniformly smooth) if there is a constant $c > 0$ such that

$$\delta_X(\epsilon) \geq c\epsilon^p \left(\text{resp. } \rho_X(t) \leq ct^q\right), \tag{6}$$

where δ_X and ρ_X are defined, respectively, by

$$\begin{aligned} \delta_X(\epsilon) &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon\right\}, \quad 0 \leq \epsilon \leq 2, \\ \rho_X(t) &= \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1,\ \|y\| = t\right\}, \quad t > 0. \end{aligned} \tag{7}$$

Notice that the constants p and q in the previous definition always satisfy $p \geq 2$ and $q \in (1, 2]$. Also we need to recall from [1] the concept of *V-proximal subdifferential* $\partial^\pi f(x)$ (called in [1] *generalised proximal subdifferential*). An element $x^* \in X^*$ belongs to $\partial^\pi f(x)$ provided that there exists $\sigma > 0$ so that

$$\left\langle x^*, x' - x\right\rangle \leq f\left(x'\right) - f(x) + \sigma V\left(J(x), x'\right), \tag{8}$$

for x' very close to x, where $J : X \to X^*$ is the normalised duality mapping and $V : X^* \times X \to \mathbf{R}$ is a functional defined by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2, \quad \text{for any } x^* \in X^*,\ x \in X. \tag{9}$$

For a closed nonempty set S in X and $\overline{x} \in S$, the authors in [1] defined the concept of *V-proximal normal cone* $N^\pi(S; \overline{x})$ (called in [1] *generalised proximal normal cone*) by $N^\pi(S; \overline{x}) = \partial^\pi \psi_S(\overline{x})$, where ψ_S denotes the indicator function associated with S, that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ if $x \notin S$. We recall, respectively, the concepts of limiting Fréchet subdifferential ∂^{LF} and limiting V-proximal subdifferential $\partial^{L\pi}$ (see [11]):

$$\begin{aligned} \partial^{L\pi} f(x) &= \limsup_{x' \to x} \partial^\pi f\left(x'\right) \\ &:= \left\{w - \lim_n x_n^* : x_n^* \in \partial^\pi f(x_n) \text{ with } x_n \longrightarrow^f x\right\}, \\ \partial^{LF} f(x) &= \limsup_{x' \to x} \partial^F f\left(x'\right) \\ &:= \left\{w - \lim_n x_n^* : x_n^* \in \partial^F f(x_n) \text{ with } x_n \longrightarrow^f x\right\}, \end{aligned} \tag{10}$$

where $x_n \to^f x$ means $x_n \to x$ with $f(x_n) \to f(x)$ and

$$\begin{aligned} \partial^F f(x) = \Big\{x^* \in X^* : \forall\epsilon > 0,\ \exists\delta > 0 : \left\langle x^*, x' - x\right\rangle \\ \leq f\left(x'\right) - f(x) + \epsilon\left\|x' - x\right\|,\ \forall x' \in x + \delta\mathbf{B}\Big\}. \end{aligned} \tag{11}$$

The limiting Fréchet normal cone is defined similarly, that is,

$$\begin{aligned} \partial^{LF} N(S; x) &= \limsup_{x' \to x} \partial^F N\left(S; x'\right) \\ &:= \left\{w - \lim_n x_n^* : x_n^* \in N^F(S; x_n) \text{ with } x_n \longrightarrow^S x\right\}, \end{aligned} \tag{12}$$

where $x_n \to^S x$ denotes $x_n \to x$ with $x_n \in S$ and $N^F(S;x)$ is the Fréchet normal cone which is defined by $N^F(S;\overline{x}) = \partial^F \psi_S(\overline{x})$.

These all nonconvex objects coincide with their analogues defined in Convex Analysis whenever the data are convex as the following proposition shows (see [1]).

Proposition 1. *Let X be a reflexive Banach space.*

(1) *Let $f : X \to \mathbf{R} \cup +\infty$ be a l.s.c. convex function and $\overline{x} \in X$ with $f(\overline{x}) < \infty$. Then*

$$\partial^\pi f(x) = \partial^{Con.} f(x) := \{x^* \in X^* : \langle x^*; x - \overline{x}\rangle \leq f(x) - f(\overline{x}),\ \forall x \in X\}. \tag{13}$$

(2) *Let S be a closed convex subset in of X and $\overline{x} \in S$. Then*

$$N^\pi(S;\overline{x}) = N^{Con.}(S;\overline{x}) := \{x^* \in X^* : \langle x^*; x - \overline{x}\rangle \leq 0,\ \forall x \in S\}. \tag{14}$$

The following result is needed in our study. It has been proved in [11].

Theorem 2. *Let X be a q-uniformly smooth and p-uniformly convex Banach space. Assume that X admits an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^s$ (for some $s \geq 2$) is C^2-differentiable on $X \setminus \{0\}$ and let V be the functional associated with that norm $\|\cdot\|$.*

(1) *Let $f : X \to \mathbf{R}\cup\{\infty\}$, be a l.s.c. function at $\overline{x} \in \operatorname{dom} f$. Then*

$$\partial^{L\pi} f(\overline{x}) = \partial^{FL} f(\overline{x}). \tag{15}$$

(2) *Let S be any closed nonempty set of X. Then*

$$N^{FL}(S;\overline{x}) = N^{L\pi}(S;\overline{x}). \tag{16}$$

We notice that the class of spaces satisfying the assumptions of the previous theorem is very large; it contains obviously any Hilbert space and L^p spaces and Sobolev spaces $W^{p,m}$ with $p \geq 2$ (see Theorem 1.1 in Section 5 in [10, 12]) and for more examples and discussions we refer to [10, 12]. We close this section with the following two concepts of uniform V-prox-regularity for functions and sets (see [13]).

Definition 3. Let X be a reflexive smooth Banach space. For a given $r \in (0,\infty]$, a subset S is V-uniformly prox-regular with respect to r provided that for all $x \in S$ and all nonzero $x^* \in N^\pi(S;x)$ we have

$$\left\langle \frac{x^*}{\|x^*\|}, x' - x \right\rangle \leq \frac{1}{2r} V\left(J(x), x'\right), \quad \forall x' \in S. \tag{17}$$

We use the convention $1/r = 0$ for $r = +\infty$.

Obviously, this class contains the class of uniformly prox-regular sets ([14, 15]) from Hilbert spaces to Banach spaces since in Hilbert spaces we have $V(J(x), x') = \|x - x'\|^2$ and the V-proximal normal cone $N^\pi(S;x)$ coincides with the usual proximal normal cone $N^P(S;x)$.

Definition 4. Let X be a reflexive smooth Banach space. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a l.s.c. function and let $S \subset \operatorname{dom} f := \{x \in X : f(x) < \infty\}$ be a nonempty closed set in X. We recall from [13] that f is said to be uniformly V-prox-regular over S provided that for all $x \in S$ and all $x^* \in \partial^\pi f(x)$ we have

$$\left\langle x^*, x' - x\right\rangle \leq f\left(x'\right) - f(x) + \frac{1}{2r} V\left(J(x), x'\right), \quad \forall x' \in S. \tag{18}$$

We say that f is uniformly V-prox-regular around $\overline{x} \in \operatorname{dom} f$ provided that f is uniformly V-prox-regular over some closed neighborhood of $\overline{x}$; that is, there exists a closed neighborhood $V_{\overline{x}}$ of $\overline{x}$ such that $\forall x \in V_{\overline{x}}$, $\forall x^* \in \partial^\pi f(x)$ the inequality (18) holds for any $x' \in V_{\overline{x}}$.

The following example is quoted from [13]. For its proof we refer the reader to [13].

Example 5. (1) Any l.s.c. proper convex function is uniformly V-prox-regular over any nonempty closed set S in its domain with $r = +\infty$.

(2) Both the indicator function ψ_S and the distance function d_S of uniformly V-prox-regular set S are uniformly V-prox-regular over S with respect to the same constant r.

(3) Any lower-C^2 function f over convex strongly compact K in X is uniformly V-prox-regular over K with some $r \in (0,+\infty]$ (see [13] for the definition of lower-C^2 functions).

The following two lemmas are needed in our proofs in Section 4. The proof of the first one is proved in [1]. The second one is proved in [16].

Lemma 6. *Let X be a p-uniformly convex and q-uniformly smooth Banach space and S be a bounded set. Then for some $\eta, \kappa > 0$ we have*

$$\eta^{-1}\|x - y\|^p \leq V\left(J(x), y\right) \leq \kappa^{-1}\|x - y\|^q, \quad \forall x, y \in S. \tag{19}$$

Lemma 7. *If X is a uniformly convex Banach space, then the inequality*

$$V\left(J(x), y\right) \geq 8C^2 \delta_X\left(\frac{\|x - y\|}{4C}\right) \tag{20}$$

holds for all x and y in X, where $C = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

3. Main Results

First we show that in the convex case (NQVP$[C,F]$) coincides with the quasi-equilibrium problem (QEP$[C,F]$).

Proposition 8. *Let X be a reflexive Banach space. Assume that C is a closed convex set-valued mapping and F is a convex bifunction satisfying $F(x,x) = 0$ for any $x \in \operatorname{Fix}(C)$. Then we have (NQVP$[C,F]$) $\Leftrightarrow$ (QEP$[C,F]$).*

Proof.

⇒?. Let $\bar{x}$ be a solution of (NQVP$[C,F]$); that is, there exists $y^* \in \partial^\pi F(\bar{x},\cdot)(\bar{x})$ such that $-y^* \in N^\pi(C(\bar{x}),\bar{x})$. Since $C(\bar{x})$ is a closed convex set, the V-proximal normal cone $N^\pi(C(\bar{x}),\bar{x})$ coincides with the convex normal cone $N^{\text{Con.}}(C(\bar{x}),\bar{x})$ (by Proposition 1) and so

$$\langle y^*; x-\bar{x}\rangle \geq 0, \quad \forall x \in C(\bar{x}). \tag{21}$$

On the other hand, the convexity of the bifunction F and Proposition 1 yield

$$\langle y^*; x-\bar{x}\rangle \leq F(\bar{x},x) - F(\bar{x},\bar{x}), \quad \forall x \in X. \tag{22}$$

Since $\bar{x} \in C(\bar{x})$ we have $F(\bar{x},\bar{x}) = 0$ (by assumption) and hence the previous two inequalities ensure

$$F(\bar{x},x) \geq 0, \quad \forall x \in C(\bar{x}); \tag{23}$$

that is, $\bar{x}$ is a solution of (QEP$[C,F]$).

⇐?. Let $\bar{x}$ be a solution of (NQEP$[C,F]$), that is, $F(\bar{x},x) \geq 0$, $\forall x \in C(\bar{x})$. Since $C(\bar{x})$ is a closed convex set and $F(\bar{x},\cdot)$ is a convex function, the function $x \mapsto h(x) := F(\bar{x},x) + \psi_{C(\bar{x})}(x)$ admits at $\bar{x}$ a global minimum on X. It follows that

$$\begin{aligned} 0 \in \partial^{\text{Con.}} h(\bar{x}) &= \partial^{\text{Con.}} F(\bar{x},\cdot)(\bar{x}) + \partial^{\text{Con.}} \psi_{C(\bar{x})}(\bar{x}) \\ &= \partial^{\text{Con.}} F(\bar{x},\cdot)(\bar{x}) + N^{\text{Con.}}(C(\bar{x});\bar{x}). \end{aligned} \tag{24}$$

which is equivalent to $[-\partial^{\text{Con.}} F(\bar{x},\cdot)(\bar{x})] \cap N^{\text{Con.}}(C(\bar{x});\bar{x}) \neq \emptyset$ and hence the proof is complete since $\partial^\pi F(\bar{x},\cdot)(\bar{x}) = \partial^{\text{Con.}} F(\bar{x},\cdot)(\bar{x})$ and $N^\pi(C(\bar{x}),\bar{x}) = N^{\text{Con.}}(C(\bar{x}),\bar{x})$. □

In the next proposition we establish an inequality characterisation of the proposed nonconvex quasi-variational problem (NQVP$[C,F]$) whenever the data C and F are uniformly V-prox-regular.

Proposition 9. *Let X be a reflexive Banach space and $\bar{x} \in X$. Assume that $C(\bar{x})$ is uniformly V-prox-regular with ratio $r \in (0,\infty]$ and that $F(\bar{x},\cdot)$ is uniformly V-prox-regular over $C(\bar{x})$ with ratio $r' \in (0,\infty]$. Assume also that $F(\bar{x},\cdot)$ is γ-Lipschitz around $\bar{x}$ and $F(x,x) = 0$ for any $x \in \text{Fix}(C)$. If $\bar{x}$ is a solution of (NQVP$[C,F]$), then $\bar{x}$ is a solution of the following nonconvex quasi-equilibrium problem. Find $\bar{x} \in C(\bar{x})$ such that*

$$F(\bar{x},x) + \rho V(J\bar{x},x) \geq 0, \quad \forall x \in C(\bar{x}), \tag{NQEP[C, F]}$$

for some nonnegative $\rho \geq 0$.

Proof. Assume that $\bar{x}$ is a solution of (NQVP$[C,F]$); that is, $y^* \in \partial^\pi F(\bar{x},\cdot)(\bar{x})$ such that $-y^* \in N^\pi(C(\bar{x});\bar{x})$. By uniform V-prox-regularity of the set $C(\bar{x})$ we have

$$\langle -y^*, x-\bar{x}\rangle \leq \frac{\|y^*\|}{2r} V(J\bar{x},x), \quad \forall x \in C(\bar{x}). \tag{25}$$

The γ-Lipschitz continuity of $F(\bar{x},\cdot)$ ensures that $\|y^*\| \leq \gamma$ and so we obtain

$$\langle -y^*, x-\bar{x}\rangle \leq \frac{\gamma}{2r} V(J\bar{x},x), \quad \forall x \in C(\bar{x}). \tag{26}$$

On the other hand the uniform V-prox-regularity of $F(\bar{x},\cdot)$ over $C(\bar{x})$ with ratio $r' > 0$; we have

$$\langle y^*, x-\bar{x}\rangle \leq \frac{1}{2r'} V(J\bar{x},x) + F(\bar{x},x) - F(\bar{x},\bar{x}), \quad \forall x \in C(\bar{x}). \tag{27}$$

Combining this inequality (27) with (26) we obtain

$$F(\bar{x},x) - F(\bar{x},\bar{x}) + \frac{1}{2r'} V(J\bar{x},x) \geq -\frac{\gamma}{2r} V(J\bar{x},x) \quad \forall x \in C(\bar{x}). \tag{28}$$

Since $\bar{x} \in C(\bar{x})$ we have $F(\bar{x},\bar{x}) = 0$ and so (28) becomes

$$F(\bar{x},x) + \rho V(J\bar{x},x) \geq 0 \quad \forall x \in C(\bar{x}), \tag{29}$$

with $\rho := \gamma/2r + 1/2r' \geq 0$. Thus the proof is complete. □

It is a natural question to ask whether the converse in the previous proposition is true or not. The answer is positive provided that the space X and the data C and F satisfy some additional assumptions as the following proposition shows.

Proposition 10. *Let X be a q-uniformly smooth and p-uniformly convex Banach space. Assume that X admits an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^s$ (for some $s \geq 2$) is C^2-differentiable on $X \setminus \{0\}$ and let V be the functional associated with that norm $\|\cdot\|$. Assume that $C(\bar{x})$ is V-proximal normally regular at $\bar{x}$, that is, $N^\pi(C(\bar{x}),\bar{x}) = N^{L\pi}(C(\bar{x}),\bar{x})$ and that $F(\bar{x},\cdot)$ is V-proximal subdifferentially regular at $\bar{x}$, that is, $\partial^\pi F(\bar{x},\cdot)(\bar{x}) = \partial^{L\pi} F(\bar{x},\cdot)(\bar{x})$. Assume that $F(x,x) = 0$ for any $x \in \text{Fix}(C)$. If $\bar{x}$ is a solution of (NQEP$[C,F]$) for some $\rho \geq 0$, then $\bar{x}$ is a solution of (NQVP$[C,F]$).*

Proof. Let $\bar{x}$ be a solution of (NQEP$[C,F]$) for some $\rho \geq 0$; that is,

$$F(\bar{x},x) + \rho V(J\bar{x},x) \geq 0 \quad \forall x \in C(\bar{x}). \tag{30}$$

Then $\bar{x}$ is a global minimum of the function $x \mapsto h(x) = F(\bar{x},x) + \rho V(J\bar{x},x) + \psi_{C(\bar{x})}(x)$ over X and hence

$$\begin{aligned} 0 \in \partial^\pi h(\bar{x}) &\subset \partial^{L\pi} h(\bar{x}) \\ &= \partial^{L\pi}\left[F(\bar{x},\cdot) + \rho V(J\bar{x},\cdot) + \psi_{C(\bar{x})}(\cdot)\right](\bar{x}). \end{aligned} \tag{31}$$

Note that the function $x \mapsto V(J(\bar{x}),x)$ is differentiable and its gradient is given by $\text{grad}\,(V(J(\bar{x}),\cdot))(x) = 2(J(x) - J(\bar{x}))$. Using the fact that the limiting V-proximal subdifferential coincides with the limiting Fréchet subdifferential (by Theorem 2) and the exact sum rules for the limiting Fréchet subdifferential (see for instance [17]) we can write

$$\begin{aligned} 0 &\in \partial^{L\pi}\left[F(\overline{x},\cdot)+\rho V(J\overline{x},\cdot)+\psi_{C(\overline{x})}(\cdot)\right](\overline{x}) \\ &\in \partial^{LF}\left[F(\overline{x},\cdot)+\rho V(J\overline{x},\cdot)+\psi_{C(\overline{x})}(\cdot)\right](\overline{x}) \\ &\in \partial^{LF}F(\overline{x},\cdot)(\overline{x})+\partial^{LF}(\rho V(J\overline{x},\cdot))(\overline{x}) \\ &\quad+\partial^{LF}\psi_{C(\overline{x})}(\cdot)(\overline{x}) \\ &\in \partial^{LF}F(\overline{x},\cdot)(\overline{x})+2\rho(J(\overline{x})-J(\overline{x})) \\ &\quad+N^{LF}(C(\overline{x});\overline{x}) \\ &\in \partial^{LF}F(\overline{x},\cdot)(\overline{x})+N^{LF}(C(\overline{x});\overline{x}). \end{aligned} \tag{32}$$

This is equivalent to say that $[-\partial^{LF}F(\overline{x},\cdot)(\overline{x})]\cap N^{LF}(C(\overline{x});\overline{x})\neq\emptyset$. Thus completing the proof since $\partial^{\pi}F(\overline{x},\cdot)(\overline{x})=\partial^{L\pi}F(\overline{x},\cdot)(\overline{x})=\partial^{LF}F(\overline{x},\cdot)(\overline{x})$ and $N^{\pi}(C(\overline{x}),\overline{x})=N^{L\pi}(C(\overline{x});\overline{x})=N^{LF}(C(\overline{x});\overline{x})$. □

The following proposition has its own interest and is needed to prove the equivalence between (NQVP$[C,F]$) and (NQEP$[C,F]$) whenever C and F are uniformly V-prox-regular.

Proposition 11. *Let X be a reflexive Banach space and let $f: X\to\mathbf{R}\cup\{\infty\}$ be a l.s.c. function and let $\overline{x}\in\operatorname{dom} f$. If f is uniformly V-prox-regular around $\overline{x}$, then $\partial^{\pi}f(\overline{x})=\partial^{L\pi}f(\overline{x})$; that is, f is V-proximal subdifferentially regular at $\overline{x}$. Consequently, for any uniformly V-prox-regular closed set S at $\overline{x}\in S$ we have $N^{\pi}(S,\overline{x})=N^{L\pi}(S;\overline{x})$; that is, S is V-proximal normally regular at $\overline{x}$.*

Proof. We only prove the first assertion; the second one follows directly from the first one and Example 5 Part (2). Since we always have the inclusion $\partial^{\pi}f(\overline{x})\subset\partial^{L\pi}f(\overline{x})$, it is enough to prove the reverse one, that is, $\partial^{L\pi}f(\overline{x})\subset\partial^{\pi}f(\overline{x})$. Let $x^*\in\partial^{L\pi}f(\overline{x})$; that is, there exists $x_n\to^f x$ and $x_n^*\in\partial^{\pi}f(x_n)$ such that $x^*=w-\lim_n x_n^*$. By the uniform V-prox-regularity of f around $\overline{x}$, there exist $r>0$ and $\delta>0$ such that for any $x\in\overline{x}+\delta\mathbf{B}$ and any $y^*\in\partial^{\pi}f(x)$

$$\left\langle y^*,x'-x\right\rangle\le\frac{1}{2r}V\left(Jx,x'\right)+f\left(x'\right)-f(x),\qquad \forall x'\in x+\delta\mathbf{B}. \tag{33}$$

Since $x_n\to\overline{x}$ we can write for n large enough that $x_n\in\overline{x}+(\delta/2)\mathbf{B}$ and hence by (33) we have

$$\left\langle x_n^*,x'-x_n\right\rangle\le\frac{1}{2r}V\left(Jx_n,x'\right)+f\left(x'\right)-f(x_n),\qquad \forall x'\in x_n+\delta\mathbf{B}. \tag{34}$$

Fix any $y\in\overline{x}+(\delta/2)\mathbf{B}$. Clearly $y\in x_n+(\delta/2)\mathbf{B}+(\delta/2)\mathbf{B}\subset x_n+\delta\mathbf{B}$ and hence (34) ensures

$$\begin{aligned}\left\langle x^*,y-\overline{x}\right\rangle &= \left\langle x^*-x_n^*,y-\overline{x}\right\rangle+\left\langle x_n^*,y-x_n\right\rangle \\ &\quad+\left\langle x_n^*;x_n-\overline{x}\right\rangle \\ &\le \left\langle x^*-x_n^*,y-\overline{x}\right\rangle+\left\langle x_n^*;x_n-\overline{x}\right\rangle \\ &\quad+\frac{1}{2r}V\left(Jx_n,y\right)+f(y)-f(x_n).\end{aligned} \tag{35}$$

Using now the fact that $x_n\to^f\overline{x}$, the continuity of J and V, and the weak convergence of x_n^* to x^* to pass to the limit as n goes to ∞ and to get

$$\left\langle x^*,y-\overline{x}\right\rangle\le\frac{1}{2r}V\left(J\overline{x},y\right)+f(y)-f(\overline{x}), \tag{36}$$

for any $y\in\overline{x}+(\delta/2)\mathbf{B}$, this means by definition that $x^*\in\partial^{\pi}f(\overline{x})$ and the proof is complete. □

Using this result together with Propositions 9 and 10 we obtain the equivalence between (NQVP$[C,F]$) and (NQEP$[C,F]$).

Proposition 12. *Let X be a q-uniformly smooth and p-uniformly convex Banach space and $\overline{x}\in X$. Assume that X admits an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^s$ (for some $s\ge 2$) is C^2-differentiable on $X\setminus\{0\}$ and let V be the functional associated with that norm $\|\cdot\|$. Assume that $C(\overline{x})$ is uniformly V-prox-regular with ratio $r\in(0,\infty]$ and that $F(\overline{x},\cdot)$ is uniformly V-prox-regular over $C(\overline{x})$ with ratio $r'\in(0,\infty]$. Assume also that $F(\overline{x},\cdot)$ is γ-Lipschitz around $\overline{x}$ and $F(x,x)=0$ for any $x\in\operatorname{Fix}(C)$. Then (NQVP$[C,F]$) is equivalent to (NQEP$[C,F]$) for some $\rho\ge 0$.*

4. Convergence Analysis

4.1. Case 1: C Is a Constant Set-Valued Mapping. In this case the proposed problem becomes as follows:

$$\begin{aligned}&\text{Find}\quad \overline{x}\in C\\ &\text{s.t.}\quad \left[-\partial^{\pi}F(\overline{x},\cdot)(\overline{x})\right]\cap N^{\pi}(C;\overline{x})\neq\emptyset.\end{aligned} \tag{NVP$[C,F]$}$$

In this subsection we propose the following algorithm.

Algorithm 13. Let $\rho\ge 0$ and $\lambda_n>0$ for all $n\ge 1$;

(1) Select $x_0\in C$;

(2) For $n\ge 1$ select $x_{n+1}\in C$ such that

$$\begin{aligned}&\lambda_n^{-1}\left\langle J(x_n)-J(x_{n+1}),x-x_{n+1}\right\rangle\\ &\quad\le F(x_n,x)+\rho V(J(x_n),x),\quad \forall x\in C.\end{aligned} \tag{37}$$

Theorem 14. *Let X be a q-uniformly convex Banach space. Let C be a closed nonempty subset of X and let $F: C\times C\to\mathbf{R}$ be a bifunction satisfying $F(x,x)=0$ for any $x\in\operatorname{Fix}(C)$. Let $\{x_n\}_n$ be a sequence generated by Algorithm 13. Assume that*

(1) *C is V-uniformly prox-regular with some* $r \in (0, \infty]$;

(2) *C is ball compact; that is,* $C \cap \eta B$ *is compact for any* $\eta > 0$;

(3) *The solution set of* (*NQVP*[C, F]) *is nonempty;*

(4) *F is W-strongly monotone over C for some* $\sigma \geq 0$; *that is,*

$$F(x, y) + F(y, x) \leq -\sigma W(x, y), \quad \forall x, y \in C, \tag{38}$$

where $W(x, y) := (1/2)[V(J(x), y) + V(J(y), x)]$;

(5) *F is upper semicontinuous with respect to the first variable over C; that is,*

$$\limsup_{x' \to x} F(x', y) \leq F(x, y) \quad \forall x, y \in C; \tag{39}$$

(6) *The bifurcation F is* γ*-Lipschitz with respect to the second variable and* $F(x_{n+1}, \cdot)$ *is V-uniformly prox-regular over C with some* $r' \in (0, +\infty]$;

(7) *There exists* $\lambda > 0$ *such that* $\lambda_n \geq \lambda$ *for all n;*

(8) *The parameters* $r, r', \gamma, \rho, \sigma$ *satisfy the inequalities* $2\rho \leq \gamma/2r + 1/2r' \leq \sigma/3$.

Then, there exists subsequence of $\{x_n\}$ *converges to* $\tilde{x} \in C$ *which solves* (*NVP*[C, F]).

Proof. Let $\overline{x} \in C$ be a solution of (NVP[C, F]). Then by Proposition 9 we have

$$F(\overline{x}, x) + \rho_0 V(J(\overline{x}), x) \geq 0, \quad \forall x \in C, \tag{40}$$

for $\rho_0 := \gamma/2r + 1/2r'$. By the W-strong monotonicity of F over C we have

$$F(x, \overline{x}) + F(\overline{x}, x) \leq -\sigma W(x, \overline{x}), \quad \forall x \in C. \tag{41}$$

By setting $x = x_n$ in these two inequalities we get

$$\begin{aligned} F(x_n, \overline{x}) + F(\overline{x}, x_n) &\leq -\sigma W(x_n, \overline{x}), \\ -F(\overline{x}, x_n) &\leq \rho_0 V(J(\overline{x}), x_n). \end{aligned} \tag{42}$$

Combining these two inequalities we obtain

$$\begin{aligned} F(x_n, \overline{x}) &\leq \rho_0 V(J(\overline{x}), x_n) - \sigma W(x_n, \overline{x}) \\ &\leq (2\rho_0 - \sigma) W(x_n, \overline{x}). \end{aligned} \tag{43}$$

Using the 8th assumption in Theorem 14 we have $2\rho_0 - \sigma \leq -\rho_0$ and hence

$$F(x_n, \overline{x}) \leq -\rho_0 W(x_n, \overline{x}). \tag{44}$$

This combined with Algorithm 13 gives

$$\begin{aligned} \langle x^*_{n+1}, \overline{x} - x_{n+1} \rangle &\leq F(x_n, \overline{x}) + \rho V(J(x_n), \overline{x}) \\ &\leq -\rho_0 W(x_n, \overline{x}) + \rho V(J(x_n), \overline{x}) \\ &\leq (2\rho - \rho_0) W(x_n, \overline{x}), \end{aligned} \tag{45}$$

with $x^*_{n+1} := \lambda_n^{-1}[J(x_n) - J(x_{n+1})]$. Therefore,

$$\begin{aligned} &\langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1} \rangle \\ &\quad \leq \lambda_n (2\rho - \rho_0) W(x_n, \overline{x}). \end{aligned} \tag{46}$$

Define now a sequence of nonnegative real numbers $\phi_n = (1/2)V(J(x_n), \overline{x})$. It is not hard to verify that

$$\begin{aligned} &2[\phi_{n+1} - \phi_n] + V(J(x_n), x_{n+1}) \\ &\quad = 2\langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1} \rangle. \end{aligned} \tag{47}$$

Indeed,

$$\begin{aligned} 2[\phi_{n+1} - \phi_n] &= V(J(x_{n+1}), \overline{x}) - V(J(x_n), \overline{x}) \\ &= \left[\|J(x_{n+1})\|^2 - 2\langle J(x_{n+1}), \overline{x} \rangle + \|\overline{x}\|^2\right] \\ &\quad - \left[\|J(x_n)\|^2 - 2\langle J(x_n), \overline{x} \rangle + \|\overline{x}\|^2\right] \\ &= \|J(x_{n+1})\|^2 + 2\langle J(x_n) - J(x_{n+1}), \overline{x} \rangle \\ &\quad - \|J(x_n)\|^2 \\ &= 2\langle J(x_n) - J(x_{n+1}), \overline{x} \rangle - \|J(x_{n+1})\|^2 \\ &\quad - \|J(x_n)\|^2 + 2\langle J(x_{n+1}), x_{n+1} \rangle \\ &= 2\langle J(x_n) - J(x_{n+1}), \overline{x} \rangle - V(J(x_n), x_{n+1}) \\ &\quad - 2\langle J(x_n), x_{n+1} \rangle + 2\langle J(x_{n+1}), x_{n+1} \rangle \\ &= 2\langle J(x_n) - J(x_{n+1}), \overline{x} \rangle - V(J(x_n), x_{n+1}) \\ &\quad - 2\langle J(x_n) - J(x_{n+1}), x_{n+1} \rangle \\ &= 2\langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1} \rangle \\ &\quad - V(J(x_n), x_{n+1}). \end{aligned} \tag{48}$$

It follows that

$$\phi_{n+1} - \phi_n \leq \langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1} \rangle, \tag{49}$$

which ensures with (46) that

$$\phi_{n+1} - \phi_n \leq \lambda_n (2\rho - \rho_0) W(x_n, \overline{x}). \tag{50}$$

Using the assumption $\rho_0 \geq 2\rho$ in the 8th assumption we obtain

$$\phi_{n+1} \leq \phi_n. \tag{51}$$

Therefore, the sequence $\{\phi_n\}$ is a nonincreasing converging sequence to some limit and so it is bounded by some $\alpha > 0$. Thus by the properties of the functional V we obtain

$$(\|\overline{x}\| - \|x_n\|)^2 \leq V(J(x_n), \overline{x}) = 2\phi_n \leq 2\alpha \tag{52}$$

and so

$$\|x_n\| \leq \|\overline{x}\| + \sqrt{2\alpha}; \tag{53}$$

that is, $\{x_n\}$ is bounded and so by the q'-uniform convexity of X^* (by Lemma 6) we have for some $\eta > 0$ depending on α and on the space X^* the inequality

$$\begin{aligned}\|J(x_{n+1}) - J(x_n)\|^{q'} &\le \eta V_*(J^*(J(x_{n+1})), J(x_n)) \\ &= \eta V(J(x_n), x_{n+1}),\end{aligned} \tag{54}$$

where $J^* : X^* \to X^{**}(=X)$ is the normalised duality mapping on X^* and $V_* : X^{**} \times X^* \to \mathbf{R}$ is the functional defined by

$$V_*(x^{**}; x^*) := \|x^{**}\|^2 - 2\langle x^{**}; x^*\rangle + \|x^*\|^2, \quad \forall x^* \in X^*,\ x^{**} \in X^{**}. \tag{55}$$

Using now (46) and (47) and the assumption $\rho_0 \ge 2\rho$ we obtain

$$\frac{1}{2} V(J(x_n), x_{n+1}) \le \phi_n - \phi_{n+1}. \tag{56}$$

Therefore, it follows from the 7th assumption of Theorem 14 that

$$\begin{aligned}\|x^*_{n+1}\|^{q'} &= \lambda_n^{-q'} \|J(x_{n+1}) - J(x_n)\|^{q'} \\ &\le \lambda^{-q'} \|J(x_{n+1}) - J(x_n)\|^{q'} \\ &\le \lambda^{-q'} \eta V(J(x_n), x_{n+1}) \\ &\le \frac{2\eta}{\lambda^{q'}} [\phi_n - \phi_{n+1}] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{57}$$

which ensures that $\lim_{n\to\infty} x^*_{n+1} = 0$. On the other hand, since $\{x_n\}$ is bounded in C and C is ball compact then there exists a subsequence $\{x_{n_k}\}$ which converges to some limit $\tilde{x} \in C$. By Algorithm 13 this subsequence satisfies

$$\left\langle x^*_{n_k+1}, x - x_{n_k+1}\right\rangle \le F\left(x_{n_k}, x\right) + \rho V\left(J\left(x_{n_k}\right), x\right), \quad \forall k, \forall x \in C. \tag{58}$$

Thus, by letting $k \to \infty$ in the inequality (58) and by taking into account the upper semicontinuity of F and the continuity of V and J, we obtain

$$0 \le F(\tilde{x}, x) + \rho V(J(\tilde{x}), x), \quad \forall x \in C. \tag{59}$$

This means that $\tilde{x}$ is a solution of (NEP$[C,F]$). Finally, using now Proposition 12 we get $\tilde{x}$ is a solution of (NVP$[C,F]$) and so the proof is complete. □

4.2. Case 2: C Is a General Set-Valued Mapping. In this general case we propose the following algorithm.

Algorithm 15. Let $\rho \ge 0$ and $\lambda_n > 0$ for all $n \ge 1$;

(1) Select $x_0 \in C(x_0)$;

(2) For $n \ge 1$ select $x_{n+1} \in C(x_n)$ such that

$$\begin{aligned}&\lambda_n^{-1} \langle J(x_n) - J(x_{n+1}), x - x_{n+1}\rangle \\ &\quad \le F(x_n, x) + \rho V(J(x_n), x), \quad \forall x \in \operatorname{Im} C,\end{aligned} \tag{60}$$

where $M > 0$ is a given positive number and $\operatorname{Im} C$ is the image of C, that is, $\operatorname{Im} C := \{y \in X : \exists x \in X \text{ such that } y \in C(x)\}$.

Obviously Algorithm 15 coincides with Algorithm 13 when C is a constant set-valued mapping. However the assumptions assumed on F in the previous subsection are not sufficient to prove the convergence of the sequence $\{x_n\}$ generated by Algorithm 15 to a solution of (NQVP$[C,F]$). We need to replace the W-strong monotonicity by a relaxed W-strong monotonicity of the bifunction F over $\operatorname{Im} C$ and we do not assume the nonemptiness of the solution set of the proposed problem. We will say that F is relaxed W-strongly monotone over $\operatorname{Im} C$ provided that for some $\sigma \ge 0$ we have

$$F(x, y) \le -\sigma W(x, y), \quad \forall x, y \in \operatorname{Im} C. \tag{61}$$

By symmetry of W, it is clear that any W-relaxed strongly monotone bifunction with respect to $\sigma \ge 0$ is W-strongly monotone with respect to 2σ. This relaxed assumption on F has been used in Hilbert spaces in [4] and in Banach spaces in [13]. The following theorem is our main result in this subsection.

Theorem 16. *Let X be a q-uniformly convex Banach space. Let C be a closed nonempty subset of X and let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying $F(x,x) = 0$ for any $x \in \operatorname{Fix}(C)$. Let $\{x_n\}_n$ be a sequence generated by Algorithm 15. Assume that*

(1) *The values of C are V-uniformly prox-regular with some ratio $r \in (0, \infty]$;*

(2) *The image of C is ball compact in X and its graph is closed;*

(3) *F is relaxed W-strongly monotone over $\operatorname{Im} C$ with some $\sigma > 0$;*

(4) *F is upper semicontinuous with respect to the first variable over $\operatorname{Im} C$;*

(5) *$F(x_n, \cdot)$ is V-uniformly prox-regular over $\operatorname{Im} C$ with some $r' \in (0, +\infty]$;*

(6) *There exists $\lambda > 0$ such that $\lambda_n \ge \lambda$ for all n;*

(7) *The nonnegative parameter ρ is taken in the interval $[0, \sigma/2]$.*

Then, there exists subsequence of $\{x_n\}$ converging to a solution of (NQVP$[C,F]$).

Proof. Let $\overline{x} \in \operatorname{Im} C$. By the relaxed W-strong monotonicity of F over $\operatorname{Im} C$ we have

$$F(x_n, \overline{x}) \le -\sigma W(x_n, \overline{x}), \quad \forall n \ge 1. \tag{62}$$

By Algorithm 15 we have

$$\langle x^*_{n+1}, \overline{x} - x_{n+1}\rangle \le F(x_n, \overline{x}) + \rho V(J(x_n), \overline{x}), \tag{63}$$

with $x^*_{n+1} := \lambda_n^{-1}[J(x_n) - J(x_{n+1})]$. Combining these two inequalities we get

$$\begin{aligned}\langle x^*_{n+1}, \overline{x} - x_{n+1}\rangle &\le \rho V(J(x_n), \overline{x}) - \sigma W(x_n, \overline{x}) \\ &\le (2\rho - \sigma) W(x_n, \overline{x}).\end{aligned} \tag{64}$$

Therefore,

$$\langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1}\rangle \leq \lambda_n (2\rho - \sigma) W(x_n, \overline{x}). \quad (65)$$

Define now the same nonnegative real sequence $\phi_n = (1/2)V(J(x_n), \overline{x})$ used in the proof of Theorem 14. Then we have

$$\phi_{n+1} - \phi_n \leq \langle J(x_n) - J(x_{n+1}), \overline{x} - x_{n+1}\rangle, \quad (66)$$

which ensures with (65) that

$$\phi_{n+1} - \phi_n \leq \lambda_n (2\rho - \sigma) W(x_{n+1}, \overline{x}). \quad (67)$$

Using the assumption $\sigma \geq 2\rho$ yields

$$\phi_{n+1} \leq \phi_n. \quad (68)$$

Following the same reasoning in the proof of Theorem 14 and the ball compactness of the image of C, we get a subsequence $\{x_{n_k}\}$ which converges to some limit $\tilde{x}$ satisfying $\tilde{x} \in C(\tilde{x})$ by closedness of the graph of C. By Algorithm 15 this subsequence satisfies

$$\left\langle x^*_{n_k+1}, x - x_{n_k+1}\right\rangle \leq F\left(x_{n_k+1}, x\right) + \rho V\left(J\left(x_{n_k+1}\right), x\right), \quad \forall k, \forall x \in \operatorname{Im} C. \quad (69)$$

Thus, by letting $k \to \infty$ in the inequality (69) and by taking into account the upper semicontinuity of F and the continuity of V and J, we obtain

$$0 \leq F(\tilde{x}, x) + \rho V(J(\tilde{x}), x), \quad \forall x \in C(\tilde{x}). \quad (70)$$

This means that $\tilde{x}$ is a solution of (NQEP$[C,F]$) which ensures by Proposition 12 that under the assumptions of our theorem the solution $\tilde{x}$ is also a solution of (NQVP$[C,F]$). Thus completing the proof. □

4.3. Case 3: F Has the Form: $F(x,y) = \langle T(x), y - x\rangle$. In this subsection we restrict our attention to the following form of the bifunction F:

$$F(x,y) = \langle T(x), y - x\rangle, \quad (71)$$

where $T : X \to X^*$ is a nonlinear operator. In this case $\partial^{\pi} F(\overline{x}, \cdot)(\overline{x}) = \{T(\overline{x})\}$ and so (NQVP$[C,F]$) becomes:

$$\text{Find } \overline{x} \in C(\overline{x}), \quad \text{such that } T(\overline{x}) \in -N^{\pi}(C(\overline{x}), \overline{x}). \quad (\text{NQVP}[C,T])$$

We suggest the following algorithm to solve (NQVP$[C,T]$) under some natural and appropriate assumptions on C and T.

Algorithm 17. Let $\delta_n \downarrow 0$ with δ_0 be too small.

(i) Select $x_0 \in C(x_0)$, $y_0^* = T(x_0)$ and $\rho > 0$;

(ii) For $n \geq 0$,

  (a) Compute $z_{n+1} := J^*(J(x_n) - \rho y_n^*)$;

  (b) Compute $x_{n+1} := \pi_{C(x_n)}(J(z_{n+1}))$ and $y_{n+1}^* := T(x_{n+1})$,

where π_S is the generalised projection defined in terms of the functional V instead of the norm square (introduced and studied in the convex case in [16] and for the nonconvex case we refer to the recent paper [11]). A point $\overline{x} \in S$ is called the generalised projection of a given $x^* \in X^*$ provided that

$$V(x^*, \overline{x}) = \inf_{s \in S} V(x^*, s). \quad (72)$$

The following characterisation of the V-proximal normal cone in terms of the generalised projection is proved in [1].

Proposition 18. *For any closed nonempty set S in a reflexive Banach space X and for any point $\overline{x} \in S$ we have*

$$N^{\pi}(S; \overline{x}) = \left\{x^* \in X^* : \exists \lambda > 0 \text{ such that } \overline{x} \in \pi_S(J(\overline{x}) + \lambda x^*)\right\}. \quad (73)$$

We need the following lemma:

Lemma 19. *Let S be a closed set in X, $\overline{x} \in S$, $y^* \in X^*$, and $r > 0$. If $\overline{x} \in \pi_S(J(\overline{x}) - r y^*)$; then $\overline{x} \in \pi_S(J(\overline{x}) - \rho y^*)$, for any $\rho \in [0, r]$.*

Proof. Let $r > 0$, $y^* \in X^*$, and let $\overline{x}$ be a point satisfying $\overline{x} \in \pi_S(J(\overline{x}) - r y^*)$. Assume that $\rho \in [0, r]$. Let $\lambda := \rho / r \in [0,1]$. We claim that

$$V(J(\overline{x}) - \rho y^*, \overline{x}) = \inf_{s \in S} V(J(\overline{x}) - \rho y^*, s). \quad (74)$$

First, observe that for any $s \in S$ we have

$$\begin{aligned} 2\langle J(\overline{x}) - \rho y^* - J\overline{x}; s - \overline{x}\rangle &= 2\langle \lambda(J(\overline{x}) - r y^*) + (1 - \lambda) J(\overline{x}) - J\overline{x}; s - \overline{x}\rangle \\ &= 2\lambda \langle (J(\overline{x}) - r y^*) - J(\overline{x}); s - \overline{x}\rangle. \end{aligned} \quad (75)$$

If $\langle (J(\overline{x}) - r y^*) - J(\overline{x}); s - \overline{x}\rangle < 0$, then obviously we have

$$2\langle J(\overline{x}) - \rho y^* - J\overline{x}; s - \overline{x}\rangle < 0 \leq V(J(\overline{x}), s). \quad (76)$$

Otherwise, we have $\langle (J(\overline{x}) - r y^*) - J(\overline{x}); s - \overline{x}\rangle \geq 0$. Then since $0 \leq \lambda \leq 1$ we have

$$2\lambda \langle (J(\overline{x}) - r y^*) - J(\overline{x}); s - \overline{x}\rangle \leq 2\langle (J(\overline{x}) - r y^*) - J(\overline{x}); s - \overline{x}\rangle \quad (77)$$

and so we obtain

$$\begin{aligned}
&2\left\langle J(\bar{x})-\rho y^{*}-J\bar{x}; s-\bar{x}\right\rangle \\
&\quad\leq 2\left\langle\left(J(\bar{x})-ry^{*}\right)-J(\bar{x}); s-\bar{x}\right\rangle \\
&\quad\leq\left\|J(\bar{x})-ry^{*}\right\|^{2}-2\left\langle\left(J(\bar{x})-ry^{*}\right); \bar{x}\right\rangle+\|\bar{x}\|^{2} \\
&\qquad+2\left\langle\left(J(\bar{x})-ry^{*}\right); s\right\rangle-\left\|J(\bar{x})-ry^{*}\right\|^{2}-\|s\|^{2} \\
&\qquad+\|s\|^{2}-2\langle J(\bar{x}); s-\bar{x}\rangle-\|\bar{x}\|^{2} \\
&\quad\leq V\left(J(\bar{x})-ry^{*}, \bar{x}\right)-V\left(J(\bar{x})-ry^{*}, s\right) \\
&\qquad+V(J(\bar{x}), s) \\
&\quad\leq \inf_{z\in S} V\left(J(\bar{x})-ry^{*}, z\right)-V\left(J(\bar{x})-ry^{*}, s\right) \\
&\qquad+V(J(\bar{x}), s) \leq V(J(\bar{x}), s);
\end{aligned} \tag{78}$$

that is,

$$2\left\langle J(\bar{x})-\rho y^{*}-J\bar{x}; s-\bar{x}\right\rangle \leq V(J(\bar{x}), s). \tag{79}$$

Therefore, from (76) and (79) we have in both cases

$$2\left\langle J(\bar{x})-\rho y^{*}-J\bar{x}; s-\bar{x}\right\rangle \leq V(J(\bar{x}), s), \quad \forall s \in S. \tag{80}$$

Hence

$$2\left\langle J(\bar{x})-\rho y^{*}-J\bar{x}; s-\bar{x}\right\rangle-V(J(\bar{x}), s) \leq 0, \quad \forall s \in S. \tag{81}$$

On the other hand we have the decomposition

$$\begin{aligned}
&2\left\langle J(\bar{x})-\rho y^{*}-J\bar{x}; s-\bar{x}\right\rangle-V(J(\bar{x}), s) \\
&\quad=2\left\langle J(\bar{x})-\rho y^{*}; s\right\rangle-2\left\langle J(\bar{x})-\rho y^{*}; \bar{x}\right\rangle+2\|\bar{x}\|^{2} \\
&\qquad-2\langle J\bar{x}; s\rangle-\left[\|\bar{x}\|^{2}-2\langle J\bar{x}; s\rangle+\|s\|^{2}\right] \\
&\quad=\left[\left\|J(\bar{x})-\rho y^{*}\right\|^{2}-2\left\langle J(\bar{x})-\rho y^{*}; \bar{x}\right\rangle+\|\bar{x}\|^{2}\right] \\
&\qquad-\left[\left\|J(\bar{x})-\rho y^{*}\right\|^{2}-2\left\langle J(\bar{x})-\rho y^{*}; s\right\rangle+\|s\|^{2}\right] \\
&\quad=V\left(J(\bar{x})-\rho y^{*}, \bar{x}\right)-V\left(J(\bar{x})-\rho y^{*}, s\right).
\end{aligned} \tag{82}$$

Consequently, we have

$$V\left(J(\bar{x})-\rho y^{*}, \bar{x}\right)-V\left(J(\bar{x})-\rho y^{*}, s\right) \leq 0, \quad \text{for any } s \in S, \tag{83}$$

that is,

$$V\left(J(\bar{x})-\rho y^{*}, \bar{x}\right)=\inf_{s\in S} V\left(J(\bar{x})-\rho y^{*}, s\right); \tag{84}$$

which means that $\bar{x} \in \pi_S(J(\bar{x})-\rho y^*)$ and hence the proof is complete. □

Now, we state and prove our main theorem for (NQVP$[C,T]$).

Theorem 20. *Let X be a 2-uniformly smooth Banach space. Let $C : X \rightrightarrows X$ be a set-valued mapping with closed nonempty values and $T : X \to X^*$. Let $\{x_n\}_n$ be a sequence generated by Algorithm 17. Assume that*

(1) *The solution set of (NQVP$[C,T]$) is nonempty;*
(2) *T is bounded by some constant $L > 0$;*
(3) *T is J-Lipschitz, with constant $\beta > 0$; that is,*

$$\left\|T(x_1)-T(x_2)\right\| \leq \beta\left\|J(x_1)-J(x_2)\right\|, \quad \forall x_i \in X,\ i=1,2; \tag{85}$$

(4) *T is J-strongly monotone with constant $\alpha > 0$; that is,*

$$\begin{aligned}
&\left\langle J^{*}\left(T(x_1)-T(x_2)\right); J(x_1)-J(x_2)\right\rangle \\
&\quad\geq \alpha\left\|J(x_1)-J(x_2)\right\|^{2}, \quad \forall x_1, x_2 \in X;
\end{aligned} \tag{86}$$

(5) *The values of C satisfy for some $r \in (0,\infty]$:*

$$\bar{u} \in \pi_{C(\bar{u})}\left(J(\bar{u})+ru^{*}\right), \quad \forall u^{*} \in X^{*} \tag{87}$$

for any unit vector u^* in X^* and any $\bar{u}$ solution of (NQVP$[C,T]$);

(6) *There exists some constant $k \in (0,1)$ and $\xi > 0$ such that*

$$\begin{aligned}
&\left\|J\left(\pi_{C(x_1)}(x_1^{*})\right)-J\left(\pi_{C(x_2)}(x_2^{*})\right)\right\| \\
&\quad\leq \xi\left\|x_1^{*}-x_2^{*}\right\|+k\left\|J(x_1)-J(x_2)\right\|,
\end{aligned} \tag{88}$$

for all $x_i \in X$, $x_i^* \in X^*$, $i = 1,2$;

(7) *The positive constants α and β satisfy the inequality $\alpha > \beta\sqrt{1-(1-k)^2/c\xi^2}$;*
(8) *The parameter ρ in Algorithm 17 satisfies*

$$\begin{aligned}
&\frac{\alpha}{\beta^{2}}-\bar{\epsilon}<\rho<\min\left\{\frac{\mu-\delta_0}{L}, \frac{\alpha}{\beta^{2}}+\bar{\epsilon}\right\}, \\
&\bar{\epsilon} \coloneqq \frac{\sqrt{\alpha^{2}-\beta^{2}\left(1-(1-k)^{2}/c\xi^{2}\right)}}{\beta^{2}}.
\end{aligned} \tag{89}$$

Then, the sequence $\{x_n\}_n$ generated by Algorithm 17 converges to a solution of (NQVP$[C,T]$).

Proof. Let $\bar{x} \in C(\bar{x})$ be a solution of (NQVP$[C,T]$), that is, $-T(\bar{x}) \in N^{\pi}(C(\bar{x});\bar{x})$. Then by the characterisation of the V-proximal normal cone in Proposition 18, there exists $\lambda > 0$ such that $\bar{x} \in \pi_{C(\bar{x})}(J(\bar{x})-\lambda T(\bar{x}))$. Using Lemma 19 we obtain $\bar{x} \in \pi_{C(\bar{x})}(J(\bar{x})-\tau T(\bar{x}))$, for any $\tau \in [0,\lambda]$. By assumption (5) we may assume that $\lambda \leq r/L$ and so we get $\rho \leq r/L$. Hence $\bar{x} \in \pi_{C(\bar{x})}(J(\bar{z}))$ for $\bar{z} \coloneqq J^*(J(\bar{x})-\rho T(\bar{x}))$. Since X is 2-uniformly smooth we have X^* is 2-uniformly convex; that is,

$$\delta_{X^{*}}(\epsilon) \geq 2c^{-1}\epsilon^{2}, \tag{90}$$

for some constant $c > 0$ (depending only on the space X^*) and so by Lemma 7 we get

$$\begin{aligned} V_*\left(J^*x^*, y^*\right) &\geq 8C^2\delta_{X^*}\left(\frac{\|x^* - y^*\|}{4C}\right) \\ &\geq c^{-1}\|x^* - y^*\|^2, \quad \forall x^*, y^* \in X^*. \end{aligned} \tag{91}$$

Thus we can write

$$\begin{aligned} &\left\|\rho\left[T\left(x_n\right) - T\left(\overline{x}\right)\right] - \left(J\left(x_n\right) - J\left(\overline{x}\right)\right)\right\|^2 \\ &\quad \leq c\left[V_*\left(\rho J^*\left(T\left(x_n\right) - T\left(\overline{x}\right)\right); J\left(x_n\right) - J\left(\overline{x}\right)\right)\right]. \end{aligned} \tag{92}$$

Therefore,

$$\begin{aligned} &\left\|J\left(z_{n+1}\right) - J\left(\overline{z}\right)\right\|^2 \\ &\quad = \left\|J\left(x_n\right) - \rho T\left(x_n\right) - J\left(\overline{x}\right) + \rho T\left(\overline{x}\right)\right\|^2 \\ &\quad \leq c\left[V_*\left(\rho J^*\left(T\left(x_n\right) - T\left(\overline{x}\right)\right); J\left(x_n\right) - J\left(\overline{x}\right)\right)\right] \\ &\quad \leq c\left[\rho^2\left\|T\left(x_n\right) - T\left(\overline{x}\right)\right\|^2 + \left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2\right] \\ &\qquad - 2c\rho\left\langle J^*\left(T\left(x_n\right) - T\left(\overline{x}\right)\right); J\left(x_n\right) - J\left(\overline{x}\right)\right\rangle. \end{aligned} \tag{93}$$

Using the J-Lipschitz continuity of T with ratio β we have

$$\left\|T\left(x_n\right) - T\left(\overline{x}\right)\right\|^2 \leq \beta^2\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2 \tag{94}$$

and by the J-strong monotonicity of T with ratio α we have

$$\begin{aligned} &\left\langle J^*\left(T\left(x_n\right) - T\left(\overline{x}\right)\right); J\left(x_n\right) - J\left(\overline{x}\right)\right\rangle \\ &\quad \geq \alpha\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2. \end{aligned} \tag{95}$$

Thus, we get

$$\begin{aligned} &\left\|J\left(z_{n+1}\right) - J\left(\overline{z}\right)\right\|^2 \\ &\quad \leq c\left[\rho^2\beta^2\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2 + \left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2\right] \\ &\qquad - 2c\rho\alpha\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2 \\ &\quad \leq c\left(1 + \rho^2\beta^2 - 2\rho\alpha\right)\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|^2 \end{aligned} \tag{96}$$

and so

$$\begin{aligned} &\left\|J\left(z_{n+1}\right) - J\left(\overline{z}\right)\right\| \\ &\quad \leq \sqrt{c\left(1 + \rho^2\beta^2 - 2\rho\alpha\right)}\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|. \end{aligned} \tag{97}$$

On the other hand we have by the 6th assumption

$$\begin{aligned} &\left\|J\left(x_{n+1}\right) - J\left(\overline{x}\right)\right\| \\ &\quad = \left\|J\left(\pi_{C(x_n)}\left(J\left(z_{n+1}\right)\right)\right) - J\left(\pi_{C(\overline{x})}\left(J\left(\overline{z}\right)\right)\right)\right\| \\ &\quad \leq \xi\left\|J\left(z_{n+1}\right) - J\left(\overline{z}\right)\right\| + k\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|. \end{aligned} \tag{98}$$

Thus

$$\begin{aligned} &\left\|J\left(x_{n+1}\right) - J\left(\overline{x}\right)\right\| \\ &\quad \leq \left(k + \xi\sqrt{c\left(1 + \rho^2\beta^2 - 2\rho\alpha\right)}\right)\left\|J\left(x_n\right) - J\left(\overline{x}\right)\right\|. \end{aligned} \tag{99}$$

Our assumptions and the choice of ρ ensure that $(k + \xi\sqrt{c(1 + \rho^2\beta^2 - 2\rho\alpha)}) < 1$ and hence $\|J(x_n) - J(\overline{x})\| \to 0$ which means that $x_n \to \overline{x}$ by the uniform continuity of J^* and thus completing the proof. □

Remark 21. A simple inspection of the proof of the previous theorem shows that the result is valid in the case when T is taken a general set-valued mapping instead of a single-valued operator defined from X to X^* and of course the assumptions on T should be adapted naturally for the set-valued case.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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