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Research Article

Common Fixed Point Results for Generalized α_{s^p} Contractive Mappings and Applications

Kastriot Zoto [1] **and Ilir Vardhami** [2]

[1] *Department of Mathematics and Computer Sciences, Faculty of Natural Sciences, University of Gjirokastra, Gjirokastra, Albania*
[2] *Department of Mathematics, Faculty of Natural Sciences, University of Tirana, Tirana, Albania*

Correspondence should be addressed to Kastriot Zoto; zotokastriot@yahoo.com

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A class of $g - \alpha_{s^p}$-admissible mappings and general type $g - \alpha_{s^p}$ contraction mappings on b-metric-like space are defined. Some fixed point results dealing with such a class of contractions are obtained. The generalized contractions considered in this work cover and unify many particular types of contractions. Finally, we present application to the existence of solutions for a system of integral equations by means of b-metric-like spaces.

1. Introduction

The study of new classes of spaces and their basic properties are always favorite topics of interest. Recently some authors have introduced some generalizations of metric spaces in several ways and have studied fixed point problems in these spaces, as well as their applications. In this context, Matthews [1] introduced the notion of partial metric space where self-distance of an arbitrary point needs not be equal to zero. Hitzler and Seda [2] and Amini-Harandi [3] made further generalizations under the name of dislocated, respectively, metric-like space. Further, Shukla et al. introduced in [4] the notion of $0 - \sigma$ complete metric space. This concept was further extended by Alghamdi et al. [5] under the name of b-metric-like spaces. This class of spaces has received significant attention lately.

Also these generalizations have been associated with new and generalized classes of contractive mappings. In this direction Samet et al. [6] introduced the concept of α-admissible and $\alpha - \psi$-contractive mappings, further extended to the (α, β)-contractive mappings, (α, ψ, ϕ)-contractive mappings, and $f - \alpha$-admissible mappings. Many papers dealing with the above notions have been considered to prove fixed point results (e.g., see [7–25]).

In this paper, following the above discussion, we introduce the notion of $g - \alpha_s$-admissible mapping and also introduce the concepts of $(g - \alpha_{s^p}, \lambda)$ quasi-contractive mappings and rational $(g - \alpha_{s^p}, \psi, \phi)$ contractive mappings in the larger framework of b-metric-like space. The considered contractive conditions not only generalize the known ones but also include and unify a huge number of existing results on the topic in the corresponding literature. Finally, we apply the given results to obtain existence of solutions of integral equations.

2. Preliminaries

Definition 1 (see [8]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, \infty)$ is called a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space with parameter s.

Definition 2 (see [26]). Let X be a nonempty set. A mapping $\sigma : X \times X \to [0, \infty)$ is called metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $\sigma(x, y) = 0$ implies $x = y$;

(ii) $\sigma(x, y) = \sigma(y, x)$;

(iii) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Definition 3 (see [5]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $\sigma_b : X \times X \to [0, \infty)$ is called b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $\sigma_b(x, y) = 0$ implies $x = y$;

(ii) $\sigma_b(x, y) = \sigma_b(y, x)$;

(iii) $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.

The pair (X, σ_b) is called a b-metric-like space.

In a b-metric-like space (X, σ_b), if $x, y \in X$ and $\sigma_b(x, y) = 0$, then $x = y$, but the converse needs not be true, and $\sigma_b(x, x)$ may be positive for $x \in X$.

Example 4 (see [18]). Let $X = [0, +\infty)$ and $m > 1$ be a constant. Define a function $\sigma_b : X^2 \to [0, \infty)$ by $\sigma_b(x, y) = (x + y)^m$ or $\sigma_b(x, y) = (\max\{x, y\})^m$. Then (X, σ_b) is a b-metric-like space with parameter $s = 2^{m-1}$. Clearly, (X, σ_b) is neither a b-metric, nor metric-like, nor partial b-metric space.

Definition 5 (see [5]). Let (X, σ_b) be a b-metric-like space with parameter s, and let $\{x_n\}$ be any sequence in X and $x \in X$. Then

(1) The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n\to\infty}\sigma_b(x_n, x) = \sigma_b(x, x)$;

(2) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, σ_b) if $\lim_{n,m\to\infty}\sigma_b(x_n, x_m)$ exists and is finite;

(3) (X, σ_b) is said to be a complete b-metric-like space if, for every Cauchy sequence $\{x_n\}$ in X, there exists an $x \in X$ such that $\lim_{n,m\to\infty}\sigma_b(x_n, x_m) = \lim_{n\to\infty}\sigma_b(x_n, x) = \sigma_b(x, x)$.

The limit of a sequence in a b-metric-like space need not be unique.

Proposition 6 (see [5]). *Let (X, σ_b) be a b-metric-like space with parameter s, and let $\{x_n\}$ be any sequence in X with $x \in X$ such that $\lim_{n\to\infty}\sigma_b(x_n, x) = 0$. Then*

(i) *x is unique,*

(ii) *$\sigma_b(x, y)/s \leq \lim_{n\to\infty}\sigma_b(x_n, y) \leq s\sigma_b(x, y)$ for all $y \in X$.*

In 2012, Samet et al. [6] introduced the class of α-admissible mappings.

Definition 7. Let X be a nonempty set, $f : X \to X$, and $\alpha : X \times X \to \mathbb{R}^+$ be mappings. We say that f is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies that $\alpha(fx, fy) \geq 1$, for all $x, y \in X$.

In 2014 a new notion of $g - \alpha$-admissible mapping was introduced by Rosa and Vetro [27].

Definition 8. For a nonempty set X, let $f, g : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be mappings. The mapping f is called $g - \alpha$-admissible if, for all $x, y \in X$ such that $\alpha(gx, gy) \geq 1$, we have $\alpha(fx, fy) \geq 1$.

Definition 9 (see [28]). Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function and arbitrary constants q, p such that $q \geq 1$ and $p \geq 1$. A self-mapping $f : X \to X$ is α_{qs^p}-admissible if $\alpha(x, y) \geq qs^p$ implies $\alpha(fx, fy) \geq qs^p$, for all $x, y \in X$.

Examples 3.3 and 3.4 in [28] illustrate Definition 9.

Lemma 10 (see [9]). *Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$ and $f : X \to X$ be a given mapping. Suppose that f is continuous at $u \in X$. Then for all sequence $\{x_n\}$ in X such that $x_n \to u$, we have $fx_n \to fu$; that is,*

$$\lim_{n\to\infty}\sigma_b(fx_n, fu) = \sigma_b(fu, fu). \tag{1}$$

Lemma 11. *Let $\{x_n\}$ be a sequence in a b-metric-like space (X, σ_b) with parameter $s \geq 1$, such that*

$$\sigma_b(x_n, x_{n+1}) \leq \lambda\sigma_b(x_{n-1}, x_n) \quad \forall n \in \mathbb{N}, \tag{2}$$

for some λ, where $0 \leq \lambda < 1/s$. Then

(1) $\lim_{n\to\infty}\sigma_b(x_n, x_{n+1}) = 0$,

(2) *$\{x_n\}$ is a Cauchy sequence in (X, σ_b) and $\lim_{n,m\to\infty}\sigma_b(x_n, x_m) = 0$.*

Proof. For the proof of the lemma, one can use the following clear inequalities:

$$\begin{aligned}
\sigma_b(x_{n+1}, x_{n+2}) &\leq \lambda\sigma_b(x_n, x_{n+1}) \leq \lambda^2\sigma_b(x_{n-1}, x_n) \\
&\leq \cdots \leq \lambda^{n+1}\sigma_b(x_0, x_1), \\
\sigma_b(x_m, x_n) &\leq s\sigma_b(x_m, x_{m+1}) + s^2\sigma_b(x_{m+1}, x_{m+2}) \\
&\quad + \cdots + s^{n-m-1}\sigma_b(x_{n-2}, x_{n-1}) \\
&\quad + s^{n-m}\sigma_b(x_{n-1}, x_n),
\end{aligned} \tag{3}$$

where $m, n \in \mathbb{N}$ and $n > m$. □

Lemma 12 (see [5]). *Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are σ_b-convergent to x and y, respectively. Then we have*

$$\begin{aligned}
&\frac{1}{s^2}\sigma_b(x, y) - \frac{1}{s}\sigma_b(x, x) - \sigma_b(y, y) \\
&\quad \leq \liminf_{n\to\infty}\sigma_b(x_n, y_n) \leq \limsup_{n\to\infty}\sigma_b(x_n, y_n) \\
&\quad \leq s\sigma_b(x, x) + s^2\sigma_b(y, y) + s^2\sigma_b(x, y).
\end{aligned} \tag{4}$$

In particular, if $\sigma_b(x, y) = 0$, then we have $\lim_{n\to\infty}\sigma_b(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\begin{aligned}\frac{1}{s}\sigma_b(x,z) - \sigma_b(x,x) &\le \liminf_{n\to\infty} \sigma_b(x_n,z) \\ &\le \limsup_{n\to\infty} \sigma_b(x_n,z) \\ &\le s\sigma_b(x,z) + s\sigma_b(x,x).\end{aligned} \tag{5}$$

In particular, if $\sigma_b(x,x) = 0$, then

$$\begin{aligned}\frac{1}{s}\sigma_b(x,z) &\le \liminf_{n\to\infty} \sigma_b(x_n,z) \le \limsup_{n\to\infty} \sigma_b(x_n,z) \\ &\le s\sigma_b(x,z).\end{aligned} \tag{6}$$

The following result is useful.

Lemma 13 (see [28]). *Let (X,σ_b) be a b-metric-like space with parameter $s \ge 1$. Then*

(1) *If $\sigma_b(x,y) = 0$, then $\sigma_b(x,x) = \sigma_b(y,y) = 0$;*

(2) *If $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} \sigma_b(x_n,x_{n+1}) = 0$, then we have*

$$\lim_{n\to\infty} \sigma_b(x_n,x_n) = \lim_{n\to\infty} \sigma_b(x_{n+1},x_{n+1}) = 0; \tag{7}$$

(3) *If $x \ne y$, then $\sigma_b(x,y) > 0$.*

Lemma 14 (see [29]). *Let (X,σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and let $\{x_n\}$ be a sequence such that*

$$\lim_{n\to\infty} \sigma_b(x_n,x_{n+1}) = 0. \tag{8}$$

If for the sequence $\{x_n\}$, $\lim_{n,m\to\infty}\sigma_b(x_n,x_m) \ne 0$, then there exist $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n_k > m_k > k$, such that

$$\begin{aligned}\varepsilon \le \sigma_b(x_{m_k},x_{n_k}) &\le \varepsilon s, \\ \sigma_b(x_{m_k},x_{n_k-1}) &< \varepsilon, \\ \frac{\varepsilon}{s^2} \le \limsup_{k\to\infty} \sigma_b(x_{m_k-1},x_{n_k-1}) &\le \varepsilon s, \\ \frac{\varepsilon}{s} \le \limsup_{k\to\infty} \sigma_b(x_{n_k-1},x_{m_k}) &\le \varepsilon, \\ \frac{\varepsilon}{s} \le \limsup_{k\to\infty} \sigma_b(x_{m_k-1},x_{n_k}) &\le \varepsilon s^2.\end{aligned} \tag{9}$$

3. Main Results

We begin this section with the following definition.

Definition 15. Let (X,σ_b) be a b-metric-like space with parameter $s \ge 1$, and let $f,g : X \to X$ and $\alpha : X\times X \to [0,\infty)$ be given mappings and arbitrary constant p such that $p \ge 2$. The mapping $f : X \to X$ is $g - \alpha_{s^p}$-admissible if, for all $x,y \in X$, $\alpha(gx,gy) \ge s^p$ implies $\alpha(fx,fy) \ge s^p$.

Remark 16. Taking $s = 1$ in definition we obtain an $g - \alpha$-admissible mapping defined in [27]. Taking $g = I_x$ as the identity mapping on X, we deduce the definition of $f - \alpha_{s^p}$-admissible mapping as in [28]. For $s = 1$ and $g = I_x$ the definition reduces to the definition of an α-admissible mapping in a metric space [6].

In the sequel, according to [28] we shall consider the following properties in case of $q = 1$.

Let (X,σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and $\alpha : X \times X \to [0,\infty)$ be a function. Then

(H_{s^p}) If $\{x_n\}$ is a sequence in X such that $gx_n \to gx \in X$ as $n \to \infty$ and $\alpha(gx_n,gx_{n+1}) \ge s^p$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n_k},gx) \ge s^p$ for all $k \in \mathbb{N}$.

(U_{s^p}) For all $u,v \in C(f,g)$, we have $\alpha(gu,gv) \ge s^p$ or $\alpha(gv,gu) \ge s^p$, where $C(f,g)$ denotes the set of all coincidence points of f and g.

Example 17. Let $X = [0,+\infty)$. We define the mappings $f,g : X \to X$ by $fx = e^x$; $gx = 1/2x^2$ for all $x \in X$, and $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} s^2 & x,y \in [0,1] \\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

Then f is $g - \alpha_{s^p}$-admissible.

Extending the well known definition of quasi-contraction from Cirić, we introduce the notion of a generalized $(\alpha_{s^p} - g,\lambda)$-quasi-contraction in the setting of a b-metric-like space.

Definition 18. Let (X,σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and $f,g : X \to X$ be given mappings. We say that f is a generalized $(\alpha_{s^p} - g,\lambda)$-quasi-contraction if f is a $g - \alpha_{s^p}$-admissible mapping satisfying

$$\begin{aligned}\alpha(gx,gy)\sigma_b(fx,fy) \le \lambda \max\{&\sigma_b(gx,gy), \\ &\sigma_b(gx,fx), \sigma_b(gy,fy), \sigma_b(gx,fy), \sigma_b(gy,fx), \\ &\sigma_b(gx,gx), \sigma_b(gy,gy)\}\end{aligned} \tag{11}$$

for all $x,y \in X$ and $\lambda \in [0,1/2)$.

Remark 19. If we take $\alpha(gx,gy) = s^2$ ($p = 2$), the definition reduces to the definition of an $s - \lambda$ quasi-contraction. If we take $s = 1$, the definition reduces to the λ-quasi-contraction in the setting of metric spaces.

Theorem 20. *Let (X,σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and f,g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0,+\infty[$ is a given mapping. Suppose that the following conditions are satisfied:*

(i) *f is $g - \alpha_{s^p}$-admissible mapping;*

(ii) *f is a generalized $(g - \alpha_{s^p},\lambda)$ contractive mapping;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(gx_0, fx_0) \geq s^p$;

(iv) *properties* H_{s^p} *and* U_{s^p} *are satisfied.*

Then f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. By hypothesis (iii), there exists an $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq s^p$. We define a sequence $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $gx_{n+1} = y_n = y_{n+1} = fx_{n+1}$ and f and g have a point of coincidence. Without loss of generality, one can suppose that $y_n \neq y_{n+1}$ (by Lemma 13; i.e., $\sigma_b(y_n, y_{n+1}) > 0$) for each $n \in \mathbb{N}$.

Since f is $g - \alpha_{s^p}$-admissible mapping, we have

$$\begin{aligned} \alpha(gx_0, gx_1) &= \alpha(gx_0, fx_0) \geq s^p, \\ \alpha(gx_1, gx_2) &= \alpha(fx_0, fx_1) \geq s^p, \\ \alpha(gx_2, gx_3) &= \alpha(fx_1, fx_2) \geq s^p. \end{aligned} \tag{12}$$

Hence, by induction, we get $\alpha(y_n, y_{n+1}) = \alpha(gx_n, gx_{n+1}) \geq s^p$ for all $n \in \mathbb{N}$.

By condition (11), we have

$$\begin{aligned} s^p\sigma_b(y_n, y_{n+1}) &= s^p\sigma_b(fx_n, fx_{n+1}) \leq \alpha(gx_n, gx_{n+1})\sigma_b(fx_n, fx_{n+1}) \leq \lambda \\ &\cdot \max\{\sigma_b(gx_n, gx_{n+1}), \sigma_b(gx_n, fx_n), \sigma_b(gx_{n+1}, fx_{n+1}), \sigma_b(gx_n, fx_{n+1}), \sigma_b(gx_{n+1}, fx_n), \sigma_b(gx_n, gx_n), \sigma_b(gx_{n+1}, gx_{n+1})\} \\ &= \lambda \max\{\sigma_b(y_{n-1}, y_n), \sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1}), \sigma_b(y_{n-1}, y_{n+1}), \sigma_b(y_n, y_n), \sigma_b(y_{n-1}, y_{n-1}), \sigma_b(y_n, y_n)\} \leq \lambda \\ &\cdot \max\{\sigma_b(y_{n-1}, y_n), \sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1}), s[\sigma_b(y_{n-1}, y_n) + \sigma_b(y_n, y_{n+1})], 2s\sigma_b(y_{n-1}, y_n), 2s\sigma_b(y_{n-1}, y_n), 2s\sigma_b(y_{n-1}, y_n)\} \\ &= \lambda \max\{\sigma_b(y_{n-1}, y_n), \sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1}), s[\sigma_b(y_{n-1}, y_n) + \sigma_b(y_n, y_{n+1})], 2s\sigma_b(y_{n-1}, y_n)\}. \end{aligned} \tag{13}$$

If $\sigma_b(y_{n-1}, y_n) < \sigma_b(y_n, y_{n+1})$ for some $n \in \mathbb{N}$, then, from inequality (13), we have

$$\sigma_b(y_n, y_{n+1}) \leq \frac{2\lambda}{s^{p-1}}\sigma_b(y_n, y_{n+1}), \tag{14}$$

a contradiction since $2\lambda/s^{p-1} < 1$.

Hence, for all $n \in \mathbb{N}$, $\sigma_b(y_n, y_{n+1}) \leq \sigma_b(y_{n-1}, y_n)$, and also by inequality (13), we get

$$\sigma_b(y_n, y_{n+1}) \leq \frac{2\lambda}{s^{p-1}}\sigma_b(y_{n-1}, y_n) = c\sigma_b(y_{n-1}, y_n), \tag{15}$$

where $0 \leq c = 2\lambda/s^{p-1} < 1/s$.

Then, by Lemma 11, we have

$$\lim_{n\to\infty}\sigma_b(y_n, y_{n+1}) = 0, \tag{16}$$

and the sequence $\{y_n\}$ is a Cauchy sequence, and $\lim_{n,m\to\infty}\sigma_b(y_n, y_m) = 0$. By completeness of (X, σ_b) the sequence $\{y_n\} = \{fx_n\} = \{gx_{n+1}\}$ converges to a point u. By hypothesis, since $g(X)$ is closed, then $u \in g(X)$. Therefore, there exists $z \in X$ such that $u = gz$. That is

$$\begin{aligned} 0 &= \lim_{n,m\to\infty}\sigma_b(y_n, y_m) = \lim_{n\to\infty}\sigma_b(y_n, gz) \\ &= \lim_{n\to\infty}\sigma_b(fx_n, gz) = \lim_{n\to\infty}\sigma_b(gx_{n+1}, gz) \\ &= \sigma_b(gz, gz). \end{aligned} \tag{17}$$

Since property H_{s^p} is satisfied, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_k-1}, gz) \geq s^p$ (that is $\alpha(gx_{n_k}, gz) \geq s^p$) for all $k \in \mathbb{N}$. If $fz \neq gz$, applying contractive condition (ii) of theorem, with $x = x_{n_k}$ and $y = u$, we obtain

$$\begin{aligned} s^p\sigma_b(fx_{n_k}, fz) &\leq \alpha(gx_{n_k}, gz)\sigma_b(fx_{n_k}, fz) \leq \lambda \\ &\cdot \max\{\sigma_b(gx_{n_k}, gz), \sigma_b(gx_{n_k}, fx_{n_k}), \sigma_b(gz, fz), \\ &\sigma_b(gx_{n_k}, fz), \sigma_b(gz, fx_{n_k}), \sigma_b(gx_{n_k}, gx_{n_k}), \\ &\sigma_b(gz, gz)\} = \lambda \max\{\sigma_b(y_{n_k-1}, gz), \\ &\sigma_b(y_{n_k-1}, y_{n_k}), \sigma_b(gz, fz), \sigma_b(y_{n_k-1}, fz), \\ &\sigma_b(gz, y_{n_k}), \sigma_b(y_{n_k-1}, y_{n_k-1}), \sigma_b(gz, gz)\}. \end{aligned} \tag{18}$$

Taking the upper limit as $k \to \infty$ in the above inequality, and using (16), (17), and Lemmas 12 and 13, we have

$$s^{p-1}\sigma_b(gz, fz) = s^p\frac{1}{s}\sigma_b(gz, fz) \leq 2\lambda s\sigma_b(gz, fz). \tag{19}$$

Hence, from above we get $\sigma_b(gz, fz) = 0$; that is, $fz = gz$ and $fz = gz = u$ is a point of coincidence of f and g. By using condition (11) and property U_{s^p} one can be convinced that the point of coincidence is unique and by weak compatibility of f and g it follows that z is a unique common fixed point. □

Example 21. Let $X = [0, +\infty)$ and $\sigma_b(x, y) = (x + y)^2$; for all $x, y \in X$ is a b-metric-like space with parameter $s = 2$. Define the mappings f and g on X by

$$fx = \begin{cases} \frac{1}{8}x & \text{for } x \in [0, 1] \\ 3x & \text{for } x > 1 \end{cases}$$

$$gx = \begin{cases} \frac{1}{2}x & \text{for } x \in [0,1] \\ \frac{3}{2}x & \text{for } x > 1, \end{cases}$$

$$\alpha : g(X) \times g(X) \longrightarrow [0, +\infty[$$

$$\text{by } \alpha(x,y) = \begin{cases} s^2 & x, y \in \left[0, \frac{1}{2}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

It is clear that $f(X) \subseteq g(X)$. For $x, y \in X$ such that $\alpha(gx, gy) \geq s^2$; then $gx, gy \in [0, 1/2]$ and this implies that $x, y \in [0, 1]$. By definitions we have $fx, fy \in [0, 1/2]$ and $\alpha(fx, fy) \geq s^2$; that is, f is $g - \alpha_{s^p}$-admissible mapping;

For $x, y \in [0, 1]$, we have

$$\begin{aligned} \alpha(gx, gy)\sigma_b(fx, fy) &= s^2\sigma_b(fx, fy) = 4\sigma_b\left(\frac{1}{8}x, \frac{1}{8}y\right) = 4\left(\frac{1}{8}x + \frac{1}{8}y\right)^2 = \frac{4}{16}\left(\frac{1}{2}x + \frac{1}{2}y\right)^2 = \frac{1}{4}\sigma_b(gx, gy) \\ &\leq \lambda \max\{\sigma_b(gx, gy), \sigma_b(gx, fx), \sigma_b(gy, fy), \sigma_b(gx, fy), \sigma_b(gy, fx), \sigma_b(gx, gx), \sigma_b(gy, gy)\}, \end{aligned} \tag{21}$$

where $1/4 \leq \lambda < 1/2$.

Obviously the other assumptions of theorem can be verified and $x = 0$ is the unique common fixed point of f and g.

Theorem 22. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$ and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0, +\infty[$ a given mapping. Suppose that the following conditions are satisfied:*

(i) *f is $g - \alpha_{s^p}$-admissible mapping;*

(ii) *for all $x, y \in X$, and constants $a_i \geq 0$, $i = 1, \ldots, 5$,*

$$\begin{aligned} &\alpha(gx, gy)\sigma_b(fx, fy) \\ &\quad \leq a_1\sigma_b(gx, gy) + a_2\sigma_b(gx, fx) + a_3\sigma_b(gy, fy) \\ &\qquad + a_4\sigma_b(gx, fy) + a_5\sigma_b(gy, fx), \end{aligned} \tag{22}$$

where $a_1 + a_2 + a_3 + a_4 + a_5 < 1/2$;

(iii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq s^p$;*

(iv) *properties H_{s^p} and U_{s^p} are satisfied.*

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. This theorem can be considered as a corollary of Theorem 20, since, for all $x, y \in X$, inequality (22) is a special case of (11). □

If we consider $\alpha(gx, gy) = s^2$ (where $p = 2$) in Theorem 20, then we deduce the following corollary.

Corollary 23. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X and satisfy the condition*

$$\begin{aligned} s^2\sigma_b(fx, fy) \leq \lambda \max\{&\sigma_b(gx, gy), \sigma_b(gx, fx), \\ &\sigma_b(gy, fy), \sigma_b(gx, fy), \sigma_b(gy, fx), \sigma_b(gx, gx), \\ &\sigma_b(gy, gy)\} \end{aligned} \tag{23}$$

for all $x, y \in X$ and $\lambda \in [0, 1/2)$.

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Remark 24. Theorem 20 generalizes Theorem 18 in [30]. For $\alpha(x, y) = s^2$ and, for each $x, y \in X$, Theorems 20 and 22 reduce to Theorems 3.2 and 3.13 of [31]. In Theorem 22, by choosing the constants a_i in certain manner, we obtain, as particular cases, certain classes of $g - \alpha_{s^p}$ types of classical contractions (such as Kannan, Chatterjea, Reich, and Zamfirescu contractions).

Contraction-type mappings have been generalized in several directions. A series of generalizations start with Samet et al. [6] with the concept of α-admissible mappings and α–ψ-contractive mapping. Later many authors used these classes of mappings under weakly and generalized weakly contractivity conditions and discussed fixed point results in various spaces.

The contractivity conditions considered in the second part of this section are constructed via auxiliary functions defined with the families Ψ, Φ, and $\mathbb{S}$, respectively:

$\psi : [0, \infty) \to [0, \infty)$ is an increasing and continuous function;

$\phi : [0, \infty) \to [0, \infty)$ is continuous function and $\phi(t) < \psi(t)$ for all $t > 0$;

$\beta : [0, \infty) \to [0, 1)$ satisfying the condition $\beta(t_n) \to 1$ as $n \to \infty$ implies that $t_n \to 0$ as $n \to \infty$.

Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$. For two self-mappings $f, g : X \to X$, we define the set $N(x, y)$ with the involvement of rational terms.

$$N(x,y) = \max\left\{\sigma_b(gx,gy), \sigma_b(gx,fx), \sigma_b(gy,fy), \frac{\sigma_b(gx,fy)+\sigma_b(gy,fx)}{4s}, \frac{\sigma_b(gx,fx)\sigma_b(gy,fy)}{1+\sigma_b(fx,fy)}, \frac{\sigma_b(gy,fy)[1+\sigma_b(gx,fx)]}{1+\sigma_b(gx,gy)}\right\} \tag{24}$$

for all $x, y \in X$.

We introduce now the notion of rational $(g - \alpha_{s^p}, \psi, \phi)$ contraction in the setting of b-metric-like spaces.

Definition 25. Let (X, σ_b) be a b-metric-like space with parameter $s \geq 1$ and $f, g : X \to X$ two self-mappings. Also, let $\alpha : X \times X \to [0, \infty)$ and $p \geq 2$. We say that f is called a generalized $(g - \alpha_{s^p}, \psi, \phi)$ contractive mapping, if there exist $\psi \in \Psi, \phi \in \Phi$ such that

$$\psi(\alpha(gx,gy)\sigma_b(fx,fy)) \leq \phi(N(x,y)) \tag{25}$$

for all $x, y \in X$ with $\alpha(x, y) \geq s^p$, where $N(x, y)$ is defined as in (24).

Remark 26. (1) If we take $g = I_X$ as the identity mapping on X, then we obtain the definition of $(\alpha_{s^p} - \psi, \phi)$ contractive mapping as in [28].

(2) Taking $s = 1$ in the Definition 25, we obtain $(g - \alpha - \psi, \phi)$-contractive mappings.

(3) For $s = 1$ and $g = I_X$ the definition reduces to the definition of an $(\alpha - \psi, \phi)$-contractive mapping.

(4) The definition reduces to a $(\psi, \phi, s - g)$-contractive mapping if we take $\alpha(gx, gy) = s^p$.

(5) The definition reduces to an $g - \alpha_{s^p} - \phi$ contractive mapping if we take $\psi(t) = t$.

Theorem 27. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \geq 1$ and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0, +\infty[$ a given mapping. Suppose that the following conditions are satisfied:*

(i) *f is $g - \alpha_{s^p}$-admissible mapping;*

(ii) *f is a generalized $(g - \alpha_{s^p} - \psi, \phi)$ contractive mapping;*

(iii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq s^p$;*

(iv) *properties H_{s^p} and U_{s^p} are satisfied.*

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. From the similar arguments as in proof of Theorem 20, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. Supposing that $y_n \neq y_{n+1}$ (which by Lemma 13 implies $\sigma_b(y_n, y_{n+1}) > 0$) for each $n \in \mathbb{N}$, we get

$$\alpha(y_n, y_{n+1}) = \alpha(gx_n, gx_{n+1}) \geq s^p. \tag{26}$$

By (26) and condition (25) we have

$$\begin{aligned}\psi(s^p\sigma_b(y_n, y_{n+1})) &= \psi(s^p\sigma_b(fx_n, fx_{n+1})) \\ &\leq \psi(\alpha(gx_n, gx_{n+1})\sigma_b(fx_n, fx_{n+1})) \\ &\leq \phi(N(x_n, x_{n+1})),\end{aligned} \tag{27}$$

where

$$\begin{aligned}N(x_n, x_{n+1}) &= \max\left\{\sigma_b(gx_n, gx_{n+1}), \sigma_b(gx_n, fx_n), \sigma_b(gx_{n+1}, fx_{n+1}), \frac{\sigma_b(gx_n, fx_{n+1})+\sigma_b(gx_{n+1}, fx_n)}{4s}, \frac{\sigma_b(gx_n, fx_n)\sigma_b(gx_{n+1}, fx_{n+1})}{1+\sigma_b(fx_n, fx_{n+1})}, \frac{\sigma_b(gx_{n+1}, fx_{n+1})[1+\sigma_b(gx_n, fx_n)]}{1+\sigma_b(gx_n, gx_{n+1})}\right\} \\ &= \max\left\{\sigma_b(y_{n-1}, y_n), \sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1}), \frac{\sigma_b(y_{n-1}, y_{n+1})+\sigma_b(y_n, y_n)}{4s}, \frac{\sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1})}{1+\sigma_b(y_n, y_{n+1})}, \frac{\sigma_b(y_n, y_{n+1})[1+\sigma_b(y_{n-1}, y_n)]}{1+\sigma_b(y_{n-1}, y_n)}\right\} \\ &\leq \max\left\{\sigma_b(y_{n-1}, y_n), \sigma_b(y_n, y_{n+1}), \frac{s[\sigma_b(y_{n-1}, y_n)+\sigma_b(y_n, y_{n+1})]+2s\sigma_b(y_{n-1}, y_n)}{4s}\right\}.\end{aligned} \tag{28}$$

If we assume that, for some $n \in \mathbb{N}$,

$$\sigma_b(y_{n-1}, y_n) < \sigma_b(y_n, y_{n+1}), \tag{29}$$

then, from inequality (28), we get

$$N(x_n, x_{n+1}) \leq \sigma_b(y_n, y_{n+1}). \tag{30}$$

Again, by (30) and using condition (27) and property of ψ, we obtain

$$\sigma_b(y_n, y_{n+1}) \leq N(x_n, x_{n+1}). \tag{31}$$

Hence as a result we have

$$N(x_n, x_{n+1}) = \sigma_b(y_n, y_{n+1}). \tag{32}$$

Then by (27) using (32) we obtain

$$\psi(s^p\sigma_b(y_n, y_{n+1})) \leq \phi(\sigma_b(y_n, y_{n+1})) \tag{33}$$

which gives a contradiction, since we have assumed that $\sigma_b(x_n, x_{n+1}) > 0$ and property $\phi(t) < \psi(t)$ for all $t > 0$. So $\sigma_b(y_n, y_{n+1}) \leq \sigma_b(y_{n-1}, y_n)$, for all $n \in \mathbb{N}$. Hence, the sequence of nonnegative numbers $\{\sigma_b(y_n, y_{n+1})\}$ is nonincreasing. Thus it converges to a nonnegative number, say $r \geq 0$. That is $\lim_{n\to\infty}\sigma_b(y_n, y_{n+1}) = r$ and also $\lim_{n\to\infty}\sigma_b(y_n, y_{n+1}) = \lim_{n\to\infty}N(x_n, x_{n+1}) = r$. If $r > 0$, then letting $n \to +\infty$ in (27) we get $\psi(s^p r) \leq \phi(r)$ which implies $r = 0$; that is,

$$\lim_{n\to\infty}\sigma_b\left(y_n, y_{n+1}\right) = \lim_{n\to\infty}N\left(x_n, x_{n+1}\right) = 0. \tag{34}$$

Now for the sequence $\{y_n\}$ we shall show that $\lim_{n,m\to\infty}\sigma_b(y_n, y_m) = 0$. Suppose, on the contrary, that $\lim_{n,m\to\infty}\sigma_b(y_n, y_m) \neq 0$. Then, using Lemma 14, we get that there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$, with $n_k > m_k > k$, such that the following hold:

$$\begin{aligned}
&\sigma_b\left(y_{m_k}, y_{n_k}\right) \geq \varepsilon,\\
&\sigma_b\left(y_{m_k}, y_{n_k-1}\right) < \varepsilon;\\
&\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty}\sigma_b\left(y_{m_k-1}, y_{n_k-1}\right) \leq \varepsilon s;\\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty}\sigma_b\left(y_{n_k-1}, y_{m_k}\right) \leq \varepsilon;\\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty}\sigma_b\left(y_{m_k-1}, y_{n_k}\right) \leq \varepsilon s^2
\end{aligned} \tag{35}$$

From the definition of $N(x, y)$, we have

$$\begin{aligned}
&N\left(x_{m_k-1}, x_{n_k-1}\right) = N\left(x_{m_k}, x_{n_k}\right) = N\left(gx_{m_k}, gx_{n_k}\right)\\
&= \max\left\{\sigma_b\left(gx_{m_k}, gx_{n_k}\right), \sigma_b\left(gx_{m_k}, fx_{m_k}\right), \sigma_b\left(gx_{n_k}, fx_{n_k}\right), \frac{\sigma_b\left(gx_{m_k}, fx_{n_k}\right) + \sigma_b\left(gx_{n_k}, fx_{m_k}\right)}{4s}, \frac{\sigma_b\left(gx_{m_k}, fx_{m_k}\right)\sigma_b\left(gx_{n_k}, fx_{n_k}\right)}{1 + \sigma_b\left(fx_{m_k}, fx_{n_k}\right)}, \frac{\sigma_b\left(gx_{n_k}, fx_{n_k}\right)\left[1 + \sigma_b\left(gx_{m_k}, fx_{m_k}\right)\right]}{1 + \sigma_b\left(gx_{m_k}, gx_{n_k}\right)}\right\}\\
&= \max\left\{\sigma_b\left(y_{m_k-1}, y_{n_k-1}\right), \sigma_b\left(y_{m_k-1}, y_{m_k}\right), \sigma_b\left(y_{n_k-1}, y_{n_k}\right), \frac{\sigma_b\left(y_{m_k-1}, y_{n_k}\right) + \sigma_b\left(y_{n_k-1}, y_{m_k}\right)}{4s} \frac{\sigma_b\left(y_{m_k-1}, y_{m_k}\right)\sigma_b\left(y_{n_k-1}, y_{n_k}\right)}{1 + \sigma_b\left(y_{m_k}, y_{n_k}\right)}, \frac{\sigma_b\left(y_{n_k-1}, y_{n_k}\right)\left[1 + \sigma_b\left(y_{m_k-1}, y_{m_k}\right)\right]}{1 + \sigma_b\left(y_{m_k-1}, y_{n_k-1}\right)}\right\}.
\end{aligned} \tag{36}$$

Taking the upper limit as $k \to \infty$ in (36), and using (34), (35), we get

$$\begin{aligned}
&\limsup_{k\to\infty} N\left(x_{m_k-1}, x_{n_k-1}\right)\\
&\quad \leq \max\left\{\varepsilon s, 0, 0, \frac{\varepsilon s^2 + \varepsilon}{4s}, 0, 0\right\} = \varepsilon s.
\end{aligned} \tag{37}$$

Applying (25) with $x = x_{n_k}$ and $y = x_{m_k}$, we obtain

$$\begin{aligned}
\psi\left(\sigma_b\left(y_{m_k}, y_{n_k}\right)\right) &= \psi\left(\sigma_b\left(fx_{m_k}, fx_{n_k}\right)\right)\\
&\leq \psi\left(s^p\sigma_b\left(fx_{m_k}, fx_{n_k}\right)\right)\\
&\leq \psi\left(\alpha\left(gx_{m_k}, gx_{n_k}\right)\sigma_b\left(fx_{m_k}, fx_{n_k}\right)\right)\\
&\leq \phi\left(N\left(x_{m_k}, x_{n_k}\right)\right).
\end{aligned} \tag{38}$$

Taking the upper limit in (38), using (35) and (37), we obtain

$$\begin{aligned}
\psi(\varepsilon s) &\leq \psi\left(\varepsilon s^{p-1}\right) = \psi\left(s^p\frac{\varepsilon}{s}\right)\\
&\leq \psi\left(s^p\limsup_{k\to\infty}\sigma_b\left(fx_{m_k}, fx_{n_k}\right)\right)\\
&\leq \phi\left(\limsup_{k\to\infty}\left(N\left(x_{m_k-1}, x_{n_k-1}\right)\right)\right) \leq \phi(\varepsilon s),
\end{aligned} \tag{39}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Hence, $\lim_{n,m\to\infty}\sigma_b(y_n, y_m) = 0$; that is, $\{y_n\}$ is a Cauchy sequence in X. From the completeness of (X, σ_b), there exists $u \in X$ such that

$$\begin{aligned}
\lim_{n\to\infty}\sigma_b\left(y_n, u\right) &= \lim_{n\to\infty}\sigma_b\left(fx_n, u\right) = \lim_{n\to\infty}\sigma_b\left(gx_{n+1}, u\right)\\
&= \lim_{n,m\to\infty}\sigma_b\left(y_n, y_m\right) = \sigma_b(u, u) = 0.
\end{aligned} \tag{40}$$

By hypothesis, since $g(X)$ is closed, by (40), $u \in g(X)$. Therefore, there exists $z \in X$ such that $u = gz$. And (40) can be written as

$$\begin{aligned}
0 &= \lim_{n,m\to\infty}\sigma_b\left(y_n, y_m\right) = \lim_{n\to\infty}\sigma_b\left(y_n, gz\right)\\
&= \lim_{n\to\infty}\sigma_b\left(fx_n, gz\right) = \lim_{n\to\infty}\sigma_b\left(gx_{n+1}, gz\right)\\
&= \sigma_b\left(gz, gz\right).
\end{aligned} \tag{41}$$

Since property H_{s^p} is satisfied there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_k-1}, gz) \geq s^p$ (that is $\alpha(gx_{n_k}, gz) \geq s^p$) for all $k \in \mathbb{N}$. If $fz \neq gz$, applying contractive condition (26), with $x = x_{n_k}$ and $y = z$, we obtain

$$\begin{aligned}
\psi\left(s^p\sigma_b\left(y_{n_k}, fz\right)\right) &= \psi\left(s^p\sigma_b\left(fx_{n_k}, fz\right)\right)\\
&\leq \psi\left(\alpha\left(gx_{n_k}, gz\right)\sigma_b\left(fx_{n_k}, fz\right)\right)\\
&\leq \phi\left(N\left(x_{n_k}, z\right)\right)\\
&= \phi\left(N\left(gx_{n_k}, gz\right)\right),
\end{aligned} \tag{42}$$

where

$$\begin{aligned} N\left(x_{n_k}, z\right) = N\left(gx_{n_k}, gz\right) = \max\Bigg\{&\sigma_b\left(gx_{n_k}, gz\right), \\ &\sigma_b\left(gx_{n_k}, fx_{n_k}\right), \sigma_b\left(gz, fz\right), \\ &\frac{\sigma_b\left(gx_{n_k}, fz\right) + \sigma_b\left(gz, fx_{n_k}\right)}{4s}, \\ &\frac{\sigma_b\left(gx_{n_k}, fx_{n_k}\right)\sigma_b\left(gz, fz\right)}{1+\sigma_b\left(fx_{n_k}, fz\right)} \\ &\cdot \frac{\sigma_b\left(gz, fz\right)\left[1+\sigma_b\left(gx_{n_k}, fx_{n_k}\right)\right]}{1+\sigma_b\left(gx_{n_k}, gz\right)}\Bigg\} \\ = \max\Bigg\{&\sigma_b\left(y_{n_k-1}, gz\right), \sigma_b\left(y_{n_k-1}, y_{n_k}\right), \sigma_b\left(gz, fz\right), \\ &\frac{\sigma_b\left(y_{n_k-1}, fz\right) + \sigma_b\left(gz, y_{n_k}\right)}{4s}, \\ &\frac{\sigma_b\left(y_{n_k-1}, y_{n_k}\right)\sigma_b\left(gz, fz\right)}{1+\sigma_b\left(y_{n_k}, fz\right)}, \\ &\frac{\sigma_b\left(gz, fz\right)\left[1+\sigma_b\left(y_{n_k-1}, y_{n_k}\right)\right]}{1+\sigma_b\left(y_{n_k-1}, gz\right)}\Bigg\}. \end{aligned} \tag{43}$$

Taking the upper limit in (43) and using Lemma 13 and (40), we obtain

$$\begin{aligned} \limsup_{n\to\infty} N\left(x_{n_k}, u\right) \le \max\Bigg\{&0, 0, \sigma_b\left(gz, fz\right), \\ &\frac{s\sigma_b\left(gz, fz\right) + 0}{4s}, 0, \frac{\sigma_b\left(gz, fz\right)[1+0]}{1+0}\Bigg\} = \sigma_b\left(gz, \right. \\ &\left. fz\right). \end{aligned} \tag{44}$$

Taking the upper limit as $k \to \infty$ in (42), and using (44) and Lemma 13, we obtain

$$\begin{aligned} \psi\left(s^{p-1}\sigma_b\left(gz, fz\right)\right) &= \psi\left(s^p \frac{1}{s}\sigma_b\left(gz, fz\right)\right) \\ &\le \psi\left(s^p \limsup_{k\to\infty} \sigma_b\left(fx_{n_k}, fz\right)\right) \\ &\le \phi\left(\limsup_{k\to\infty} N\left(x_{n_k}, z\right)\right) \\ &\le \phi\left(\sigma_b\left(gz, fz\right)\right). \end{aligned} \tag{45}$$

In view of property of ψ, ϕ from (45), we get $\sigma_b(gz, fz) = 0$ which implies that $fz = gz$. Hence, $u = fz = gz$ is a point of coincidence for f and g. Similarly as in Theorem 20 by using condition (26) and property U_{s^p} and weak compatibility it can be shown that z is a unique common fixed point. □

By taking $\phi(t) = \psi(t) - \varphi(t)$, where $\varphi \in \Psi$, in Theorem 27, we obtain the following result.

Corollary 28. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0, +\infty[$ a given mapping. Suppose that the following conditions are satisfied:*

(i) *f is $g - \alpha_{s^p}$-admissible mapping;*

(ii) *there exist functions $\psi, \varphi \in \Psi$ such that*

$$\begin{aligned} &\psi\left(\alpha\left(gx, gy\right)\sigma_b\left(fx, fy\right)\right) \\ &\quad \le \psi\left(N\left(x, y\right)\right) - \varphi\left(N\left(x, y\right)\right); \end{aligned} \tag{46}$$

(iii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \ge s^p$;*

(iv) *properties H_{s^p} and U_{s^p} are satisfied.*

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 29. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0, +\infty[$ a given mapping. Suppose that the following conditions are satisfied:*

(i) *f is $g - \alpha_{s^p}$-admissible mapping;*

(ii) *there exist functions $\psi \in \Psi, \phi \in \Phi, \beta \in \mathbb{S}$ such that*

$$\begin{aligned} &\psi\left(\alpha\left(gx, gy\right)\sigma_b\left(fx, fy\right)\right) \\ &\quad \le \beta\left(N\left(x, y\right)\right)\phi\left(N\left(x, y\right)\right); \end{aligned} \tag{47}$$

(iii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \ge s^p$;*

(iv) *properties H_{s^p} and U_{s^p} are satisfied.*

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. This corollary is a special case of Theorem 27 since inequality (47) implies inequality (25). □

By taking $\psi(t) = t$ and $\phi(t) = \beta(t)t$, where $\beta \in \mathbb{S}$ in Theorem 27, we obtain the following result.

Corollary 30. *Let (X, σ_b) be a complete b-metric-like space with parameter $s \ge 1$ and f, g be self-mappings on X such that $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is a closed subset of X, and $\alpha : X \times X \to [0, +\infty[$. Suppose that the following conditions are satisfied:*

(i) *f is a $g - \alpha_{s^p}$-admissible mapping;*

(ii) *there exists function $\beta \in \mathbb{S}$ such that*

$$\alpha\left(gx, gy\right)\sigma_b\left(fx, fy\right) \le \beta\left(N\left(x, y\right)\right)N\left(x, y\right); \tag{48}$$

(iii) *there exists an* $x_0 \in X$ *such that* $\alpha(gx_0, fx_0) \geq s^p$;

(iv) *properties* H_{s^p} *and* U_{s^p} *are satisfied.*

Then f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

If we take $\psi(t) = t$ in Theorem 27, then we get the following result.

Corollary 31. *Let* (X, σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$ *and* f, g *be self-mappings on* X *such that* $f(X) \subset g(X)$, $f(X)$ *or* $g(X)$ *is a closed subset of* X, *and* $\alpha : X \times X \to [0, +\infty[$. *Suppose that the following conditions are satisfied:*

(i) f *is a* $g - \alpha_{s^p}$*-admissible mapping;*

(ii) *there exist function* $\phi \in \Psi$ *such that*

$$\alpha(gx, gy)\sigma_b(fx, fy) \leq \phi(N(x, y)); \tag{49}$$

(iii) *there exists an* $x_0 \in X$ *such that* $\alpha(gx_0, fx_0) \geq s^p$;

(iv) *properties* H_{s^p} *and* U_{s^p} *are satisfied.*

Then f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Corollary 32. *Let* (X, σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$ *and* f, g *be self-mappings on* X *such that* $f(X) \subset g(X)$, $f(X)$ *or* $g(X)$ *is a closed subset of* X, *and* $\alpha : X \times X \to [0, +\infty[$. *Suppose that the following conditions are satisfied:*

(i) f *is a* $g - \alpha_{s^p}$*-admissible mapping;*

(ii) *there exist functions* $\psi, \varphi \in \Psi$ *such that*

$$\alpha(gx, gy)\sigma_b(fx, fy) \leq N(x, y) - \varphi(N(x, y)); \tag{50}$$

(iii) *there exists an* $x_0 \in X$ *such that* $\alpha(gx_0, fx_0) \geq s^p$;

(iv) *properties* H_{s^p} *and* U_{s^p} *are satisfied.*

Then f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. It follows from Corollary 28 by taking $\psi(t) = t$. □

Corollary 33. *Let* (X, σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$ *and* f, g *be self-mappings on* X *such that* $f(X) \subset g(X)$, $f(X)$ *or* $g(X)$ *is a closed subset of* X, *and* $\alpha : X \times X \to [0, +\infty[$. *Suppose that the following conditions are satisfied:*

(i) f *is a* $g - \alpha_{s^p}$*-admissible mapping;*

(ii) *there exists function* $\psi \in \Psi$ *such that*

$$\psi(\alpha(gx, gy)\sigma_b(fx, fy)) \leq \lambda\psi(N(x, y)); \tag{51}$$

for all $x, y \in X$, *where* $N(x, y)$ *is defined as in (32) and* $0 < \lambda < 1$;

(iii) *there exists an* $x_0 \in X$ *such that* $\alpha(gx_0, fx_0) \geq s^p$;

(iv) *properties* H_{s^p} *and* U_{s^p} *are satisfied.*

Then f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. In Theorem 27 take $\phi(t) = \lambda\psi(t)$, where $0 < \lambda < 1$. □

Corollary 34. *Let* (X, σ_b) *be a complete b-metric-like space with parameter* $s \geq 1$ *and* f, g *be two self-maps of* X *and* $\psi \in \Psi$ *such that they satisfy the condition*

$$\psi(s^p\sigma_b(fx, fy)) \leq \lambda\psi(N(x, y)) \tag{52}$$

for all $x, y \in X$, *where* $N(x, y)$ *is defined as in (32) and* $0 < \lambda < 1$. *Then* f *and* g *have a unique common fixed point in* X.

Proof. In Theorem 27 take $\phi(t) = \lambda\psi(t)$, where $0 < \lambda < 1$ and $\alpha(gx, gy) = s^p$. □

Remark 35. Our results generalize, extend, and improve the results appearing in the literature [3, 7, 10, 12, 16, 21, 26, 28, 29, 31–34]. It is clear that several more corresponding results can be derived by our main theorems by choosing constant p and the mappings ψ, ϕ, in a suitable way.

4. Application

In this section we will use Theorem 27 to show that there is a solution to the following system of integral equations:

$$\begin{aligned} x(t) &= \int_0^t K(r, x(r))\,dr, \\ x(t) &= \int_0^t x(r)\,dr. \end{aligned} \tag{53}$$

Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ for $T > 0$. We define a b-metric-like $\sigma_b : X \times X \to [0, \infty)$ by

$$\sigma_b(x, y) = \max_{t \in [0,T]} (|x(t)| + |y(t)|)^m \tag{54}$$

for all $x, y \in X$, where $m > 1$.

It is evident that (X, σ_b, s^{m-1}) is a complete b-metric-like space.

Consider the mappings $f, g : X \to X$ by

$$\begin{aligned} fx(t) &= \int_0^t K(r, x(r))\,dr, \\ gx(t) &= \int_0^t x(r)\,dr, \end{aligned} \tag{55}$$

and let $\zeta : R \times R \to R$ be a given function.

Theorem 36. *Consider the system of integral equations (53) and suppose that the following assertions hold:*

(i) $K : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ *(that is* $K(r, x(r)) \geq 0$, $\int_0^t x(r)dr \geq 0$*) is continuous;*

(ii) *if* $K(r, x(r)) = x(r)$ *for all* $r \in [0, T]$, *then we have*

$$K\left(r, \int_0^t x(\omega)\, d\omega\right) = \int_0^t K(\omega, x(\omega))\, d\omega \quad \forall r \in [0, T]; \tag{56}$$

(iii) *there exists* $x_0 \in X$ *such that* $\zeta(gx_0(t), fx_0(t)) \geq 0$ *for all* $t \in [0, T]$;

(iv) *for all* $t \in [0, T]$ *and* $x, y \in X$,

$$\zeta(gx(t), gy(t)) \geq 0 \text{ implies that } \zeta(fx(t), fy(t)) \geq 0; \tag{57}$$

(v) *properties* H_{s^p} *and* U_{s^p} *are satisfied;*

(vi) *there is a continuous function* $\gamma : [0, T] \to R^+$ *such that*

$$(|K(r, x(r))| + |K(r, y(r))|) \leq \gamma(r)(|x(r)| + |y(r)|) \tag{58}$$

for all $r \in [0, T]$, $x, y \in X$;

(vii) *there exist constant* $L \in (0, 1)$, $p \geq 2$ *such that for all* $r \in [0, T]$

$$\sup_{t\in[0,T]} \int_0^T \gamma(r)\, dr \leq \sqrt[m]{\frac{L}{s^p}}. \tag{59}$$

Then the system of integral equations (53) has a unique solution in $x \in X$.

Proof. We define a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} s^p & \text{if } \xi(x(t), y(t)) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{60}$$

It is easy to see that the function f is $g - \alpha_{s^p}$-admissible. By condition (ii) f and g are weakly compatible.

Let $x, y \in X = C([0, T], \mathbb{R})$ be such that $\alpha(gx, gy) \geq s^p$, that is, $\zeta(gx(t), gy(t)) \geq 0$ for all $t \in [0, T]$, then from conditions (iv), (vi), and (vii), for all $t \in [0, T]$, we have

$$\begin{aligned}
\sigma_b(fx(t), fy(t)) &= (|fx(t)| + |fy(t)|)^m \\
&= \left(\left|\int_0^t K(r, x(r))\, dr\right| + \left|\int_0^t K(r, y(r))\, dr\right|\right)^m \\
&\leq \left(\int_0^t |K(r, x(r))|\, dr + \int_0^t |K(r, y(r))|\, dr\right)^m \\
&= \left(\int_0^t (|K(r, x(r))| + |K(r, y(r))|)\, dr\right)^m \\
&\leq \left(\int_0^t \gamma(r)(|x(r)| + |y(r)|)\, dr\right)^m \\
&\leq \left(\sup_{t\in[0,T]} \gamma(r)\right)^m \left(\int_0^t (|x(r)| + |y(r)|)\, dr\right)^m \\
&\leq \left(\sup_{t\in[0,T]} \gamma(r)\right)^m (|gx(t)| + |gy(t)|)^m \\
&\leq \left(\left(\frac{L}{s^p}\right)^{1/m}\right)^m \sigma_b(gx(t), gy(t)) \\
&= \frac{L}{s^p}\sigma_b(gx(t), gy(t)) \leq \frac{L}{s^p} N(x(t), y(t)),
\end{aligned} \tag{61}$$

which implies that

$$\alpha(g(x), g(y))\, \sigma_b(fx(t), fy(t)) \leq LN(x(t), y(t)). \tag{62}$$

Therefore, taking $\psi(x) = x$ and $\phi(x) = Lx$, where $L \in (0, 1)$, and in view of assertion (v) all of the conditions of Theorem 27 are satisfied, and, as a result, the mappings f and g have a unique common fixed point in X, which is a solution of the system of integral equations in (53). □

5. Conclusions

In this manuscript, we defined new rational contraction using a larger class of α_{s^p}-admissible mappings and auxiliaries functions in the framework of b-metric-like spaces. The presented main theorems of the paper cover and unify a huge number of published results and also complement the previous work on the topic in the related literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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