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Research Article

Boundedness of the Segal-Bargmann Transform on Fractional Hermite-Sobolev Spaces

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Let $s \in \mathbb{R}$ and $2 \leq p \leq \infty$. We prove that the Segal-Bargmann transform $\mathscr{B}$ is a bounded operator from fractional Hermite-Sobolev spaces $W_H^{s,p}(\mathbb{R}^n)$ to fractional Fock-Sobolev spaces $F_{\mathscr{R}}^{s,p}$.

1. Introduction

In quantum mechanics, the Schrödinger equation is a partial differential equation that describes how the quantum state of some physical system changes with time. The most famous example is the nonrelativistic Schrödinger equation for a single particle moving in a potential:

$$\sqrt{-1}\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[\frac{-\hbar^2}{2m}\Delta + V(x,t)\right]\Psi(x,t), \quad (1)$$

where m is the particle's mass, $\hbar$ is the Planck constant, V is its potential energy, and Ψ is the wave function.

Let H be the most basic Schrödinger operator in $\mathbb{R}^n$, $n \geq 1$, the Hermite operator (or the harmonic oscillator):

$$H = -\Delta + |x|^2. \quad (2)$$

Then the Schrödinger equation can be written by

$$\sqrt{-1}\frac{\partial\Psi}{\partial t} = H\Psi. \quad (3)$$

This is an important model in quantum mechanics (see, e.g., [1]).

For $s \in \mathbb{R}$, we define the fractional Hermite operator $H^s = (-\Delta + |x|)^s$ of order s. Let $0 < p \leq \infty$. The Hermite-Sobolev space $W_H^{s,p}(\mathbb{R}^n)$ of fractional order s is the space of all tempered distributions for which the distribution $H^{s/2}f$ is given by an L^p function on $\mathbb{R}^n$.

Let $\mathbb{C}^n$ be the complex n-space and let dV be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j, \qquad |z| = (z \cdot \overline{z})^{1/2}. \quad (4)$$

For any $0 < p \leq \infty$ the Fock space F^p denotes the space of entire functions f on $\mathbb{C}^n$ such that the function $f(z)e^{-(1/4)|z|^2}$ is in $L^p(\mathbb{C}^n, dV)$. We define

$$\|f\|_{F^p} = \left[\left(\frac{p}{4\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-(1/4)|z|^2}\right|^p dV(z)\right]^{1/p}. \quad (5)$$

For $p = \infty$ the norm in F^∞ is defined by

$$\|f\|_{F^\infty} = \sup\left\{\left|f(z)\right| e^{-(1/4)|z|^2} : z \in \mathbb{C}^n\right\}. \quad (6)$$

Let

$$A_j f(z) = 2\frac{\partial}{\partial z_j} f(z), \qquad A_j^* f(z) = z_j f(z), \qquad 1 \leq j \leq n, \; f \in F^p. \quad (7)$$

Both A_j and A_j^*, as defined above, are densely defined linear operators on F^p (unbounded though). We consider the radial derivative $\mathscr{R}$ defined by

$$\mathscr{R} := \frac{1}{2}\sum_{j=1}^{n}\left(A_jA_j^* + A_j^*A_j\right). \tag{8}$$

Let s be a real number and $0 < p \leq \infty$. The fractional Fock-Sobolev space $F_{\mathscr{R}}^{s,p}$ of order s is the space of all entire functions for which $\mathscr{R}^{s/2}f$ is given by an F^p function.

The Segal-Bargmann transform $\mathscr{B}$ is defined by

$$\mathscr{B}f(z) = \frac{1}{\pi^{n/4}}\int_{\mathbb{R}^n} f(x)\, e^{x\cdot z-(1/2)|x|^2-(1/4)z\cdot z}dV(x), \tag{9}$$

where $dV(x)$ is the volume measure on $\mathbb{R}^n$. It is well-known that the Segal-Bargmann transform is a unitary isomorphism between $L^2(\mathbb{R}^n)$ and F^2 [2, 3].

We prove that the radial derivative $\mathscr{R}$ has a parallel behavior to the Hermite operator H. In particular, $\mathscr{R}$ is densely defined, positive, self-adjoint and has the discrete spectrum; it generates a diffusion semigroup. Moreover, we show that the Segal-Bargmann transform intertwines fractional Hermite-Sobolev spaces with fractional Fock-Sobolev spaces as follows.

Theorem 1. *Let $s \in \mathbb{R}$ and $2 \leq p \leq \infty$. Then the Segal-Bargmann transform $\mathscr{B} : W_H^{s,p}(\mathbb{R}^n) \to F_{\mathscr{R}}^{s,p}$ is bounded.*

2. Fractional Hermite-Sobolev Spaces

In one dimension, the Hermite polynomials H_k are defined by

$$H_k(x) = e^{x^2}\frac{d^k}{dx^k}\left(e^{-x^2}\right), \quad x \in \mathbb{R}, \tag{10}$$

and by normalization we obtain the Hermite functions

$$h_k(x) = \left(\sqrt{\pi}2^k k!\right)^{-1/2} e^{-x^2/2}(-1)^k H_k(x), \quad x \in \mathbb{R}. \tag{11}$$

Note that

$$\begin{aligned}\left(-\frac{d^2}{dx^2} + x^2\right)&\left[e^{-(1/2)x^2}H_k(x)\right] \\ &= (2k+1)\left[e^{-(1/2)x^2}H_k(x)\right], \quad x \in \mathbb{R}.\end{aligned} \tag{12}$$

In higher dimensions, for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, the Hermite functions h_α are defined by

$$h_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{13}$$

Here, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integer. By (12), we know that these are the eigenfunctions of the Hermite operator defined in (2). In fact,

$$Hh_\alpha = (2|\alpha| + n)h_\alpha. \tag{14}$$

Moreover, $\{h_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Let $\mathscr{H}$ be the space of finite linear combinations of Hermite functions

$$f = \sum_{|\alpha|\leq N}\langle f, h_\alpha\rangle h_\alpha, \tag{15}$$

where

$$\langle f, h_\alpha\rangle = \int_{\mathbb{R}^n} f(x)\, h_\alpha(x)\, dV(x). \tag{16}$$

The space $\mathscr{H}$ is dense in $L^2(\mathbb{R}^n)$, and so, by the orthonormality of the Hermite functions,

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\sum_{\alpha\in\mathbb{N}_0^n}\left|\langle f, h_\alpha\rangle\right|^2\right)^{1/2}. \tag{17}$$

For $s \in \mathbb{R}$, we define the fractional Hermite operator $H^s = (-\Delta + |x|)^s$ of order s. For $f \in \mathscr{S}(\mathbb{R}^n)$, the Hermite series expansion

$$\sum_{\alpha\in\mathbb{N}_0^n}\langle f, h_\alpha\rangle h_\alpha \tag{18}$$

converges to f uniformly in $\mathbb{R}^n$ (and also in $L^2(\mathbb{R}^n)$), since $\|h_\alpha\|_{L^\infty(\mathbb{R}^n)} \leq C$, for all $\alpha \in \mathbb{N}_0^n$, and each $m \in \mathbb{N}$, and we have (see [4])

$$\left|\langle f, h_\alpha\rangle\right| \leq \left\|H^m f\right\|_{L^2(\mathbb{R}^n)}(2|\alpha| + n)^{-m}. \tag{19}$$

Definition 2. Let $s \in \mathbb{R}$ and $f \in \mathscr{S}(\mathbb{R}^n)$. One defines the fractional Hermite operator H^s by

$$H^s f = \sum_{\alpha\in\mathbb{N}_0^n}(2|\alpha| + n)^s\langle f, h_\alpha\rangle h_\alpha. \tag{20}$$

The fractional Hermite operators H^s were introduced in [5].

Definition 3. Let $s \in \mathbb{R}$ and $0 < p \leq \infty$. The fractional Hermite-Sobolev space $W_H^{s,p}(\mathbb{R}^n)$ of order s is the space of all tempered distributions for which the distribution $H^{s/2}f$ is given by an L^p function on $\mathbb{R}^n$. The fractional Hermite-Sobolev norm of order s is defined accordingly,

$$\|f\|_{W_H^{s,p}(\mathbb{R}^n)} = \left\|H^{s/2}f\right\|_{L^p(\mathbb{R}^n)}. \tag{21}$$

The fractional Hermite-Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ of order s were introduced in [6].

3. Radial Derivative

We consider the radial derivative $\mathscr{R}$ defined on

$$\mathscr{D}om(\mathscr{R}) = \left\{f \in F^2 : \mathscr{R}f \in F^2\right\} \tag{22}$$

by

$$\mathscr{R} := \frac{1}{2}\sum_{j=1}^{n}\left(A_jA_j^* + A_j^*A_j\right), \tag{23}$$

where

$$A_j f(z) = 2\frac{\partial}{\partial z_j} f(z),$$
$$A_j^* f(z) = z_j f(z), \qquad (24)$$
$$1 \le j \le n, \quad f \in F^2.$$

We have

$$\mathscr{R} = 2\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n. \qquad (25)$$

The following example tells us that $\mathscr{Dom}(\mathscr{R}) \subsetneq F^2$. Thus $\mathscr{R}$ is an unbounded operator on F^2.

Example 4. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{z_1^k}{\sqrt{2}^k (k+1)\sqrt{k!}}. \qquad (26)$$

Then $f \in F^2$, but $\mathscr{R} f \notin F^2$.

Proof. Note that

$$\begin{aligned} \|f\|_{F^2}^2 &= \frac{1}{(2\pi)^n} \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} \left| \frac{z_1^k}{\sqrt{2}^k (k+1)\sqrt{k!}} \right|^2 e^{-(1/2)|z|^2} dV(z) \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) < \infty, \end{aligned} \qquad (27)$$

where $\zeta(\cdot)$ is the Riemann zeta function. However, we have

$$\begin{aligned} \|\mathscr{R} f\|_{F^2}^2 &= \left\| \sum_{k=0}^{\infty} \frac{(2k+n) z_1^k}{\sqrt{2}^k (k+1)\sqrt{k!}} \right\|_{F^2}^2 = \sum_{k=0}^{\infty} \frac{(2k+n)^2}{(k+1)^2} \\ &= \infty. \end{aligned} \qquad (28)$$

□

Lemma 5. *$\mathscr{R}$ is a positive, self-adjoint operator on $\mathscr{Dom}(\mathscr{R})$.*

Proof. Let $\mathscr{P}(\mathbb{C}^n)$ be the set of all holomorphic polynomials on $\mathbb{C}^n$. We know that $\mathscr{P}(\mathbb{C}^n)$ is dense in F^2 and $\mathscr{R}$ is self-adjoint on $\mathscr{P}(\mathbb{C}^n)$. Hence $\mathscr{Dom}(\mathscr{R})$ is the domain of its unique self-adjoint extension.

Note that

$$\langle f, \mathscr{R} f \rangle_{F^2} = 2\sum_{j=1}^{n} \left\| \frac{\partial f}{\partial z_j} \right\|_{F^2}^2 + n\|f\|_{F^2}^2 \ge n\|f\|_{F^2}^2, \quad \forall f \in \mathscr{Dom}(\mathscr{R}). \qquad (29)$$

Thus $\mathscr{R}$ is positive. □

Lemma 6. *$\mathscr{R}$ has the discrete spectrum $\sigma(\mathscr{R}) = \{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$.*

Proof. By (29), we have $\sigma(\mathscr{R}) \subseteq [n, \infty)$.

We define

$$e_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|_{F^2}} = \frac{z^\alpha}{\sqrt{2^{|\alpha|}\alpha!}}. \qquad (30)$$

Then $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2. It is easy to see that $\{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$ is the set of all eigenvalues.

Let $\lambda \in [n, \infty) \setminus \{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$. First, we show that $\lambda I - \mathscr{R} : \mathscr{Dom}(\mathscr{R}) \to F^2$ is injective and surjective.

Suppose that $(\lambda I - \mathscr{R}) f = (\lambda I - \mathscr{R})\tilde{f}$. Then

$$\begin{aligned} 0 &= (\lambda I - \mathscr{R}) f - (\lambda I - \mathscr{R})\tilde{f} \\ &= \sum_{\alpha \in \mathbb{N}_0^n} \{\lambda - (2|\alpha| + n)\} \langle f - \tilde{f}, e_\alpha \rangle e_\alpha. \end{aligned} \qquad (31)$$

This implies $f = \tilde{f}$. Thus $\lambda I - \mathscr{R} : \mathscr{Dom}(\mathscr{R}) \to F^2$ is injective.

For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z) \qquad (32)$$

be the orthonormal decomposition of f. We define

$$g = \frac{1}{\lambda} f + \frac{1}{\lambda} \sum_{\alpha \in \mathbb{N}_0^n} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} c_\alpha e_\alpha(z). \qquad (33)$$

Since

$$\varphi_N = \sum_{|\alpha|=0}^{N} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} c_\alpha e_\alpha(z) \qquad (34)$$

is a Cauchy sequence in F^2, the series in (33) converges in F^2. Hence

$$g = \frac{1}{\lambda} f + \frac{1}{\lambda} \sum_{|\alpha|=0}^{\infty} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} c_\alpha e_\alpha(z) \qquad (35)$$

is a well-defined element of F^2 and it satisfies $(\lambda I - \mathscr{R}) g = f$. This means that $\lambda I - \mathscr{R} : \mathscr{Dom}(\mathscr{R}) \to F^2$ is surjective.

Moreover,

$$\begin{aligned} \left\| (\lambda I - \mathscr{R})^{-1} f \right\|_{F^2} &\le \frac{1}{\lambda} \|f\|_{F^2} + \frac{1}{\lambda} \beta \|f\|_{F^2} \\ &= \frac{1}{\lambda} (1 + \beta) \|f\|_{F^2}, \end{aligned} \qquad (36)$$

where $\beta = \sup_{\alpha \in \mathbb{N}_0^n} |(2|\alpha|+n)/(\lambda-(2|\alpha|+n))|$. Hence $(\lambda I - \mathscr{R})^{-1}$ is bounded and so $\sigma(\mathscr{R}) = \{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$. □

For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z) \qquad (37)$$

be the orthonormal decomposition of f. Associated with the operator $\mathscr{R}$ is a semigroup $\{B_t\}_{t \ge 0}$ defined by the expansion

$$B_t f(z) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)t} c_\alpha e_\alpha(z). \qquad (38)$$

We can check that $u(z,t) := B_t f(z)$ is the solution of the heat-type equation:

$$\begin{aligned}(\partial_t + \mathscr{R})\, u &= 0 \quad \text{on } \mathbb{C}^n \times (0,\infty),\\ u(\cdot,0) &= f \quad \text{on } \mathbb{C}^n.\end{aligned} \tag{39}$$

It is easy to see that

$$\|B_t f\|_{F^2}^2 \le e^{-2nt} \|f\|_{F^2}^2. \tag{40}$$

Thus B_t is contractive.

Proposition 7. *$\{B_t\}_{t\ge 0}$ is a strongly continuous semigroup.*

Proof. We note that

$$\begin{aligned}\|B_t f - f\|_{F^2}^2 &= \sum_{\alpha\in\mathbb{N}_0^n} \left|e^{-(2|\alpha|+n)t} - 1\right|^2 |c_\alpha|^2\\ &= \sum_{k=0}^{\infty} \left|e^{-(2k+n)t} - 1\right|^2 \sum_{|\alpha|=k} |c_\alpha|^2.\end{aligned} \tag{41}$$

For $k \in \mathbb{N}_0$ and $X \subset \mathbb{N}_0$ we define $\delta_k(X)$ by

$$\delta_k(X) = \begin{cases} 1, & \text{if } k \in X,\\ 0, & \text{if } k \notin X.\end{cases} \tag{42}$$

Then

$$\begin{aligned}\lim_{t\to 0^+} \|B_t f - f\|_{F^2}^2 &= \lim_{t\to 0^+} \sum_{k=0}^{\infty} \left|e^{-(2k+n)t} - 1\right|^2 \sum_{|\alpha|=k} |c_\alpha|^2\\ &= \lim_{t\to 0^+} \int_0^{\infty} \left|e^{-(2\lambda+n)t} - 1\right|^2 d\nu(\lambda),\end{aligned} \tag{43}$$

where ν is a discrete measure defined by

$$\nu = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} |c_\alpha|^2\right) \delta_k. \tag{44}$$

By Lebesgue dominate convergence theorem, we have

$$\begin{aligned}\lim_{t\to 0^+} \|B_t f - f\|_{F^2}^2 &= \int_0^{\infty} \lim_{t\to 0^+} \left|e^{-(2\lambda+n)t} - 1\right|^2 d\nu(\lambda)\\ &= 0.\end{aligned} \tag{45}$$

Hence $\{B_t\}_{t\ge 0}$ is a strongly continuous semigroup. □

Proposition 8. *$-\mathscr{R}$ is the infinitesimal generator of $\{B_t\}_{t\ge 0}$. That is,*

$$\lim_{t\to 0^+} \frac{B_t f - f}{t} = -\mathscr{R} f. \tag{46}$$

Proof. By using the previous discrete measure ν, it follows that

$$\begin{aligned}&\left\|\frac{B_t f - f}{t} - (-\mathscr{R} f)\right\|_{F^2}^2\\ &\quad = \int_0^{\infty} \left|\frac{e^{-(2\lambda+n)t} - 1}{t} + (2\lambda + n)\right|^2 d\nu(\lambda).\end{aligned} \tag{47}$$

Taking limit on both sides and by Lebesgue dominate convergence theorem,

$$\begin{aligned}&\lim_{t\to 0^+} \left\|\frac{B_t f - f}{t} - (-\mathscr{R} f)\right\|_{F^2}^2\\ &\quad = \lim_{t\to 0^+} \int_0^{\infty} \left|\frac{e^{-(2\lambda+n)t} - 1}{t} + (2\lambda + n)\right|^2 d\nu(\lambda)\\ &\quad = \int_0^{\infty} \lim_{t\to 0^+} \left|\frac{e^{-(2\lambda+n)t} - 1}{t} + (2\lambda + n)\right|^2 d\nu(\lambda) = 0.\end{aligned} \tag{48}$$

Thus we get the result. □

By Proposition 8, we have

$$B_t = e^{-t\mathscr{R}}. \tag{49}$$

4. Fractional Fock-Sobolev Spaces

Since $\mathscr{R}$ has discrete spectrum $\{2|\alpha|+n : \alpha \in \mathbb{N}_0^n\}$, by using the spectral theorem, we define the fractional radial derivative $\mathscr{R}^s$ for $s \in \mathbb{R}$ as follows.

Definition 9. Let $s \in \mathbb{R}$. For $f \in F^2$ let

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha e_\alpha(z) \tag{50}$$

be the orthonormal decomposition of f. By the spectral theorem, $\mathscr{R}^s$ is given by

$$\mathscr{R}^s f(z) = \sum_{\alpha\in\mathbb{N}_0^n} (2|\alpha| + n)^s c_\alpha e_\alpha(z), \quad f \in \mathscr{Dom}(\mathscr{R}^s). \tag{51}$$

Definition 10. Let s be a real number and $0 < p \le \infty$. The fractional Fock-Sobolev space $F_{\mathscr{R}}^{s,p}$ of order s is the space of all entire functions for which $\mathscr{R}^{s/2} f$ is given by an F^p function. The fractional Fock-Sobolev norm of f of order s is defined accordingly,

$$\|f\|_{F_{\mathscr{R}}^{s,p}} = \left\|\mathscr{R}^{s/2} f\right\|_{F^p}. \tag{52}$$

We refer the reader to [7–10] for other Fock-Sobolev spaces.

5. L^p-Boundedness of the Segal-Bargmann Transform

The Hermite operator H is self-adjoint on the set of infinitely differentiable functions with compact support $C_c^{\infty}(\mathbb{R}^n)$, and it can be factorized as

$$H = \frac{1}{2}\sum_{j=1}^{n} \left(a_j a_j^{\dagger} + a_j^{\dagger} a_j\right), \tag{53}$$

where

$$a_j = \frac{\partial}{\partial x_j} + x_j, \quad a_j^\dagger = -\frac{\partial}{\partial x_j} + x_j, \quad 1 \le j \le n. \tag{54}$$

Lemma 11. *For each $j = 1, \ldots, n$, one has*

$$\mathscr{B}\left(a_j f\right) = A_j \mathscr{B}(f), \quad \mathscr{B}\left(a_j^\dagger f\right) = A_j^* \mathscr{B}(f). \tag{55}$$

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$. By the integration by parts, we have

$$\begin{aligned}\mathscr{B}\left(\frac{\partial}{\partial x_j} f\right)(z) &= \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{x\cdot z-(1/2)|x|^2-(1/4)z\cdot z} dV(x) \\ &= -z_j \mathscr{B}(f) + \mathscr{B}\left(x_j f\right).\end{aligned} \tag{56}$$

This gives

$$\mathscr{B}\left(a_j^\dagger f\right) = A_j^* \mathscr{B}(f). \tag{57}$$

We differentiate

$$\mathscr{B} f(z) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} f(x) e^{x\cdot z-(1/2)|x|^2-(1/4)z\cdot z} dV(x) \tag{58}$$

under the integral sign to obtain

$$A_j \mathscr{B} f(z) = \frac{1}{\pi^{n/4}} \cdot \int_{\mathbb{R}^n} \left(2x_j - z_j\right) f(x) e^{x\cdot z-(1/2)|x|^2-(1/4)z\cdot z} dV(x). \tag{59}$$

This gives

$$A_j \mathscr{B}(f) = 2\mathscr{B}\left(x_j f\right) - A_j^* \mathscr{B}(f). \tag{60}$$

By (57) and (60), it follows that

$$A_j \mathscr{B}(f) = \mathscr{B}\left(a_j f\right). \tag{61}$$

□

Corollary 12. *Consider*

$$\mathscr{B} H = \mathscr{R} \mathscr{B}. \tag{62}$$

Proof. By Lemma 11, we have

$$\mathscr{B}(Hf) = \frac{1}{2} \sum_{j=1}^{n} \left(A_j A_j^* + A_j^* A_j\right) \mathscr{B}(f) = \mathscr{R} \mathscr{B}. \tag{63}$$

□

Proposition 13. *Let $s \in \mathbb{R}$. Then*

$$\mathscr{B} H^s = \mathscr{R}^s \mathscr{B}. \tag{64}$$

Proof. We define

$$e_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|_{F^2}}. \tag{65}$$

Then $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2 and $\mathscr{B}(h_\alpha) = e_\alpha$. For $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$H^s f = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle f, h_\alpha \rangle h_\alpha \tag{66}$$

and so

$$\mathscr{B}(H^s f) = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle f, h_\alpha \rangle e_\alpha. \tag{67}$$

Since $\mathscr{B}$ is a unitary isomorphism, we have $\langle f, h_\alpha \rangle = \langle \mathscr{B}(f), e_\alpha \rangle$. Hence

$$\begin{aligned}\mathscr{B}(H^s f) &= \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle \mathscr{B}(f), e_\alpha \rangle e_\alpha \\ &= \mathscr{R}^s \mathscr{B}(f).\end{aligned} \tag{68}$$

Thus we get the result. □

We consider the mapping property of the Segal-Bargmann transform $\mathscr{B}$ as a map from $L^p(\mathbb{R}^n)$ to F^p for $p \in [2, \infty]$. Note that one-dimensional case is in [11].

Theorem 14. *Consider*

$$\|\mathscr{B} f\|_{F^\infty} \le (4\pi)^{n/4} \|f\|_{L^\infty(\mathbb{R}^n)}. \tag{69}$$

Proof. We have

$$\begin{aligned}\left|\mathscr{B} f(z)\right| &\le \frac{1}{\pi^{n/4}} e^{|z|^2/4} \sup_{x \in \mathbb{R}^n} |f(x)| \\ &\quad \cdot \int_{\mathbb{R}^n} e^{\operatorname{Re}(z\cdot x)-(1/2)|x|^2-(1/4)\operatorname{Re}(z\cdot z)-|z|^2/4} dV(x).\end{aligned} \tag{70}$$

Note that

$$|\operatorname{Re}(z)|^2 = \frac{1}{2}\left\{|z|^2 + \operatorname{Re}(z \cdot z)\right\}. \tag{71}$$

Hence

$$\begin{aligned}&\operatorname{Re}(z \cdot x) - \frac{1}{2}|x|^2 - \frac{1}{4}\operatorname{Re}(z \cdot z) - \frac{|z|^2}{4} \\ &\quad = \operatorname{Re}(z \cdot x) - \frac{1}{2}|x|^2 - \frac{1}{2}|\operatorname{Re}(z)|^2 \\ &\quad = -\frac{1}{2}|\operatorname{Re}(z) - x|^2\end{aligned} \tag{72}$$

and so

$$\begin{aligned} &\left|\mathscr{B}f(z)\right| \\ &\leq \frac{1}{\pi^{n/4}} e^{|z|^2/4} \sup_{x\in\mathbb{R}^n} \left|f(x)\right| \int_{\mathbb{R}^n} e^{-(1/2)|\operatorname{Re}(z)-x|^2} dV(x) \\ &= (4\pi)^{n/4} e^{|z|^2/4} \sup_{x\in\mathbb{R}^n} \left|f(x)\right|. \end{aligned} \quad (73)$$

Thus we get the result. □

The following Stein-Weiss interpolation theorem is well-known. See, for example, [3, 12].

Lemma 15. *Let w, w_0, and w_1 be positive weight functions on a measure space $(X, d\lambda)$. If $1 \leq p_0 \leq p_1 \leq \infty$ and $0 \leq \theta \leq 1$, then*

$$\left[L^{p_0}\left(X, w_0 d\lambda\right), L^{p_1}\left(X, w_1 d\lambda\right)\right]_\theta = L^p\left(X, w d\lambda\right) \quad (74)$$

with equal norms, where

$$\begin{aligned} \frac{1}{p} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \\ w^{1/p} &= w_0^{(1-\theta)/p_0} w_1^{\theta/p_1}. \end{aligned} \quad (75)$$

Theorem 16. *Let $2 \leq p \leq \infty$. There exists $C > 0$ such that*

$$\left\|\mathscr{B}f\right\|_{F^p} \leq C\left\|f\right\|_{L^p(\mathbb{R}^n)}. \quad (76)$$

Proof. The L^2-boundedness is followed by the unitary isomorphism of the Segal-Bargmann transform. In Theorem 14, we proved the L^∞-boundedness of the Segal-Bargmann transform. By Lemma 15, we have the required result. □

By Proposition 13 and Theorem 16, we have the following result.

Theorem 17. *Let $s \in \mathbb{R}$ and $2 \leq p \leq \infty$. Then the Segal-Bargmann transform $\mathscr{B} : W_H^{s,p}(\mathbb{R}^n) \to F_{\mathscr{R}}^{s,p}$ is bounded.*

Disclosure

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Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, Maidenhead, UK, 1965.

[2] B. Hall, *Quamtum Theory for Mathematicians*, vol. 267, Springer, New York, NY, USA, 2013.

[3] K. Zhu, *Analysis on Fock Spaces*, vol. 263 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2012.

[4] P. R. Stinga and J. L. Torrea, "Regularity theory for the fractional harmonic oscillator," *Journal of Functional Analysis*, vol. 260, no. 10, pp. 3097–3131, 2011.

[5] P. R. Stinga and J. L. Torrea, "Extension problem and Harnack's inequality for some fractional operators," *Communications in Partial Differential Equations*, vol. 35, no. 11, pp. 2092–2122, 2010.

[6] B. Bongioanni and J. L. Torrea, "Sobolev spaces associated to the harmonic oscillator," *Indian Academy of Sciences. Proceedings. Mathematical Sciences*, vol. 116, no. 3, pp. 337–360, 2006.

[7] H. R. Cho, B. R. Choe, and H. Koo, "Fock-Sobolev spaces of fractional order," *Potential Analysis*, vol. 43, no. 2, pp. 199–240, 2015.

[8] H. R. Cho and K. Zhu, "Fock-Sobolev spaces and their Carleson measures," *Journal of Functional Analysis*, vol. 263, no. 8, pp. 2483–2506, 2012.

[9] B. C. Hall and W. Lewkeeratiyutkul, "Holomorphic Sobolev spaces and the generalized SEGal-Bargmann transform," *Journal of Functional Analysis*, vol. 217, no. 1, pp. 192–220, 2004.

[10] R. Radha and S. Thangavelu, "Holomorphic Sobolev spaces, HERmite and special HERmite semigroups and a Paley-Wiener theorem for the windowed Fourier transform," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 2, pp. 564–574, 2009.

[11] K. Zhu, "Towards a dictionary for the Bargmann transform," https://arxiv.org/abs/1506.06326.

[12] E. M. Stein and G. Weiss, "Interpolation of operators with change of measures," *Transactions of the American Mathematical Society*, vol. 87, pp. 159–172, 1958.