Research Article

Bloch-Type Spaces of Minimal Surfaces

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We study Bloch-type spaces of minimal surfaces from the unit disk $\mathbb{D}$ into $\mathbb{R}^n$ and characterize them in terms of weighted Lipschitz functions. In addition, the boundedness of a composition operator C_ϕ acting between two Bloch-type spaces is discussed.

1. Introduction and Main Results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$ and $f = (f_1, \dots, f_n)$ be a harmonic map from $\mathbb{D}$ into $\mathbb{R}^n$ ($n \geq 2$). The set of all pairs $(z, f(z))$, $z = x + iy \in \mathbb{D}$, or simply the map f itself, is called a *minimal surface* if

$$\begin{aligned} f_x f_y &= 0, \\ f_x f_x &= f_y f_y \end{aligned} \quad (1)$$

in $\mathbb{D}$,

where

$$\begin{aligned} f_x &= (f_{1x}, \dots, f_{nx}), \\ f_y &= (f_{1y}, \dots, f_{ny}) \end{aligned} \quad (2)$$

are partial derivatives and the products are inner (cf. [1]). The set of all minimal surfaces in $\mathbb{D}$ is denoted by $h(\mathbb{D})$.

If $f = (f_1, \dots, f_n) : \mathbb{D} \rightarrow \mathbb{R}^n$ is a minimal surface, then there exist holomorphic functions g_j in $\mathbb{D}$ such that $f_j = \operatorname{Re} g_j$, $1 \leq j \leq n$. We say that $(g_1, \dots, g_n)$ is an admissible system of f (cf. [1]). Thus, by a simple computation, the formula in (1) can be rewritten as

$$\begin{aligned} \sum_{j=1}^n (g'_j)^2 &= 0, \\ |f_x(z)|^2 &= \frac{1}{2} \sum_{j=1}^n |g'_j|^2. \end{aligned} \quad (3)$$

For $\alpha > 0$, a map $f \in h(\mathbb{D})$ is called α -Bloch if

$$\|f\|_\alpha = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f_x(z)| < \infty. \quad (4)$$

It is readily seen that the set of all α -Bloch maps of $h(\mathbb{D})$ is a Banach space $\mathcal{B}^\alpha$ with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$.

Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be an increasing function with $\omega(0) = 0$; we say that ω is a *majorant* if $\omega(t)/t$ is nonincreasing for $t > 0$ (cf. [2]). Given a majorant ω and $\alpha > 0$, the ω - α -Bloch space $\mathcal{B}_\omega^\alpha$ consists of all maps $f \in h(\mathbb{D})$ such that

$$\|f\|_{\omega, \alpha} = \sup_{z \in \mathbb{D}} \omega((1 - |z|^2)^\alpha) |f_x(z)| < \infty \quad (5)$$

and the *little ω - α -Bloch space* $\mathcal{B}_{\omega, 0}^\alpha$ consists of the functions $f \in \mathcal{B}_\omega^\alpha$ such that

$$\lim_{|z| \rightarrow 1^-} \omega((1 - |z|^2)^\alpha) |f_x(z)| = 0. \quad (6)$$

In particular, when $n = 2$, $\omega(t) = t$, we remark that the space $\mathcal{B}_\omega^\alpha$ can be regarded as the holomorphic α -Bloch space $\mathcal{B}_h^\alpha$ (cf. [3]).

For each $a \in \mathbb{D}$, the Möbius transformation $\varphi_a : \mathbb{D} \rightarrow \mathbb{D}$ is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}. \quad (7)$$

If $a, z \in \mathbb{D}$ and $r \in (0, 1)$, we define the *pseudohyperbolic disk* with center a and radius r as

$$E(a, r) = \{z \in \mathbb{D} : \rho(a, z) = |\varphi_a(z)| < r\}. \quad (8)$$

It is known that $E(a, r)$ is a Euclidean disk with center at $(1 - r^2)a/(1 - |a|^2r^2)$ and the radius $(1 - |a|^2r)/(1 - |a|^2r^2)$.

Recall that, for $0 < \alpha \leq 1$, the weighted hyperbolic metric ds_α of $\mathbb{D}$, introduced in [4], is defined as

$$ds_\alpha^2 = \frac{|dz|^2}{(1 - |z|^2)^{2\alpha}}. \quad (9)$$

Suppose that $\gamma(t)$ ($0 \leq t \leq 1$) is a continuous and piecewise smooth curve in $\mathbb{D}$. Then, the length of $\gamma(t)$ with respect to the weighted hyperbolic metric ds_α is given by

$$L_{h_\alpha}(\gamma) = \int_\gamma ds_\alpha = \int_0^1 \frac{|\gamma'(t)|}{[(1 - |\gamma(t)|^2)]^\alpha} dt. \quad (10)$$

Consequently, the associated distance between z and w in $\mathbb{D}$ is

$$h_\alpha(z, w) = \inf \{L_{h_\alpha}(\gamma) : \gamma(0) = z, \gamma(1) = w\}, \quad (11)$$

where γ is a continuous and piecewise smooth curve in $\mathbb{D}$. Note that h_1 ($\alpha = 1$) is the hyperbolic distance h on $\mathbb{D}$. For more information on weighted hyperbolic metric ds_α , see [4, 5].

Let $s, t \geq 0$ and f be a continuous function in $\mathbb{D}$. If there exists a constant C such that

$$(1 - |z|^2)^s (1 - |w|^2)^t |f(z) - f(w)| \leq C |z - w| \quad (\text{resp. } \leq Ch(z, w)), \quad (12)$$

for any $z, w \in \mathbb{D}$, then we say that f is a weighted Euclidian (resp., hyperbolic) Lipschitz function of indices (s, t) . In particular, when $s = t = 0$, we say that f is a Euclidian (resp., hyperbolic) Lipschitz function (cf. [6]).

The relationship between Bloch spaces and (weighted) Lipschitz functions has attracted much attention in recent years. Holland and Walsh [7] characterized holomorphic Bloch space in $\mathbb{D}$ in terms of weighted Euclidian Lipschitz functions of indices $(1/2, 1/2)$. In [8–10], the authors extended it to the holomorphic Bloch-type spaces in the unit ball of $\mathbb{C}^n$. In [11], Zhu proved that a holomorphic function belongs to Bloch space if and only if it is hyperbolic Lipschitz. For the related results of harmonic functions, we refer to [6, 12–16] and the references therein. Recently, Huang and Wulan [3] considered the corresponding problems in the setting of Bloch minimal surface and established some analogous characterizations for Bloch minimal surfaces in terms of weighted Lipschitz functions. As the first aim of this paper, we consider the similar results of the abovementioned type for Bloch-type spaces of minimal surfaces. Our results in this line read as follows.

Theorem 1. *Let $f \in h(\mathbb{D})$ and $\alpha > 0$. Then, $f \in \mathcal{B}^\alpha$ if and only if there is a constant $C > 0$ such that*

$$|f(z) - f(w)| \leq Ch_\alpha(z, w), \quad z, w \in \mathbb{D}. \quad (13)$$

Moreover,

$$\|f\|_\alpha = \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{h_\alpha(z, w)} \quad (14)$$

for all $f \in \mathcal{B}^\alpha$.

Theorem 2. *Let $f \in h(\mathbb{D})$, $0 < \beta < 1$, $\beta \leq \alpha < 1 + \beta$, and $0 \leq \lambda \leq 1$. Then, the following are equivalent:*

- (i) $f \in \mathcal{B}_\omega^\alpha$;
- (ii) There exists a constant $C > 0$ such that

$$\sup_{z, w \in \mathbb{D}, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha - \beta} \right) \cdot \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1 - \lambda}} < \infty; \quad (15)$$

- (iii) There exists a constant $C > 0$ such that, for all $r \in (0, 1)$,

$$\begin{aligned} & \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(z) - f(w)| dA(w) \\ & \leq \frac{C(1 - |z|^2)}{\omega((1 - |z|^2)^\alpha)}, \end{aligned} \quad (16)$$

where $|E(z, r)|$ denotes the area of $E(w, r)$ and $dA(z)$ denotes the normalized Lebesgue area measure in $\mathbb{D}$.

Remark 3. If we take $\alpha = 1$, $\beta = 1/2$, $\lambda = 1$, and $\omega(t) = t$, then Theorems 1 and 2 coincide with Theorems 1 and 5 in [3], respectively.

Let ϕ be a holomorphic self-mapping of $\mathbb{D}$. The composition operator C_ϕ , induced by ϕ , is defined by $C_\phi(f) = f \circ \phi$ for $f \in h(\mathbb{D})$. During the past few years, composition operators have been studied extensively on spaces of holomorphic functions on various domains in $\mathbb{C}$ and $\mathbb{C}^n$ (see, e.g., [17–19]). As the second aim of this paper, we also discuss the boundedness of composition operators between Bloch-type spaces of minimal surfaces.

Theorem 4. *Let $\alpha, \beta > 0$ and ϕ be a holomorphic self-mapping of $\mathbb{D}$. Then, $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha} < \infty. \quad (17)$$

This paper is organized as follows. In Section 2, we shall prove Theorems 1 and 2. The proof of Theorem 4 will be presented in Section 3.

Throughout this paper, constants are denoted by C ; they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Proofs of Theorems 1 and 2

In order to prove the main results, we need some lemmas. The following lemma is proved in [11].

Lemma 5. Let $r \in (0, 1)$, $w \in E(a, r)$. Then, we have

$$(1 - |a|^2) \asymp (1 - |w|^2) \asymp |1 - \bar{a}w|. \quad (18)$$

Lemma 6 (see [15]). Let $\omega(t)$ be a majorant and $p \in (0, 1]$, $q \in (1, \infty)$. Then, for $t \in (0, \infty)$,

$$\begin{aligned} \omega(pt) &\geq p\omega(t), \\ \omega(qt) &\leq q\omega(t). \end{aligned} \quad (19)$$

As applications of Lemmas 5 and 6, we have the following.

Lemma 7. Let $r \in (0, 1)$, $w \in E(a, r)$, and $\omega(t)$ be a majorant. Then,

$$\omega((1 - |a|^2)) \asymp \omega((1 - |w|^2)). \quad (20)$$

In this section, we come to prove Theorems 1 and 2.

Proof of Theorem 1. First, we show the “if” part. For any $z, w \in \mathbb{D}$, from the definition of $h_\alpha(z, w)$, we assume that $\gamma(s)$ is the geodesic between z and w (parametrized by arc-length) with respect to h_α , that is, $\gamma(0) = z$, $\gamma(s) = w$. Since $h_\alpha(\gamma(0), \gamma(s)) = s$, we have

$$|f(z) - f(w)| \leq Cs. \quad (21)$$

Dividing both sides by s and then letting $s \rightarrow 0$ in the abovementioned inequality give

$$|f_x(z) \parallel \gamma'(0)| \leq C. \quad (22)$$

From the minimal length property of geodesics,

$$h_\alpha(\gamma(0), \gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha} dt = s, \quad (23)$$

$$0 < s < \epsilon,$$

we obtain that

$$\lim_{s \rightarrow 0} \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha} dt = \frac{|\gamma'(0)|}{(1 - |z|^2)^\alpha} = 1. \quad (24)$$

It follows that $(1 - |z|^2)^\alpha |f_x(z)| \leq C$ and hence $f \in \mathcal{B}^\alpha$ with

$$\begin{aligned} &(1 - |z|^2)^\alpha |f_x(z)| : z \in \mathbb{D} \} \\ &\leq \sup \left\{ \frac{|f(z) - f(w)|}{h_\alpha(z, w)} : z \neq w \right\}. \end{aligned} \quad (25)$$

Next, we prove the “only if” part. Assume that $f \in \mathcal{B}^\alpha$. Let $z, w \in \mathbb{D}$ and $\gamma(t)$ ($0 \leq t \leq 1$) be a smooth curve from z to w . Then, by

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_0^1 \frac{df}{dt}(\gamma(t)) dt \right| \\ &\leq \int_0^1 |f_x(\gamma(t))| |\gamma'(t)| dt \\ &\leq \|f\|_\alpha \int_0^1 \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^\alpha} dt \\ &\leq \|f\|_\alpha h_\alpha(\gamma(t)), \end{aligned} \quad (26)$$

taking the infimum over all piecewise continuous curves connecting z and w , we conclude that

$$|f(z) - f(w)| \leq \|f\|_\alpha h_\alpha(z, w) \quad (27)$$

for all $z, w \in \mathbb{D}$. This completes the proof. $\square$

Proof of Theorem 2 ((i) $\implies$ (ii)). Following [1], for $z, w \in \mathbb{D}$ and $\gamma(t) = tw + (1 - t)z$ ($0 \leq t \leq 1$), we have

$$\begin{aligned} |f(z) - f(w)| &\leq \int_0^1 |f_x(\gamma(t))| |\gamma'(t)| dt \leq C |z - w| \\ &\cdot \int_0^1 \frac{ds}{\omega((1 - |sw + (1 - s)z|^2)^\alpha)} \leq C |z - w|^\lambda \\ &\cdot |1 - \bar{z}w|^{1-\lambda} \int_0^1 \frac{ds}{\omega((1 - |sw + (1 - s)z|^2)^\alpha)}. \end{aligned} \quad (28)$$

Since for $z, w \in \mathbb{D}$ and $s \in [0, 1]$

$$\begin{aligned} (1 - |sw + (1 - s)z|^2)^\alpha &\geq (1 - |sw + (1 - s)z|)^\alpha \\ &\geq \left(s \left(\frac{1 - |w|^2}{2} \right) + (1 - s) \left(\frac{1 - |z|^2}{2} \right) \right)^\alpha \\ &\geq \left(\frac{s}{2} \right)^{\alpha-\beta} \left(\frac{1-s}{2} \right)^\beta (1 - |w|^2)^{\alpha-\beta} (1 - |z|^2)^\beta, \end{aligned} \quad (29)$$

it follows from Lemma 6 that

$$\begin{aligned} \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} &\leq C \int_0^1 \frac{ds}{\omega((1 - |sz + (1 - s)w|^2)^\alpha)} \\ &\leq C \int_0^1 \frac{ds}{\omega((s/2)^{\alpha-\beta} ((1 - s)/2)^\beta (1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta})} \\ &\leq \frac{C}{\omega((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta})} \int_0^1 \frac{ds}{s^{\alpha-\beta} (1 - s)^\beta} \\ &\leq \frac{C}{\omega((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta})}, \end{aligned} \quad (30)$$

where the last integral converges since $\alpha < 1 + \beta$. Thus,

$$\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} < \infty. \quad (31)$$

This implies that (i) $\implies$ (ii) holds.

(ii) $\implies$ (iii). Suppose that $f \in h(\mathbb{D})$ and

$$K = \sup_{z, w \in \mathbb{D}, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \cdot \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} < \infty. \quad (32)$$

By Lemma 7,

$$\begin{aligned} & \int_{E(z, r)} |f(w) - f(z)| dA(w) \\ & \leq K \int_{E(z, r)} \frac{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}}{\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)} dA(w) \\ & \leq C \int_{E(z, r)} \frac{1 - |z|^2}{\omega \left((1 - |z|^2)^\alpha \right)} dA(w), \end{aligned} \quad (33)$$

which leads to

$$\begin{aligned} & \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(z) - f(w)| dA(w) \\ & \leq \frac{C(1 - |z|^2)}{\omega \left((1 - |z|^2)^\alpha \right)}. \end{aligned} \quad (34)$$

(iii) $\implies$ (i) follows the following inequality (see [3]):

$$\begin{aligned} & |f_x(z)| (1 - |z|^2) \\ & \leq \frac{C}{|E(z, r)|} \int_{E(z, r)} |f(w) - f(z)| dA(w). \end{aligned} \quad (35)$$

This completes the proof. $\square$

By adding a restriction $w \in E(z, r)$, we characterize the space $\mathcal{B}_\omega^\alpha$ in terms of $\omega((1 - |z|^2)^\alpha)(|f(z) - f(w)|/|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda})$ as follows.

Theorem 8. Let $r \in (0, 1)$, $0 \leq \lambda \leq 1$ and $f \in h(\mathbb{D})$. Then, $f \in \mathcal{B}_\omega^\alpha$ if and only if

$$M = \sup_{w \in E(z, r), z \neq w} \omega \left((1 - |z|^2)^\alpha \right) \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} < \infty. \quad (36)$$

Proof (sufficiency). For $z \in \mathbb{D}$, by Lemma 5 and (35), we have

$$|f_x(z)| \leq \frac{C}{|E(z, r)|} \int_{E(z, r)} \frac{|f(w) - f(z)|}{(1 - |z|^2)} dA(w) \quad (37)$$

$$\leq \frac{C}{|E(z, r)|} \int_{E(z, r)} \frac{|f(w) - f(z)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} dA(w). \quad (38)$$

This gives

$$|f_x(z)| \leq \frac{CM}{\omega \left((1 - |z|^2)^\alpha \right)}, \quad (39)$$

from which we see that $f \in \mathcal{B}_\omega^\alpha$.

Conversely, let $f \in \mathcal{B}_\omega^\alpha$ and for any $w \in E(z, r)$, $z \neq w$, from the proof of Theorem 2, we have

$$\begin{aligned} & |f(z) - f(w)| \\ & \leq C |z - w| \int_0^1 \frac{ds}{\omega \left((1 - |sw + (1 - s)z|^2)^\alpha \right)} \\ & \leq \frac{C |z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}}{\omega \left((1 - |z|^2)^\alpha \right)}. \end{aligned} \quad (40)$$

Thus,

$$\sup_{w \in E(z, r), z \neq w} \omega \left((1 - |z|^2)^\alpha \right) \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} < \infty. \quad (41)$$

The proof of Theorem 8 is completed. $\square$

A similar result is also true for the little Bloch-type spaces $\mathcal{B}_{\omega, 0}^\alpha$.

Theorem 9. Let $r \in (0, 1)$, $0 \leq \lambda \leq 1$ and $f \in \mathcal{B}_\omega^\alpha$. Then, $f \in \mathcal{B}_{\omega, 0}^\alpha$ if and only if

$$\lim_{|z| \rightarrow 1^-} \sup_{w \in E(z, r), z \neq w} \omega \left((1 - |z|^2)^\alpha \right) \frac{|f(z) - f(w)|}{|z - w|^\lambda |1 - \bar{z}w|^{1-\lambda}} = 0. \quad (42)$$

The proof is almost the same as the one of Theorem 3.2 in [9]. Thus, we omit it here.

Remark 10. When $\omega(t) = t$, Li and Wulan [8] obtained the analogues of Theorems 3.7 and 3.8 for holomorphic Bloch space on the unit ball of $\mathbb{C}^n$.

3. Proof of Theorem 4

In order to prove Theorem 4, we need the following lemma which comes from [17].

Lemma 11. Let $\alpha, \beta > 0$ and ϕ be a holomorphic self-mapping of $\mathbb{D}$. Then, the composition operator $C_\phi : \mathcal{B}_h^\alpha \rightarrow \mathcal{B}_h^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha} < \infty, \quad (43)$$

where $\mathcal{B}_h^\alpha$ and $\mathcal{B}_h^\beta$ are holomorphic α -Bloch space and holomorphic β -Bloch space, respectively.

Proof of Theorem 4. Suppose that

$$L = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha}. \quad (44)$$

For $f \in \mathcal{B}^\alpha$ and $z \in \mathbb{D}$, we have

$$\begin{aligned} (1 - |z|^2)^\beta (|C_\phi(f)_x|) &= (1 - |z|^2)^\beta |(f \circ \phi)_x(z)| \\ &= \lim_{w \rightarrow z} (1 - |z|^2)^\beta \frac{|(f \circ \phi)(z) - (f \circ \phi)(w)|}{|z - w|} \\ &= \lim_{w \rightarrow z} (1 - |z|^2)^\beta \frac{|f(\phi(z)) - f(\phi(w))|}{|\phi(z) - \phi(w)|} \\ &\cdot \frac{|\phi(z) - \phi(w)|}{|z - w|} \leq L (1 - |\phi(z)|^2)^\alpha |f_x(\phi(z))| \\ &\leq C \|f\|_\alpha, \end{aligned} \quad (45)$$

and

$$|f(\phi(0))| \leq C \|f\|_{\mathcal{B}^\alpha}. \quad (46)$$

Hence, $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded.

For the converse, assume that $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is a bounded composition operator with

$$\|C_\phi(f)\|_{\mathcal{B}^\beta} \leq C \|f\|_{\mathcal{B}^\alpha} \quad (47)$$

for all $f \in \mathcal{B}^\alpha$. Let $g = u + iv \in \mathcal{B}_h^\beta$ and $h = (u, v, 0, \dots, 0) = (\text{Reg}, \text{Re}(-ig), 0, \dots, 0)$. Then, it is easy to check that $h \in \mathcal{B}^\alpha$ with an admissible system $(g, -ig, 0, \dots, 0)$.

From (3) and assumption (47), we have

$$\begin{aligned} \|C_\phi(g)\|_{\mathcal{B}_h^\beta} &\leq \sqrt{2} \|C_\phi(h)\|_{\mathcal{B}^\beta} \leq C \|h\|_{\mathcal{B}^\alpha} \\ &\leq C_1 \|g\|_{\mathcal{B}_h^\alpha}. \end{aligned} \quad (48)$$

Since g is an arbitrary element in $\mathcal{B}_h^\alpha$, we conclude that the composition operator $C_\phi : \mathcal{B}_h^\alpha \rightarrow \mathcal{B}_h^\beta$ is bounded. It follows from Lemma 11 that (17) holds.

Recall that the classical Schwarz-Pick Lemma in the unit disk gives the notion that, for a holomorphic self-mapping ϕ of $\mathbb{D}$, $(1 - |z|^2)|\phi'(z)| \leq 1 - |\phi(z)|^2$ holds for all $z \in \mathbb{D}$. As an application of this result, it is easy to derive the following corollary. $\square$

Corollary 12. *Let ϕ be a holomorphic self-mapping of $\mathbb{D}$. Then, $C_\phi : \mathcal{B}^1 \rightarrow \mathcal{B}^1$ is bounded.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors read and approved the final manuscript.

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