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Research Article

Common Fixed Points of Four Maps Satisfying F-Contraction on b-Metric Spaces

Muhammad Nazam,[1] Ma Zhenhua,[2,3] Sami Ullah Khan,[1,4] and Muhammad Arshad[1]

[1] *Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan*
[2] *Department of Mathematics and Physics, Hebei University of Architecture, Zhangjiakou 075024, China*
[3] *School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 10001, China*
[4] *Department of Mathematics, Gomal University D. I. Khan, Khyber Pakhtunkhwa 29050, Pakistan*

Correspondence should be addressed to Ma Zhenhua; mazhenghua_1981@163.com

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We manifest some common fixed point theorems for four maps satisfying Círíc type F-contraction and Hardy-Rogers type F-contraction defined on complete b-metric spaces. We apply these results to infer several new and old corresponding results. These results also generalize some results obtained previously. We dispense examples to validate our results.

1. Introduction and Preliminaries

After the famous Banach Contraction Principle, a large number of researchers revealed many fruitful generalizations of Banach's fixed point theorem. One of these generalizations is known as F-contraction presented by Wardowski [1]. Wardowski [1] evinced the fact that every F-contraction defined on complete b-metric space has a unique fixed point. The concept of F-contraction proved to be another milestone in fixed point theory and numerous research papers on F-contraction have been published (see, e.g., [2–8]). Recently, Cosentino and Vetro [9] established a fixed point result for Hardy-Rogers type F-contraction and Mınak et al. [10] presented a fixed point result for Círíc type generalized F-contraction.

In 1989, Bakhtin [11] investigated the concept of b-metric spaces; however, Czerwik [12] initiated study of fixed point of self-mappings in b-metric spaces and proved an analogue of Banach's fixed point theorem. Since then, numerous research articles have been published comprising fixed point theorems for various classes of single-valued and multivalued operators in b-metric spaces (see, e.g., [13–19]).

We shall bring into use the idea of Círíc type F-contraction and Hardy-Rogers type F-contraction comprising four self-mappings defined on b-metric space. We present common fixed point results for four self-maps satisfying Círíc type and Hardy-Rogers type F-contraction on b-metric space. We apply our results to infer several new and old results.

We denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$, and set of natural numbers by $\mathbb{N}$.

We bring back into reader's mind some definitions and properties of b-metric.

Definition 1 (see [12]). Let M be a nonempty set and $s \geq 1$ be a real number. A function $d_b : M \times M \to [0,\infty)$ is said to be a b-metric if, for all $\theta, \rho, \sigma \in M$, one has

(d_b1) $\theta = \rho$ if and only if $d_b(\theta,\rho) = 0$,

(d_b2) $d_b(\theta,\rho) = d_b(\rho,\theta)$,

(d_b3) $d_b(\theta,\rho) \leq s[d_b(\theta,\sigma) + d_b(\sigma,\rho)]$.

In this case, the pair (M, d_b, s) is called a b-metric space (with coefficient s).

Definition 1 allows us to remark that the class of b-metric spaces is effectually more general than that of metric spaces because a b-metric is a metric when $s = 1$. The following example describes the significance of a b-metric.

Example 2. Let (M, d) be a metric space and $d_b(\theta, \rho) = (d(\theta, \rho))^r$, $r > 1$ is a real number. Then d_b is b-metric space with $s = 2^{r-1}$. Apparently, (d_b1) and (d_b2) of Definition 1 are satisfied. If $1 < r < \infty$, then the convexity of the function $f(\theta) = \theta^r$ $(\theta > 0)$ implies

$$\left(\frac{j+l}{2}\right)^r \leq \frac{1}{2}\left(j^r + l^r\right) \tag{1}$$

that gives $(j+l)^r \leq 2^{r-1}(j^r + l^r)$. Thus, for all $\theta, \rho, \sigma \in M$, one has

$$\begin{aligned} d_b(\theta, \sigma) = (d(\theta, \sigma))^r &\leq (d(\theta, \rho) + d(\rho, \sigma))^r \\ &\leq 2^{r-1}\left[(d(\theta, \rho))^r + (d(\rho, \sigma))^r\right] \\ &= 2^{r-1}\left[d_b(\theta, \rho) + d_b(\rho, \sigma)\right]. \end{aligned} \tag{2}$$

Therefore, $d_b(\theta, \rho) \leq s[d_b(\theta, \rho) + d_b(\rho, \sigma)]$, where $s = 2^{r-1}$, which shows that (M, d_b, s) is a b-metric space. Nevertheless, if (M, d) is a metric space, then (M, d_b, s) may not be a metric space. Indeed, if $M = \mathbb{R}$ and $d(\theta, \rho) = |\theta - \rho|$ (a usual metric), then $d_b(\theta, \rho) = [d(\theta, \rho)]^2$ does not define a metric on M.

For the notions like convergence, completeness, and Cauchy sequence in the setting of b-metric spaces, the reader is referred to Aghajani et al. [20], Czerwik [12], Amini-Harandi [21], Huang et al. [13], Hussain et al. [14], Radenović et al. [15], Khamsi and Hussain [16], Latif et al. [22], Parvaneh et al. [17], Roshan et al. [18], and Zabihi and Razani [23].

In line with Wardowski [1], Cosentino et al. [24] investigated a nonlinear function $F : \mathbb{R}^+ \to \mathbb{R}$ complying with the following axioms:

(F_1) F is strictly increasing.

(F_2) For each sequence $\{r_n\}$ of positive numbers, the following equation holds: $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$.

(F_3) There exists $\theta \in (0, 1)$ such that $\lim_{r_n \to 0^+} (r_n)^\theta F(r_n) = 0$.

(F_4) $\tau + F(sr_n) \leq F(r_{n-1})$ implies $\tau + F(s^n r_n) \leq F(s^{n-1} r_{n-1})$, for each $n \in \mathbb{N}$ and some $\tau > 0$.

We denote by Δ_{F_s} the set of all functions satisfying the conditions (F_1)–(F_4).

Example 3. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by

(a) $F(r) = \ln(r)$,

(b) $F(r) = r + \ln(r)$.

It is easy to check that (a) and (b) are members of Δ_{F_s}.

Definition 4 (see [25]). Let (M, d_b, s) be a b-metric space. The pair $\{f, g\}$ is said to be compatible if and only if

$$\begin{aligned} &\lim_{n\to\infty} d_b(fg(r_n), gf(r_n)) = 0, \\ &\quad \text{whenever } \{r_n\} \text{ is a sequence in } M \text{ so that} \\ &\quad \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t \\ &\quad \text{for some } t \in M. \end{aligned} \tag{3}$$

Lemma 5. *Let (M, d_b, s) be a b-metric space. If there exist two sequences $\{r_n\}$, $\{s_n\}$ such that*

$$\begin{aligned} &\lim_{n\to\infty} d_b(r_n, s_n) = 0, \\ &\lim_{n\to\infty} r_n = t \quad \textit{for some } t \in M, \end{aligned} \tag{4}$$

Then $\lim_{n\to\infty} s_n = t$.

Proof. Due to the triangular inequality we have

$$d_b(s_n, t) \leq s\left[d_b(s_n, r_n) + d_b(r_n, t)\right], \tag{5}$$

and the result follows after applying limit as $n \to \infty$. □

2. Círíc Type Fixed Point Theorem

In this section we set up a fixed point theorem for four self-maps satisfying Círíc type F-contraction. We explain this theorem through an example and discuss its consequences.

Theorem 6. *Let (M, d_b, s) be a complete b-metric space and f, g, S, and T are self-mappings on (M, d_b, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If, for all $r_1, r_2 \in M$, for some $F \in \Delta_{F_s}$ and $\tau > 0$, the inequality*

$$\begin{aligned} &(d_b(f(r_1), g(r_2)) > 0 \textit{ implies } \tau \\ &\quad + F(sd_b(f(r_1), g(r_2))) \leq F(\mathscr{M}(r_1, r_2))), \end{aligned} \tag{6}$$

holds, where

$$\begin{aligned} \mathscr{M}(r_1, r_2) = \max\Bigg\{ &d_b(S(r_1), T(r_2)), \\ &d_b(f(r_1), S(r_1)), d_b(g(r_2), T(r_2)), \\ &\frac{d_b(S(r_1), g(r_2)) + d_b(f(r_1), T(r_2))}{2s}\Bigg\}, \end{aligned} \tag{7}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f, S\}$, $\{g, T\}$ are compatible.

Proof. Let $r_0 \in M$. As $f(M) \subseteq T(M)$, there exists $r_1 \in M$ such that $f(r_0) = T(r_1)$. Since $g(r_1) \in S(M)$, we can choose $r_2 \in M$ such that $g(r_1) = S(r_2)$. In general, r_{2n+1} and r_{2n+2} are chosen in M such that $f(r_{2n}) = T(r_{2n+1})$ and $g(r_{2n+1}) = S(r_{2n+2})$. Define a sequence $\{j_n\}$ in M such that $j_{2n} = f(r_{2n}) = T(r_{2n+1})$ and $j_{2n+1} = g(r_{2n+1}) = S(r_{2n+2})$ for all $n \geq 0$; we show that $\{j_n\}$ is a Cauchy sequence. Assume that $d_b(f(r_{2n}), g(r_{2n+1})) > 0$; then from contractive condition (6), we get

$$\begin{aligned} F(sd_b(j_{2n}, j_{2n+1})) &= F(sd_b(f(r_{2n}), g(r_{2n+1}))) \\ &\leq F(\mathscr{M}(r_{2n}, r_{2n+1})) - \tau, \end{aligned} \tag{8}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}\mathcal{M}(r_{2n}, r_{2n+1}) &= \max\Big\{d_b(S(r_{2n}), T(r_{2n+1})),\\ &\quad d_b(f(r_{2n}), S(r_{2n})), d_b(g(r_{2n+1}), T(r_{2n+1})),\\ &\quad \frac{d_b(S(r_{2n}), g(r_{2n+1})) + d_b(f(r_{2n}), T(r_{2n+1}))}{2s}\Big\}\\ &= \max\Big\{d_b(j_{2n-1}, j_{2n}), d_b(j_{2n}, j_{2n-1}), d_b(j_{2n+1}, j_{2n}),\\ &\quad \frac{d_b(j_{2n-1}, j_{2n+1}) + d_b(j_{2n}, j_{2n})}{2s}\Big\}\\ &= \max\{d_b(j_{2n-1}, j_{2n}), d_b(j_{2n}, j_{2n+1})\}.\end{aligned} \tag{9}$$

If $\mathcal{M}(r_{2n}, r_{2n+1}) = d_b(j_{2n}, j_{2n+1})$, then

$$F(sd_b(j_{2n}, j_{2n+1})) \le F(d_b(j_{2n}, j_{2n+1})) - \tau, \tag{10}$$

which is a contradiction due to F_1. Therefore,

$$F(sd_b(j_{2n}, j_{2n+1})) \le F(d_b(j_{2n-1}, j_{2n})) - \tau, \tag{11}$$

for all $n \in \mathbb{N}$. Similarly,

$$F(sd_b(j_{2n-1}, j_{2n})) \le F(d_b(j_{2n-2}, j_{2n-1})) - \tau, \tag{12}$$

for all $n \in \mathbb{N}$. Hence, from (11) and (12), we have

$$F(sd_b(j_n, j_{n+1})) \le F(d_b(j_{n-1}, j_n)) - \tau, \tag{13}$$

for all $n \in \mathbb{N}$. Let $b_n = d_b(j_n, j_{n+1})$ for each $n \in \mathbb{N}$, from (13) and axiom (F_4) we have

$$\tau + F(s^n b_n) \le F(s^{n-1} b_{n-1}), \quad n \in \mathbb{N}. \tag{14}$$

Repeating the process, we obtain

$$F(s^n b_n) \le F(b_0) - n\tau, \quad n \in \mathbb{N}. \tag{15}$$

On taking limit $n \to \infty$ in (15), we have $\lim_{n\to\infty} F(s^n b_n) = -\infty$; due to axiom (F_2) we get $\lim_{n\to\infty} s^n b_n = 0$ and (F_3) implies that there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n b_n)^\kappa F(s^n b_n) = 0. \tag{16}$$

From (15), for all $n \in \mathbb{N}$, we obtain

$$(s^n b_n)^\kappa F(s^n b_n) - (s^n b_n)^\kappa F(b_0) \le -(s^n b_n)^\kappa n\tau \le 0. \tag{17}$$

On taking limit $n \to \infty$ in (17), we have

$$\lim_{n\to\infty} \left(n (s^n b_n)^\kappa\right) = 0. \tag{18}$$

This implies there exists $n_1 \in \mathbb{N}$, such that $n(s^n b_n)^\kappa \le 1$ for all $n \ge n_1$ or

$$s^n b_n \le \frac{1}{n^{1/\kappa}} \quad \forall n \ge n_1. \tag{19}$$

To prove $\{j_n\}$ as a Cauchy sequence, we use (19) and, for $m > n \ge n_1$, we consider

$$d_b(j_n, j_m) \le \sum_{i=n}^{m-1} s^i b_i \le \sum_{i=n}^{\infty} s^i b_i \le \sum_{i=n}^{\infty} \frac{1}{n^{1/\kappa}}. \tag{20}$$

The convergence of the series $\sum_{i=n}^{\infty}(1/i^{1/\kappa})$ entails $\lim_{n,m\to\infty} d_b(j_n, j_m) = 0$. Hence $\{j_n\}$ is a Cauchy sequence in (M, d_b, s). Since (M, d_b, s) is a complete b-metric space, there exists $v \in M$ such that $\lim_{n\to\infty} d_b(j_n, v) = 0$. Also we have

$$\begin{aligned}\lim_{n\to\infty} f(r_{2n}) &= \lim_{n\to\infty} T(r_{2n+1}) = \lim_{n\to\infty} g(r_{2n+1})\\ &= \lim_{n\to\infty} S(r_{2n+2}) = v.\end{aligned} \tag{21}$$

We show that v is a common fixed point of f, g, S, and T. Since S is continuous, therefore,

$$\lim_{n\to\infty} Sf(r_{2n}) = S(v) = \lim_{n\to\infty} S^2(r_{2n+2}). \tag{22}$$

Since, the pair $\{f, S\}$ is compatible, so,

$$\begin{aligned}&\lim_{n\to\infty} d_b(fS(r_{2n}), Sf(r_{2n})) = 0,\\ &\text{by Lemma 5 } \lim_{n\to\infty} fS(r_{2n}) = S(v).\end{aligned} \tag{23}$$

Now put $r_1 = S(r_{2n})$ and $r_2 = r_{2n+1}$ in (6) and suppose on the contrary that $d_b(S(v), v) > 0$; we obtain

$$\begin{aligned}&F(d_b(fS(r_{2n}), g(r_{2n+1})))\\ &\quad \le F(\mathcal{M}(S(r_{2n}), r_{2n+1})) - \tau,\end{aligned} \tag{24}$$

where

$$\begin{aligned}\mathcal{M}(S(r_{2n}), r_{2n+1}) &= \max\Big\{d_b(S^2(r_{2n}), T(r_{2n+1})),\\ &\quad d_b(fS(r_{2n}), S^2(r_{2n})), d_b(g(r_{2n+1}), T(r_{2n+1})),\\ &\quad \frac{d_b(S^2(r_{2n}), g(r_{2n+1})) + d_b(fS(r_{2n}), T(r_{2n+1}))}{2s}\Big\}.\end{aligned} \tag{25}$$

Taking the upper limit in (24), we have

$$\begin{aligned}F(d_b(S(v), v)) &\le F(d_b(S(v), v)) - \tau\\ &< F(d_b(S(v), v)),\end{aligned} \tag{26}$$

a contradiction; hence, $d_b(S(v), v) = 0$ implies $S(v) = v$. Again since T is continuous, therefore,

$$\lim_{n\to\infty} Tg(r_{2n+1}) = T(v) = \lim_{n\to\infty} T^2(r_{2n+1}). \tag{27}$$

Since the pair $\{g, T\}$ is compatible, so,

$$\begin{aligned}&\lim_{n\to\infty} d_b(gT(r_{2n+1}), Tg(r_{2n+1})) = 0,\\ &\text{by Lemma 5 } \lim_{n\to\infty} gT(r_{2n+1}) = T(v).\end{aligned} \tag{28}$$

Now put $r_2 = T(r_{2n+1})$ and $r_1 = r_{2n}$ in (6) and suppose on the contrary that $d_b(v, T(v)) > 0$; we obtain

$$\begin{aligned} &F\left(d_b\left(f\left(r_{2n}\right), gT\left(r_{2n+1}\right)\right)\right) \\ &\quad \leq F\left(\mathscr{M}\left(r_{2n}, T\left(r_{2n+1}\right)\right)\right) - \tau, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathscr{M}\left(r_{2n}, T\left(r_{2n+1}\right)\right) = \max\Bigg\{ & d_b\left(S\left(r_{2n}\right), T^2\left(r_{2n+1}\right)\right), \\ & d_b\left(f\left(r_{2n}\right), S\left(r_{2n}\right)\right), d_b\left(gT\left(r_{2n+1}\right), T^2\left(r_{2n+1}\right)\right), \\ & \frac{d_b\left(S\left(r_{2n}\right), gT\left(r_{2n+1}\right)\right) + d_b\left(f\left(r_{2n}\right), T^2\left(r_{2n+1}\right)\right)}{2s}\Bigg\}. \end{aligned} \tag{30}$$

Taking the upper limit in (29), we have

$$\begin{aligned} F\left(d_b\left(v, T\left(v\right)\right)\right) &\leq F\left(d_b\left(v, T\left(v\right)\right)\right) - \tau \\ &< F\left(d_b\left(v, T\left(v\right)\right)\right), \end{aligned} \tag{31}$$

a contradiction; hence, $d_b(v, T(v)) = 0$ which then implies $T(v) = v$. Now assuming $d_b(f(v), v) > 0$ on the contrary, from (6) we get

$$F\left(d_b\left(f\left(v\right), g\left(r_{2n+1}\right)\right)\right) \leq F\left(\mathscr{M}\left(v, r_{2n+1}\right)\right) - \tau, \tag{32}$$

where

$$\begin{aligned} \mathscr{M}\left(v, r_{2n+1}\right) = \max\Bigg\{ & d_b\left(S\left(v\right), T\left(r_{2n+1}\right)\right), \\ & d_b\left(f\left(v\right), S\left(v\right)\right), d_b\left(g\left(r_{2n+1}\right), T\left(r_{2n+1}\right)\right), \\ & \frac{d_b\left(S\left(v\right), g\left(r_{2n+1}\right)\right) + d_b\left(f\left(v\right), T\left(r_{2n+1}\right)\right)}{2s}\Bigg\}. \end{aligned} \tag{33}$$

Taking the upper limit in (32) and using the fact that $S(v) = T(v) = v$, we have

$$\begin{aligned} F\left(d_b\left(f\left(v\right), v\right)\right) &\leq F\left(d_b\left(f\left(v\right), v\right)\right) - \tau \\ &< F\left(d_b\left(f\left(v\right), v\right)\right), \end{aligned} \tag{34}$$

a contradiction; hence, $d_b(f(v), v) = 0$ which then implies $f(v) = v$. Finally, assume on the contrary $d_b(v, g(v)) > 0$. From (6) and the fact that $S(v) = T(v) = f(v) = v$, we obtain

$$\begin{aligned} F\left(d_b\left(v, g\left(v\right)\right)\right) &= F\left(d_b\left(f\left(v\right), g\left(v\right)\right)\right) \\ &\leq F\left(\mathscr{M}\left(v, v\right)\right) - \tau, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \mathscr{M}\left(v, v\right) = \max\Bigg\{ & d_b\left(S\left(v\right), T\left(v\right)\right), d_b\left(f\left(v\right), S\left(v\right)\right), \\ & d_b\left(g\left(v\right), T\left(v\right)\right), \\ & \frac{d_b\left(S\left(v\right), g\left(v\right)\right) + d_b\left(f\left(v\right), T\left(v\right)\right)}{2s}\Bigg\}. \end{aligned} \tag{36}$$

This implies that

$$\begin{aligned} F\left(d_b\left(v, g\left(v\right)\right)\right) &\leq F\left(d_b\left(v, g\left(v\right)\right)\right) - \tau \\ &< F\left(d_b\left(v, g\left(v\right)\right)\right), \end{aligned} \tag{37}$$

a contradiction; hence, $d_b(g(v), v) = 0$ which then implies $g(v) = v$. Hence, v proves to be a common fixed point of the four maps f, g, S, and T. Also v is unique; indeed if ω is another fixed point of f, g, S, and T, then from (6), we have

$$\begin{aligned} F\left(d_b\left(v, \omega\right)\right) &= F\left(d_b\left(S\left(v\right), T\left(\omega\right)\right)\right) \\ &\leq F\left(\mathscr{M}\left(v, \omega\right)\right) - \tau, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \mathscr{M}\left(v, \omega\right) = \max\Bigg\{ & d_b\left(S\left(v\right), T\left(\omega\right)\right), d_b\left(f\left(v\right), S\left(v\right)\right), \\ & d_b\left(g\left(\omega\right), T\left(\omega\right)\right), \\ & \frac{d_b\left(S\left(v\right), g\left(\omega\right)\right) + d_b\left(f\left(v\right), T\left(\omega\right)\right)}{2s}\Bigg\}. \end{aligned} \tag{39}$$

Thus, from (38), we have

$$F\left(d_b\left(v, \omega\right)\right) < F\left(d_b\left(v, \omega\right)\right), \tag{40}$$

which is a contradiction. Hence, $v = \omega$ and v is a unique common fixed point of the four maps f, g, S, and T. □

The following example explains Theorem 6.

Example 7. Let $M = [0, 1]$ and define $d_b : M \times M \to \mathbb{R}_0^+$ by $d_b(r_1, r_2) = |r_1 - r_2|^2$, so $(M, d_b, 2)$ is a complete b-metric space. Define the mappings $f, g, S, T : M \to M$, for all $r \in M$ by

$$\begin{aligned} f\left(r\right) &= \left(\frac{r}{3}\right)^{16}, \\ g\left(r\right) &= \left(\frac{r}{3}\right)^{8}, \\ S\left(r\right) &= \left(\frac{r}{3}\right)^{8}, \\ T\left(r\right) &= \left(\frac{r}{3}\right)^{4}. \end{aligned} \tag{41}$$

Clearly, f, g, S, and T are self-mappings complying with $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. We note that the pair $\{f, S\}$ is compatible. If $\{r_n\}$ is a sequence in M satisfying

$$\lim_{n\to\infty} f\left(r_n\right) = \lim_{n\to\infty} S\left(r_n\right) = t, \quad \text{for some } t \in M, \tag{42}$$

then

$$\lim_{n\to\infty} \left|f\left(r_n\right) - t\right|^2 = \lim_{n\to\infty} \left|S\left(r_n\right) - t\right|^2 = 0, \tag{43}$$

and equivalently

$$\lim_{n\to\infty} \left|\left(\frac{r_n}{3}\right)^{16} - t\right|^2 = \lim_{n\to\infty} \left|\left(\frac{r_n}{3}\right)^{8} - t\right|^2 = 0 \tag{44}$$

implies

$$\lim_{n\to\infty}\left|(r_n)^{16}-3^{16}t\right|^2=\lim_{n\to\infty}\left|(r_n)^{8}-3^{8}t\right|^2=0. \tag{45}$$

Uniqueness of limit gives $t^{1/16}=t^{1/8}$ which implies $t=0,1$. Due to the continuity of f,S we have

$$\begin{aligned}\lim_{n\to\infty}d_b\left(fS(r_n),Sf(r_n)\right)&=\lim_{n\to\infty}\left|fS(r_n)-Sf(r_n)\right|^2\\&=\left|f(t)-S(t)\right|^2=|0-0|^2=0\quad\text{for } t=0\in M.\end{aligned} \tag{46}$$

Similarly, the pair $\{g,T\}$ is compatible. Now for each $r_1,r_2\in M$, consider

$$\begin{aligned}d_b\left(f(r_1),g(r_2)\right)&=\left|f(r_1)-g(r_2)\right|^2\\&=\left|\left(\frac{r_1}{3}\right)^{16}-\left(\frac{r_2}{3}\right)^{8}\right|^2\\&=\left[\left(\frac{r_1}{3}\right)^{8}+\left(\frac{r_2}{3}\right)^{4}\right]^2\left|\left(\frac{r_1}{3}\right)^{8}-\left(\frac{r_2}{3}\right)^{4}\right|^2\\&\le\left(\frac{1}{6561}+\frac{1}{81}\right)^2d_b\left(T(r_1),S(r_2)\right)\\&=\frac{6724}{43046721}d_b\left(T(r_1),S(r_2)\right)\\&\le\frac{6724}{43046721}\mathscr{M}(r_1,r_2).\end{aligned} \tag{47}$$

The above inequality can be written as

$$\begin{aligned}&\ln\left(\frac{43046721}{6724}\right)+\ln\left(d_b\left(f(r_1),g(r_2)\right)\right)\\&\quad\le\ln\left(\mathscr{M}(r_1,r_2)\right).\end{aligned} \tag{48}$$

Define the function $F:\mathbb{R}^+\to\mathbb{R}$ by $F(r)=\ln(r)$, for all $r\in R^+>0$. Hence, for all $r_1,r_2\in M$ such that $d_b(f(r_1),g(r_2))>0$, $\tau=\ln(43046721/6724)$ we obtain

$$\tau+F\left(d_b\left(f(r_1),g(r_2)\right)\right)\le F\left(\mathscr{M}(r_1,r_2)\right). \tag{49}$$

Thus, contractive condition (6) is satisfied for all $r_1,r_2\in M$. Hence, all the hypotheses of Theorem 6 are satisfied; note that f,g,S,T have a unique common fixed point $r=0$.

The following theorems can easily be obtained by chasing the proof of Theorem 6. Theorem 8 generalizes the result presented by Bianchini [26].

Theorem 8 (Bianchini type). *Let (M,d_b,s) be a complete b-metric space and f, g, S, T are self-mappings on (M,d_b,s) such that $f(M)\subseteq T(M)$, $g(M)\subseteq S(M)$. If, for all $r_1,r_2\in M$, for some $F\in\Delta_{F_s}$ $\tau>0$, the inequality*

$$\begin{aligned}&\left(d_b\left(f(r_1),g(r_2)\right)>0\ \textit{implies}\ \tau\right.\\&\quad\left.+F\left(sd_b\left(f(r_1),g(r_2)\right)\right)\le F\left(\mathscr{M}_3(r_1,r_2)\right)\right)\end{aligned} \tag{50}$$

holds, where

$$\begin{aligned}&\mathscr{M}_3(r_1,r_2)\\&\quad=k\max\left\{d_b\left(f(r_1),S(r_1)\right),d_b\left(g(r_2),T(r_2)\right)\right\},\\&\qquad\textit{with } k\in[0,s),\end{aligned} \tag{51}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f,S\}$, $\{g,T\}$ are compatible.

Theorem 9 generalizes the result presented by Sehgal [27].

Theorem 9 (Sehgal type). *Let (M,d_b,s) be a complete b-metric space and f, g, S, and T are self-mappings on (M,d_b,s) such that $f(M)\subseteq T(M)$, $g(M)\subseteq S(M)$. If, for all $r_1,r_2\in M$, for some $F\in\Delta_{F_s}$ $\tau>0$, the inequality*

$$\begin{aligned}&\left(d_b\left(f(r_1),g(r_2)\right)>0\ \textit{implies}\ \tau\right.\\&\quad\left.+F\left(sd_b\left(f(r_1),g(r_2)\right)\right)\le F\left(\mathscr{M}_4(r_1,r_2)\right)\right)\end{aligned} \tag{52}$$

holds, where

$$\begin{aligned}&\mathscr{M}_4(r_1,r_2)=k\max\left\{d_b\left(S(r_1),T(r_2)\right),\right.\\&\quad\left.d_b\left(f(r_1),S(r_1)\right),d_b\left(g(r_2),T(r_2)\right)\right\},\\&\qquad\textit{with } k\in[0,s),\end{aligned} \tag{53}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f,S\}$, $\{g,T\}$ are compatible.

Theorem 10 generalizes the result presented by Círíc [28].

Theorem 10. *Let (M,d_b,s) be a complete b-metric space and f, g, S, and T are self-mappings on (M,d_b,s) such that $f(M)\subseteq T(M)$, $g(M)\subseteq S(M)$. If, for all $r_1,r_2\in M$, for some $F\in\Delta_{F_s}$ $\tau>0$, the inequality*

$$\begin{aligned}&\left(F\left(d_b\left(f(r_1),g(r_2)\right)\right)>0\ \textit{implies}\ \tau\right.\\&\quad\left.+F\left(sd_b\left(f(r_1),g(r_2)\right)\right)\le F\left(\mathscr{M}_5(r_1,r_2)\right)\right)\end{aligned} \tag{54}$$

holds, where for all $k\in[0,s)$

$$\begin{aligned}\mathscr{M}_5(r_1,r_2)&=k\max\left\{d_b\left(S(r_1),T(r_2)\right),\right.\\&\quad d_b\left(f(r_1),S(r_1)\right),d_b\left(g(r_2),T(r_2)\right),\\&\quad\left.d_b\left(g(r_2),S(r_1)\right),d_b\left(f(r_1),T(r_2)\right)\right\},\end{aligned} \tag{55}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f,S\}$, $\{g,T\}$ are compatible.

Theorem 11. *Let (M,d_b,s) be a complete b-metric space and f, g, S, and T are self-mappings on (M,d_b,s) such that $f(M)\subseteq T(M)$, $g(M)\subseteq S(M)$. If, for all $r_1,r_2\in M$, for some $F\in\Delta_{F_s}$ $\tau>0$, the inequality*

$$\begin{aligned}&\left(F\left(d_b\left(f(r_1),g(r_2)\right)\right)>0\ \textit{implies}\ \tau\right.\\&\quad\left.+F\left(sd_b\left(f(r_1),g(r_2)\right)\right)\le F\left(\mathscr{M}_6(r_1,r_2)\right)\right)\end{aligned} \tag{56}$$

holds, where for all $k \in [0, s)$

$$\mathcal{M}_6(r_1, r_2) = k \max\left\{d_b(S(r_1), T(r_2)), \frac{d_b(f(r_1), S(r_1)) + d_b(g(r_2), T(r_2))}{2}, \frac{d_b(g(r_2), S(r_1)) + d_b(f(r_1), T(r_2))}{2s}\right\}, \tag{57}$$

then f, g, S, *and* T *have a unique common fixed point* v *provided* S, T *are continuous and pairs* $\{f, S\}$, $\{g, T\}$ *are compatible.*

Theorem 12. *Let* (M, d_b, s) *be a complete b-metric space and* $f, g, S,$ *and* T *are self-mappings on* (M, d_b, s) *such that* $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. *If for all* $r_1, r_2 \in M$, $k \in [0, s)$ *the inequality*

$$sd_b(f(r_1), g(r_2)) \le k\mathcal{M}(r_1, r_2) \tag{58}$$

holds, where

$$\mathcal{M}(r_1, r_2) = \max\left\{d_b(S(r_1), T(r_2)), d_b(f(r_1), S(r_1)), d_b(g(r_2), T(r_2)), \frac{d_b(S(r_1), g(r_2)) + d_b(f(r_1), T(r_2))}{2s}\right\}, \tag{59}$$

then f, g, S, *and* T *have a unique common fixed point* v *provided* S, T *are continuous and pairs* $\{f, S\}$, $\{g, T\}$ *are compatible.*

Proof. Since, for all $r_1, r_2 \in M$, we have

$$d_b(f(r_1), g(r_2)) \le k\mathcal{M}(r_1, r_2), \tag{60}$$

thus, if $d_b(f(r_1), g(r_2)) > 0$, we have

$$\tau + \ln(d_b(f(r_1), g(r_2))) \le \ln(\mathcal{M}(r_1, r_2)), \tag{61}$$

where $\tau = \ln(s/k) > 0$; thus, the contractive condition (58) reduces to (6) with $F(r) = \ln(r)$. Hence, Theorem 6 is a generalization of [29, Theorem 3.1]. □

3. Hardy and Rogers Type Fixed Point Theorem

In this section, we set up a fixed point theorem for four self-maps satisfying Hardy and Rogers type F-contraction. We give an example to validate this theorem and discuss its consequences.

Theorem 13. *Let* (M, d_b, s) *be a complete b-metric space and* $f, g, S,$ *and* T *are self-mappings on* (M, d_b, s) *such that* $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. *If, for all* $r_1, r_2 \in M$, *for some* $F \in \Delta_{F_s}$ $\tau > 0$, *the inequality*

$$(d_b(f(r_1), g(r_2)) > 0 \text{ implies } \tau + F(sd_b(f(r_1), g(r_2))) \le F(\mathcal{M}_1(r_1, r_2))) \tag{62}$$

holds, where

$$\begin{aligned}\mathcal{M}_1(r_1, r_2) &= a_1 d_b(S(r_1), T(r_2)) + a_2 d_b(f(r_1), S(r_1)) \\ &\quad + a_3 d_b(g(r_2), T(r_2)) \\ &\quad + a_4[d_b(S(r_1), g(r_2)) + d_b(f(r_1), T(r_2))]\end{aligned} \tag{63}$$

with $a_i \ge 0$ $(i = 1, 2, 3, 4)$ *such that* $a_1 + a_2 + a_3 + 2sa_4 = 1$,

then f, g, S, *and* T *have a unique common fixed point* v *provided* S, T *are continuous and pairs* $\{f, S\}$, $\{g, T\}$ *are compatible.*

Proof. Let $r_0 \in M$. As $f(M) \subseteq T(M)$, there exists $r_1 \in M$ such that $f(r_0) = T(r_1)$. Since $g(r_1) \in S(M)$, we can choose $r_2 \in M$ such that $g(r_1) = S(r_2)$. In general r_{2n+1} and r_{2n+2} are chosen in M such that $f(r_{2n}) = T(r_{2n+1})$ and $g(r_{2n+1}) = S(r_{2n+2})$. Define a sequence $\{j_n\}$ in M such that $j_{2n} = f(r_{2n}) = T(r_{2n+1})$ and $j_{2n+1} = g(r_{2n+1}) = S(r_{2n+2})$ for all $n \ge 0$; we show that $\{j_n\}$ is a Cauchy sequence. Assume that $d_b(f(r_{2n}), g(r_{2n+1})) > 0$; then from contractive condition (62), we get

$$\begin{aligned}F(sd_b(j_{2n}, j_{2n+1})) &= F(sd_b(f(r_{2n}), g(r_{2n+1}))) \\ &\le F(\mathcal{M}_1(r_{2n}, r_{2n+1})) - \tau,\end{aligned} \tag{64}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}\mathcal{M}_1(r_{2n}, r_{2n+1}) &= a_1 d_b(S(r_{2n}), T(r_{2n+1})) \\ &\quad + a_2 d_b(f(r_{2n}), S(r_{2n})) \\ &\quad + a_3 d_b(g(r_{2n+1}), T(r_{2n+1})) \\ &\quad + a_4[d_b(S(r_{2n}), g(r_{2n+1})) \\ &\quad + d_b(f(r_{2n}), T(r_{2n+1}))] = a_1 d_b(j_{2n-1}, j_{2n}) \\ &\quad + a_2 d_b(j_{2n}, j_{2n-1}) + a_3 d_b(j_{2n+1}, j_{2n}) \\ &\quad + a_4[d_b(j_{2n-1}, j_{2n+1}) + d_b(j_{2n}, j_{2n})] = (a_1 + a_2 \\ &\quad + sa_4) d_b(j_{2n-1}, j_{2n}) + (a_3 \\ &\quad + sa_4) d_b(j_{2n}, j_{2n+1}).\end{aligned} \tag{65}$$

Now from (64) we have

$$\begin{aligned}F(sd_b(j_{2n}, j_{2n+1})) &\le F((a_1 + a_2 + sa_4) d_b(j_{2n-1}, j_{2n}) \\ &\quad + (a_3 + sa_4) d_b(j_{2n}, j_{2n+1})) - \tau.\end{aligned} \tag{66}$$

Since F is strictly increasing, (66) implies

$$\begin{aligned}&sd_b(j_{2n}, j_{2n+1}) \\ &\quad \le (a_1 + a_2 + sa_4) d_b(j_{2n-1}, j_{2n}) \\ &\quad\quad + (a_3 + sa_4) d_b(j_{2n}, j_{2n+1}),\end{aligned}$$

$$\begin{aligned}&(1-a_3-sa_4)\,d_b(j_{2n},j_{2n+1})\\&\quad<(s-a_3-sa_4)\,d_b(j_{2n},j_{2n+1})\\&\quad\le(a_1+a_2+sa_4)\,d_b(j_{2n-1},j_{2n}),\\&d_b(j_{2n},j_{2n+1})\le\frac{a_1+a_2+sa_4}{1-a_3-sa_4}d_b(j_{2n-1},j_{2n}).\end{aligned}\tag{67}$$

Since $a_1+a_2+a_3+2sa_4=1$, therefore

$$\begin{aligned}d_b(j_{2n},j_{2n+1})&\le\frac{a_1+a_2+sa_4}{1-a_3-sa_4}d_b(j_{2n-1},j_{2n})\\&=d_b(j_{2n-1},j_{2n}).\end{aligned}\tag{68}$$

Thus, from (66) we obtain

$$F(sd_b(j_{2n},j_{2n+1}))\le F(d_b(j_{2n-1},j_{2n}))-\tau,\tag{69}$$

for all $n\in\mathbb{N}$. Similarly,

$$F(sd_b(j_{2n-1},j_{2n}))\le F(d_b(j_{2n-2},j_{2n-1}))-\tau,\tag{70}$$

for all $n\in\mathbb{N}$. Hence, from (69) and (70), we have

$$F(sd_b(j_n,j_{n+1}))\le F(d_b(j_{n-1},j_n))-\tau,\tag{71}$$

for all $n\in\mathbb{N}$. Let $b_n=d_b(j_n,j_{n+1})$ for each $n\in\mathbb{N}$; from (71) and axiom (F_4) we have

$$\tau+F(s^nb_n)\le F\left(s^{n-1}b_{n-1}\right),\quad n\in\mathbb{N}.\tag{72}$$

Repeating the process, we obtain

$$F(s^nb_n)\le F(b_0)-n\tau,\quad n\in\mathbb{N}.\tag{73}$$

On taking limit $n\to\infty$ in (73), we have $\lim_{n\to\infty}F(s^nb_n)=-\infty$; due to axiom (F_2) we get $\lim_{n\to\infty}s^nb_n=0$ and (F_3) implies that there exists $\kappa\in(0,1)$ such that

$$\lim_{n\to\infty}(s^nb_n)^\kappa F(s^nb_n)=0.\tag{74}$$

From (73), for all $n\in\mathbb{N}$, we obtain

$$(s^nb_n)^\kappa F(s^nb_n)-(s^nb_n)^\kappa F(b_0)\le-(s^nb_n)^\kappa n\tau\le0.\tag{75}$$

On taking limit $n\to\infty$ in (75), we have

$$\lim_{n\to\infty}\left(n(s^nb_n)^\kappa\right)=0.\tag{76}$$

This implies there exists $n_1\in\mathbb{N}$, such that $n(s^nb_n)^\kappa\le1$ for all $n\ge n_1$ or

$$s^nb_n\le\frac{1}{n^{1/\kappa}}\quad\forall n\ge n_1.\tag{77}$$

To prove $\{j_n\}$ as a Cauchy sequence, we use (77) and for $m>n\ge n_1$; we consider

$$d_b(j_n,j_m)\le\sum_{i=n}^{m-1}s^ib_i\le\sum_{i=n}^{\infty}s^ib_i\le\sum_{i=n}^{\infty}\frac{1}{n^{1/\kappa}}.\tag{78}$$

The convergence of the series $\sum_{i=n}^{\infty}(1/i^{1/\kappa})$ entails $\lim_{n,m\to\infty}d_b(j_n,j_m)=0$. Hence $\{j_n\}$ is a Cauchy sequence in (M,d_b,s). Since (M,d_b,s) is a complete b-metric space, there exists $v\in M$ such that $\lim_{n\to\infty}d_b(j_n,v)=0$. Also we have

$$\begin{aligned}\lim_{n\to\infty}f(r_{2n})&=\lim_{n\to\infty}T(r_{2n+1})=\lim_{n\to\infty}g(r_{2n+1})\\&=\lim_{n\to\infty}S(r_{2n+2})=v.\end{aligned}\tag{79}$$

We show that v is a common fixed point of f, g, S, and T. Since S is continuous, therefore,

$$\lim_{n\to\infty}Sf(r_{2n})=S(v)=\lim_{n\to\infty}S^2(r_{2n+2}).\tag{80}$$

Since, the pair $\{f,S\}$ is compatible, so,

$$\begin{aligned}&\lim_{n\to\infty}d_b(fS(r_{2n}),Sf(r_{2n}))=0,\\&\text{by Lemma 5 }\lim_{n\to\infty}fS(r_{2n})=S(v).\end{aligned}\tag{81}$$

Now put $r_1=S(r_{2n})$ and $r_2=r_{2n+1}$ in (62) and suppose on the contrary that $d_b(S(v),v)>0$; we obtain

$$\begin{aligned}&F(d_b(fS(r_{2n}),g(r_{2n+1})))\\&\quad\le F(\mathscr{M}_1(S(r_{2n}),r_{2n+1}))-\tau,\end{aligned}\tag{82}$$

where

$$\begin{aligned}\mathscr{M}_1(S(r_{2n}),r_{2n+1})&=a_1d_b\left(S^2(r_{2n}),T(r_{2n+1})\right)\\&+a_2d_b\left(fS(r_{2n}),S^2(r_{2n})\right)\\&+a_3d_b(g(r_{2n+1}),T(r_{2n+1}))\\&+a_4\left[d_b\left(S^2(r_{2n}),g(r_{2n+1})\right)\right.\\&\left.+d_b(fS(r_{2n}),T(r_{2n+1}))\right].\end{aligned}\tag{83}$$

Taking the upper limit in (82) and considering the fact that $a_1+2a_4<1$, we have

$$\begin{aligned}F(d_b(S(v),v))&\le F(d_b(S(v),v))-\tau\\&<F(d_b(S(v),v)),\end{aligned}\tag{84}$$

a contradiction; hence, $d_b(S(v),v)=0$ implies $S(v)=v$. Again since T is continuous, therefore,

$$\lim_{n\to\infty}Tg(r_{2n+1})=T(v)=\lim_{n\to\infty}T^2(r_{2n+1}).\tag{85}$$

Since, the pair $\{g,T\}$ is compatible,

$$\begin{aligned}&\lim_{n\to\infty}d_b(gT(r_{2n+1}),Tg(r_{2n+1}))=0,\\&\text{by Lemma 5 }\lim_{n\to\infty}gT(r_{2n+1})=T(v).\end{aligned}\tag{86}$$

Now put $r_2=T(r_{2n+1})$ and $r_1=r_{2n}$ in (62) and suppose on the contrary that $d_b(v,T(v))>0$; we obtain

$$\begin{aligned}&F(d_b(f(r_{2n}),gT(r_{2n+1})))\\&\quad\le F(\mathscr{M}_1(r_{2n},T(r_{2n+1})))-\tau,\end{aligned}\tag{87}$$

where

$$\begin{aligned}\mathcal{M}_1(r_{2n}, T(r_{2n+1})) &= a_1 d_b\left(S(r_{2n}), T^2(r_{2n+1})\right)\\ &+ a_2 d_b\left(f(r_{2n}), S(r_{2n})\right)\\ &+ a_3 d_b\left(gT(r_{2n+1}), T^2(r_{2n+1})\right)\\ &+ a_4\left[d_b\left(S(r_{2n}), gT(r_{2n+1})\right)\right.\\ &\left.+ d_b\left(f(r_{2n}), T^2(r_{2n+1})\right)\right].\end{aligned} \tag{88}$$

Taking the upper limit in (87), we have

$$\begin{aligned}F(d_b(v, T(v))) &\le F(d_b(v, T(v))) - \tau\\ &< F(d_b(v, T(v))),\end{aligned} \tag{89}$$

a contradiction; hence, $d_b(v, T(v)) = 0$ which then implies $T(v) = v$. Assume that $d_b(f(v), v) > 0$ and from (62) we get

$$F(d_b(f(v), g(r_{2n+1}))) \le F(\mathcal{M}_1(v, r_{2n+1})) - \tau, \tag{90}$$

where

$$\begin{aligned}&\mathcal{M}_1(v, r_{2n+1})\\ &= a_1 d_b(S(v), T(r_{2n+1})) + a_2 d_b(f(v), S(v))\\ &\quad + a_3 d_b(g(r_{2n+1}), T(r_{2n+1}))\\ &\quad + a_4\left[d_b(S(v), g(r_{2n+1})) + d_b(f(v), T(r_{2n+1}))\right].\end{aligned} \tag{91}$$

Taking the upper limit in (90) and using the fact that $S(v) = T(v) = v$, we have

$$\begin{aligned}F(d_b(f(v), v)) &\le F(d_b(f(v), v)) - \tau\\ &< F(d_b(f(v), v)),\end{aligned} \tag{92}$$

a contradiction; hence, $d_b(f(v), v) = 0$ which then implies $f(v) = v$. Finally, assume that $d_b(v, g(v)) > 0$. From (62) and the equation $S(v) = T(v) = f(v) = v$, we obtain

$$\begin{aligned}F(d_b(v, g(v))) &= F(d_b(f(v), g(v)))\\ &\le F(\mathcal{M}_1(v, v)) - \tau,\end{aligned} \tag{93}$$

where

$$\begin{aligned}&\mathcal{M}_1(v, v)\\ &= a_1 d_b(S(v), T(v)) + a_2 d_b(f(v), S(v))\\ &\quad + a_3 d_b(g(v), T(v))\\ &\quad + a_4\left[d_b(S(v), g(v)) + d_b(f(v), T(v))\right].\end{aligned} \tag{94}$$

This implies that

$$\begin{aligned}F(d_b(v, g(v))) &\le F(d_b(v, g(v))) - \tau\\ &< F(d_b(v, g(v))),\end{aligned} \tag{95}$$

a contradiction; hence, $d_b(g(v), v) = 0$ which then implies $g(v) = v$. Hence, v proves to be a common fixed point of the four maps f, g, S, and T. Also v is unique. Indeed if ω is another fixed point of f, g, S, and T, then, from (62), we have

$$\begin{aligned}F(d_b(v, \omega)) &= F(d_b(S(v), T(\omega)))\\ &\le F(\mathcal{M}_1(v, \omega)) - \tau,\end{aligned} \tag{96}$$

where

$$\begin{aligned}&\mathcal{M}_1(v, \omega)\\ &= a_1 d_b(S(v), T(\omega)) + a_2 d_b(f(v), S(v))\\ &\quad + a_3 d_b(g(\omega), T(\omega))\\ &\quad + a_4\left[d_b(S(v), g(\omega)) + d_b(f(v), T(\omega))\right].\end{aligned} \tag{97}$$

Thus, from (96), we have

$$F(d_b(v, \omega)) < F(d_b(v, \omega)), \tag{98}$$

which is a contradiction. Hence, $v = \omega$ and v is a unique common fixed point of the four maps f, g, S, and T. □

The following example explains Theorem 13.

Example 14. Let $M = [0, \infty)$ and define $d_b : M \times M \to \mathbb{R}_0^+$ by $d_b(r_1, r_2) = |r_1 - r_2|^2$, so, $(M, d_b, 2)$ is a complete b-metric space. Define the mappings $f, g, S, T : M \to M$, for all $r \in M$, by

$$\begin{aligned}f(r) &= \ln\left(1 + \frac{r}{6}\right),\\ g(r) &= \ln\left(1 + \frac{r}{7}\right),\\ S(r) &= e^{7r} - 1,\\ T(r) &= e^{6r} - 1.\end{aligned} \tag{99}$$

Clearly, f, g, S, and T are continuous self-mappings complying with $f(M) = T(M) = g(M) = S(M)$. We note that the pair $\{f, S\}$ is compatible. Indeed, let $\{r_n\}$ be a sequence in M satisfying

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} S(r_n) = t, \quad \text{for some } t \in M. \tag{100}$$

Then

$$\lim_{n\to\infty} |f(r_n) - t|^2 = \lim_{n\to\infty} |S(r_n) - t|^2 = 0, \tag{101}$$

and equivalently

$$\lim_{n\to\infty} \left|\ln\left(1 + \frac{r_n}{6}\right) - t\right|^2 = \lim_{n\to\infty} \left|e^{7r_n} - 1 - t\right|^2 = 0 \tag{102}$$

implies

$$\lim_{n\to\infty} \left|r_n - \left(6e^t - 6\right)\right|^2 = \lim_{n\to\infty} \left|r_n - \frac{\ln(t+1)}{7}\right|^2 = 0. \tag{103}$$

The uniqueness of limit gives $6e^t - 6 = \ln(t+1)/7$; thus, $t = 0$ is only possible solution. Due to the continuity of f, S we have

$$\begin{aligned}\lim_{n\to\infty} d_b(fS(r_n), Sf(r_n)) &= \lim_{n\to\infty} |fS(r_n) - Sf(r_n)|^2 \\ &= |f(t) - S(t)|^2 = |0-0|^2 = 0 \quad \text{for } t = 0 \in M.\end{aligned} \tag{104}$$

Similarly, the pair $\{g, T\}$ is compatible. Now for each $r_1, r_2 \in M$, consider

$$\begin{aligned}d_b(f(r_1), g(r_2)) &= |f(r_1) - g(r_2)|^2 \\ &= \left|\ln\left(1+\frac{r}{6}\right) - \ln\left(1+\frac{r}{7}\right)\right|^2 \\ &\le \left(\frac{r}{6} - \frac{r}{7}\right)^2 \le \left(\frac{1}{42}\right)^2 |7r - 6r|^2 \\ &\le \left(\frac{1}{42}\right)^2 \left|e^{7r} - e^{6r}\right|^2 \\ &= \left(\frac{1}{42}\right)^2 d_b(T(r_1), S(r_2)) \\ &\le \left(\frac{1}{42}\right)^2 \mathscr{M}_1(r_1, r_2).\end{aligned} \tag{105}$$

The above inequality can be written as

$$\begin{aligned}&2\ln(42) + \ln(d_b(f(r_1), g(r_2))) \\ &\quad \le \ln(\mathscr{M}_1(r_1, r_2)).\end{aligned} \tag{106}$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$, for all $r \in R^+ > 0$. Hence, for all $r_1, r_2 \in M$, such that $d_b(f(r_1), g(r_2)) > 0$, $\tau = 2\ln(42)$ we obtain

$$\tau + F(d_b(f(r_1), g(r_2))) \le F(\mathscr{M}(r_1, r_2)). \tag{107}$$

Thus, contractive condition (62) is satisfied for all $r_1, r_2 \in M$. Hence, all the hypotheses of Theorem 13 are satisfied. Note that f, g, S, and T have a unique common fixed point $r = 0$.

Theorem 15 generalizes the result presented by Reich [30] and can be proved by following the proof of Theorem 13.

Theorem 15 (Reich type). *Let (M, d_b, s) be a complete b-metric space and f, g, S, and T are self-mappings on (M, d_b, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If, for all $r_1, r_2 \in M$, for some $F \in \Delta_{F_s}$, $\tau > 0$, the inequality*

$$\begin{aligned}&(d_b(f(r_1), g(r_2)) > 0 \text{ implies } \tau \\ &\quad + F(sd_b(f(r_1), g(r_2))) \le F(\mathscr{M}_2(r_1, r_2)))\end{aligned} \tag{108}$$

holds, where for all $a_i \ge 0$ $(i = 1, 2, 3)$ such that $a_1 + a_2 + a_3 = 1$

$$\begin{aligned}\mathscr{M}_2(r_1, r_2) &= a_1 d_b(S(r_1), T(r_2)) \\ &\quad + a_2 d_b(f(r_1), S(r_1)) \\ &\quad + a_3 d_b(g(r_2), T(r_2)),\end{aligned} \tag{109}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f, S\}$, $\{g, T\}$ are compatible.

Theorem 16 generalizes the result presented by Chatterjea [31] and can be proved by following the proof of Theorem 13.

Theorem 16 (Chatterjea type). *Let (M, d_b, s) be a complete b-metric space and f, g, S, and T are self-mappings on (M, d_b, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If for all $r_1, r_2 \in M$ for some $F \in \Delta_{F_s}$ $\tau > 0$ the inequality*

$$\begin{aligned}&(d_b(f(r_1), g(r_2)) > 0 \text{ implies } \tau \\ &\quad + F(sd_b(f(r_1), g(r_2))) \\ &\quad \le F(a[d_b(S(r_1), g(r_2)) + d_b(f(r_1), T(r_2))]))\end{aligned} \tag{110}$$

holds for all $a \ge 0$ such that $2a = 1$, then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f, S\}$, $\{g, T\}$ are compatible.

Similarly, Theorems 17 and 18 can be proved by following the proof of Theorem 13.

Theorem 17. *Let (M, d_b, s) be a complete b-metric space and f, g, S, and T are self-mappings on (M, d_b, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If for all $r_1, r_2 \in M$ for some $F \in \Delta_{F_s}$, $\tau > 0$ the inequality*

$$\begin{aligned}&(d_b(f(r_1), g(r_2)) > 0 \text{ implies } \tau \\ &\quad + F(sd_b(f(r_1), g(r_2))) \le F(\mathscr{M}_7(r_1, r_2)))\end{aligned} \tag{111}$$

holds, where

$$\begin{aligned}\mathscr{M}_7(r_1, r_2) &= a_1(r_1, r_2) d_b(S(r_1), T(r_2)) \\ &\quad + a_2(r_1, r_2) d_b(f(r_1), S(r_1)) \\ &\quad + a_3(r_1, r_2) d_b(g(r_2), T(r_2)) + a_4(r_1, r_2) \\ &\quad \cdot [d_b(S(r_1), g(r_2)) + d_b(f(r_1), T(r_2))]\end{aligned} \tag{112}$$

with $a_i(r_1, r_2)$ $(i = 1, 2, 3, 4)$ are nonnegative functions such that

$$\begin{aligned}&\sup_{r_1, r_2 \in M} \{a_1(r_1, r_2) + a_2(r_1, r_2) + a_3(r_1, r_2) \\ &\quad + 2sa_4(r_1, r_2)\} = 1,\end{aligned} \tag{113}$$

then f, g, S, and T have a unique common fixed point υ provided S, T are continuous and pairs $\{f, S\}$, $\{g, T\}$ are compatible.

Theorem 18. *Let (M, d_b, s) be a complete b-metric space and f, g, S, and T are self-mappings on (M, d_b, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If for all $r_1, r_2 \in M$ for some $F \in \Delta_{F_s}$, $\tau > 0$ the inequality*

$$\begin{aligned}&(d_b(f(r_1), g(r_2)) > 0 \text{ implies } \tau \\ &\quad + F(sd_b(f(r_1), g(r_2))) \le F(\mathscr{M}_8(r_1, r_2)))\end{aligned} \tag{114}$$

holds, where

$$\begin{aligned}\mathcal{M}_8(r_1,r_2) &= a_1 d_b(S(r_1),T(r_2)) \\ &+ \frac{a_2+a_3}{2}[d_b(f(r_1),S(r_1)) \\ &+ d_b(g(r_2),T(r_2))] \\ &+ \frac{a_4+a_5}{2s}[d_b(S(r_1),g(r_2)) \\ &+ d_b(f(r_1),T(r_2))] \quad \text{with } a_i \geq 0,\ \sum_{i=1}^{5} a_i = 1,\end{aligned} \tag{115}$$

then f, g, S, and T have a unique common fixed point v provided S,T are continuous and pairs $\{f,S\}$, $\{g,T\}$ are compatible.

4. Application of Theorem 6 to a System of Integral Equations

Let $M = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$, $a > 0$. Let the function $d_b : M \times M \to [0,\infty)$ be defined by

$$d_b(u,v) = \left(\sup_{t\in[a,b]} |u(t) - v(t)|\right)^2, \tag{116}$$

for all $u,v \in C([a,b],\mathbb{R})$. Obviously, $(M,d_b,2)$ is a complete b-metric space. We shall apply Theorem 6 to show the existence of a common solution of the system of Volterra type integral equations given by

$$\begin{aligned}u(t) &= p(t) + \int_a^t K(t,r,S(u(t)))\,dr, \\ w(t) &= p(t) + \int_a^t J(t,r,T(v(t)))\,dr,\end{aligned} \tag{117}$$

for all t, where $p : M \to \mathbb{R}$ is a continuous function and $K, J : [a,b]\times[a,b]\times M \to \mathbb{R}$ are lower semicontinuous operators. Now we prove the following theorem to ensure the existence of a solution of the system of integral equations (117).

Theorem 19. *Let $M = C([a,b],\mathbb{R})$ and define the mappings $f,g : M \to M$ by*

$$\begin{aligned}fu(t) &= p(t) + \int_a^t K(t,r,S(u(t)))\,dr, \\ gu(t) &= p(t) + \int_a^t J(t,r,T(v(t)))\,dr,\end{aligned} \tag{118}$$

where $p : M \to \mathbb{R}$ is a continuous function and $K, J : [a,b]\times[a,b]\times M \to \mathbb{R}$ are lower semicontinuous operators. Assume the following conditions are satisfied:

(i) *There exists a continuous function $H : M \to [0,\infty)$ such that*

$$|K(t,r,S) - J(t,r,T)| \leq H(r)\,|S(u(t)) - T(v(t))| \tag{119}$$

for each $t,r \in [a,b]$ and $S,T \in M$.

(ii) *There exists $\tau > 0$ and, for each $r \in M$, one has*

$$\int_a^t H(r)\,dr \leq \sqrt{\frac{e^{-\tau}}{s}}, \quad t \in [a,b]. \tag{120}$$

(iii) *There exists a sequence $\{r_n\}$ in M such that $\lim_{n\to\infty} d_b(fS(r_n), Sf(r_n)) = 0$ and $\lim_{n\to\infty} d_b(gT(r_n), Tg(r_n)) = 0$, whenever*

$$\begin{aligned}&\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} S(r_n) = t, \\ &\lim_{n\to\infty} g(r_n) = \lim_{n\to\infty} T(r_n) = t \\ &\qquad \text{for some } t \in M.\end{aligned} \tag{121}$$

Then the system of integral equations given in (117) has a solution.

Proof. Following assumptions (i) and (ii), we have

$$\begin{aligned}d_b(fu(t),gv(t)) &= \left(\sup_{t\in[a,b]} |fu(t) - gv(t)|\right)^2 \\ &= \left(\sup_{t\in[a,b]} \int_a^t |K(t,r,S(u(t)) - J(t,r,T(v(t))))|\,dr\right)^2 \\ &\leq \left(\sup_{t\in[a,b]} \int_a^t H(r)\,|S(u(t)) - T(v(t))|\,dr\right)^2 \\ &\leq \left(\sqrt{\sup_{t\in[a,b]} |S(u(t)) - T(v(t))|^2} \int_a^t H(r)\,dr\right)^2 \\ &= d_b(S(u(t)),T(v(t)))\left(\int_a^t H(r)\,dr\right)^2 \\ &\leq d_b(S(u(t)),T(v(t)))\,\frac{e^{-\tau}}{s} \leq \mathcal{M}_1(u(t),v(t))\,\frac{e^{-\tau}}{s}.\end{aligned} \tag{122}$$

Consequently, we have

$$s d_b(fu(t),gv(t)) \leq e^{-\tau}\mathcal{M}_1(u(t),v(t)). \tag{123}$$

As the natural logarithm belongs to Δ_{F_s}, applying it on above inequality and after some simplification, we get

$$\tau + \ln(s d_b(fu(t),gv(t))) \leq \ln(\mathcal{M}_1(u(t),v(t))). \tag{124}$$

□

So taking $F(r) = \ln(r)$, all hypotheses of Theorem 6 are satisfied. Hence the system of integral equations given in (117) has a unique common solution.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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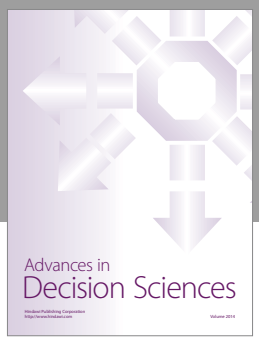

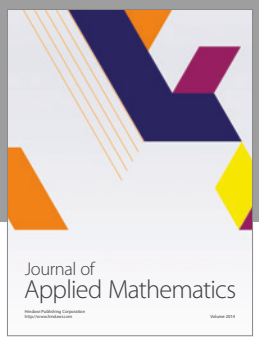

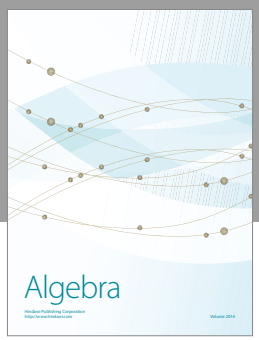

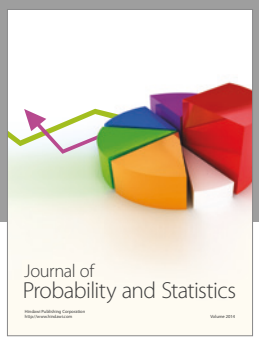

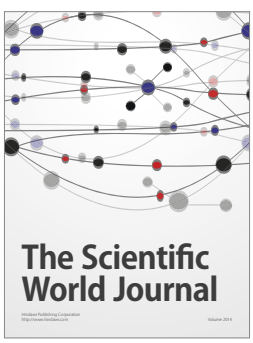

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