Hindawi
Journal of Function Spaces
Volume 2018, Article ID 1418725, 8 pages
https://doi.org/10.1155/2018/1418725

Research Article

Fixed Point Theorems for Generalized $\theta - \phi$-Contractions in G-Metric Spaces

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Received 24 September 2017; Accepted 4 December 2017; Published 9 January 2018

Academic Editor: Tomonari Suzuki

We introduce the notion of generalized $\theta - \phi$ contraction and establish some new fixed point theorems for this contraction in the setting of complete G-metric spaces. The results presented in the paper improve, extend, and unify some known results. Finally, we give an example to illustrate them.

1. Introduction and Preliminaries

In 2006, Mustafa and Sims [1] introduced the notion of G-metric space and studied the properties of it. Subsequently, many authors studied the fixed point theory in the setting of complete G-metric spaces and obtained some fixed point theorems for different contractions (see [1–10]). In 2015, Agarwal et al. [11] presented a self-contained account of the fixed point theory (techniques and results) in G-metric spaces. The book [11] contains almost all the research findings that relate to basic fixed point theorems, common fixed point theorems, and coupled fixed point theorems in G-metric spaces and partially ordered G-metric spaces (see [11] and the references therein).

In 2014, Jleli and Samet [12] introduced a new type of contraction called θ-contraction. Later, many authors have studied θ-contraction deeply (for example, see [13, 14]). Just recently, Zheng et al. [15] introduced the notion of $\theta - \phi$ contraction in metric spaces which generalized θ-contraction and other contractions (see [12, 15] and the references therein).

Inspired by [12, 15], we introduce the notion of generalized $\theta - \phi$ contraction and establish some new fixed point theorems for this contraction in the setting of complete G-metric spaces. The results presented in the paper improve and extend the corresponding results of Agarwal et al. [11], Mustafa [4], Mustafa et al. [5], Mustafa and Sims [6], and Shatanawi [9]. Also, we give an example to illustrate them.

According to [12, 15], denote by Θ the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is nondecreasing.

(Θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.

(Θ_3) θ is continuous on $(0,\infty)$.

And by Φ the set of functions $\phi : [1,\infty) \to [1,\infty)$ satisfies the following conditions:

(Φ_1) $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing.

(Φ_2) For each $t > 1$, $\lim_{n\to\infty}\phi^n(t) = 1$.

(Φ_3) ϕ is continuous on $[1,\infty)$.

Lemma 1 (see [15]). *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.*

Now we recall some basic definitions and give some lemmas that will be used in the paper.

Definition 2 (see [1, 11]). A G-metric space is a pair (X, G), where X is a nonempty set and $G : X \times X \times X \to [0,+\infty)$ is a function such that, for all $x, y, z, a \in X$, the following conditions are fulfilled:

(G_1) $G(x, y, z) = 0$ if $x = y = z$.

(G_2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$.

(G_3) $G(x,x,y) \le G(x,y,z)$ for all $x,y,z \in X$ with $z \ne y$.

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all 3).

(G_5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ (rectangle inequality).

In such a case, the function G is called a G-metric.

Example 3 (see [1, 11]). If X is a nonempty subset of R, then the function $G : X \times X \times X \to [0,+\infty)$, given by $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in X$, is a G-metric on X.

Example 4 (see [1, 11]). Let $X = [0,\infty)$ be the interval of nonnegative real numbers and let G be defined by

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}, & \text{otherwise.} \end{cases} \tag{1}$$

Then G is a complete G-metric on X.

Definition 5 (see [1, 11]). Let (X,G) be a G-metric space; let $x \in X$ and $\{x_n\} \subseteq X$ be a sequence. We say that

(i) $\{x_n\}$ G-converges to x, and we write $\{x_n\} \to x$ if $\lim_{n,m\to\infty} G(x_n,x_m,x) = 0$; that is, for all $\varepsilon > 0$ there exists $n_0 \in N$ satisfying $G(x_n,x_m,x) \le \varepsilon$ for all $n,m \in N$ such that $n,m \ge n_0$;

(ii) $\{x_n\}$ is G-Cauchy if $\lim_{n,m,k\to\infty} G(x_n,x_m,x_k) = 0$; that is, for all $\varepsilon > 0$ there exists $n_0 \in N$ satisfying $G(x_n,x_m,x_k) \le \varepsilon$ for all $n,m,k \in N$ such that $n,m,k \ge n_0$;

(iii) (X,G) is complete if every G-Cauchy sequence in X is G-convergent in X.

Lemma 6 (see [1, 11]). *Let (X,G) be a G-metric space, let $x \in X$ and $\{x_n\} \subseteq X$ be a sequence. Then the following conditions are equivalent.*

(a) $\{x_n\}$ *G-converges to* x.

(b) $\lim_{n\to\infty} G(x_n,x_n,x) = 0$.

(c) $\lim_{n\to\infty} G(x_n,x,x) = 0$.

(d) $\lim_{n,m\to\infty, m\ge n} G(x_n,x_m,x) = 0$.

(e) $\lim_{n,m\to\infty, m>n} G(x_n,x_m,x) = 0$.

Lemma 7 (see [1, 11]). *Let (X,G) be a G-metric space and $\{x_n\} \subseteq X$ be a sequence. Then the following conditions are equivalent.*

(a) $\{x_n\}$ *is G-Cauchy.*

(b) $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$.

(c) $\lim_{n,m\to\infty, m\ge n} G(x_n,x_m,x_m) = 0$.

(d) $\lim_{n,m\to\infty, m>n} G(x_n,x_m,x_m) = 0$.

(e) $\lim_{n,m\to\infty} G(x_n,x_n,x_m) = 0$.

Lemma 8 (see [11]). *Let $\{x_n\}$ be an asymptotically regular sequence in a G-metric space (X,G) and suppose that $\{x_n\}$ is not Cauchy. Then there exist a positive real number $\varepsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in N$,*

$$\begin{gathered} k \le n(k) < m(k) < n(k+1), \\ G\left(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \le \varepsilon < G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), \end{gathered} \tag{2}$$

and also, for all given $p_1, p_2, p_3 \in Z$,

$$\lim_{k\to\infty} G\left(x_{n(k)+p_1}, x_{m(k)+p_2}, x_{m(k)+p_3}\right) = \varepsilon. \tag{3}$$

Lemma 9 (see [11]). *Let (X,G) be a G-metric space; then $G(x,y,y) \le 2G(y,x,x)$ for all $x,y \in X$.*

2. Main Results

Based on the functions $\theta \in \Theta$ and $\phi \in \Phi$, we give the following definition.

Definition 10. Let (X,G) be a G-metric space. A mapping $T : X \to X$ is said to be a generalized $\theta - \phi$ contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x,y,z \in X$,

$$\begin{gathered} G(Tx,Ty,Tz) \ne 0 \Longrightarrow \\ \theta(G(Tx,Ty,Tz)) \le \phi[\theta(N(x,y,z))], \end{gathered} \tag{4}$$

where

$$\begin{aligned} N(x,y,z) = \max\Big\{ & G(x,y,z), G(x,Tx,Tx), \\ & G(y,Ty,Ty), G(z,Tz,Tz), \frac{1}{2}G(x,Ty,Ty), \frac{1}{2} \\ & \cdot G(y,Tz,Tz), \frac{1}{2}G(z,Tx,Tx), \\ & \frac{1}{3}(G(x,Ty,Ty) + G(y,Tz,Tz) + G(z,Tx,Tx))\Big\}. \end{aligned} \tag{5}$$

Theorem 11. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a generalized $\theta - \phi$ contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in N$. If $x_{n+1} = x_n$ for some $n \in N$, then $x^* = x_n$ is a fixed point for T. Next, we assume that $x_{n+1} \ne x_n$ for all $n \in N$. Then $G(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in N$. Applying inequality (4) with $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$, we obtain

$$\begin{aligned} & \theta\left(G\left(Tx_n, Tx_{n+1}, Tx_{n+1}\right)\right) \\ & \quad \le \phi\left[\theta\left(N\left(x_n, x_{n+1}, x_{n+1}\right)\right)\right], \end{aligned} \tag{6}$$

where

$$\begin{aligned} N\left(x_n, x_{n+1}, x_{n+1}\right) = \max\Big\{ & G\left(x_n, x_{n+1}, x_{n+1}\right), \\ & G\left(x_n, x_{n+1}, x_{n+1}\right), G\left(x_{n+1}, x_{n+2}, x_{n+2}\right), \end{aligned}$$

$$G\left(x_{n+1},x_{n+2},x_{n+2}\right)\frac{1}{2}G\left(x_n,x_{n+2},x_{n+2}\right),\frac{1}{2}\cdot G\left(x_{n+1},x_{n+2},x_{n+2}\right),\frac{1}{2}G\left(x_{n+1},x_{n+1},x_{n+1}\right),\frac{1}{3}\left(G\left(x_n,x_{n+2},x_{n+2}\right)+G\left(x_{n+1},x_{n+1},x_{n+1}\right)+G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right)\Big\} = \max\Big\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(x_{n+1},x_{n+2},x_{n+2}\right),\frac{1}{2}G\left(x_n,x_{n+2},x_{n+2}\right),\frac{1}{3}\left(G\left(x_n,x_{n+2},x_{n+2}\right)+G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right)\Big\} = \max\left\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right\}. \quad (\text{by } (G_5)). \tag{7}$$

If $N(x_n,x_{n+1},x_{n+1}) = G(x_{n+1},x_{n+2},x_{n+2})$, then it follows from (4) that

$$\begin{aligned}\theta\left(G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right) &= \theta\left(G\left(Tx_n,Tx_{n+1},Tx_{n+1}\right)\right)\\ &\le \phi\left[\theta\left(N\left(x_n,x_{n+1},x_{n+1}\right)\right)\right]\\ &= \phi\left[\theta\left(G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right)\right]\\ &< \theta\left(G\left(x_{n+1},x_{n+2},x_{n+2}\right)\right), \quad (\text{by Lemma 1}),\end{aligned} \tag{8}$$

which is a contradiction. Hence, for $\forall n \in N$,

$$N\left(x_n,x_{n+1},x_{n+1}\right) = G\left(x_n,x_{n+1},x_{n+1}\right). \tag{9}$$

Thus, (4) becomes

$$\begin{aligned}&\theta\left(G\left(Tx_n,Tx_{n+1},Tx_{n+1}\right)\right)\\ &\quad\le \phi\left[\theta\left(G\left(x_n,x_{n+1},x_{n+1}\right)\right)\right].\end{aligned} \tag{10}$$

Repeating this process, we get

$$\begin{aligned}\theta\left(G\left(x_n,x_{n+1},x_{n+1}\right)\right) &= \theta\left(G\left(Tx_{n-1},Tx_n,Tx_n\right)\right)\\ &\le \phi\left[\theta\left(G\left(x_{n-1},x_n,x_n\right)\right)\right]\\ &\le \phi^2\left[\theta\left(G\left(x_{n-2},x_{n-1},x_{n-1}\right)\right)\right]\\ &\le \phi^3\left[\theta\left(G\left(x_{n-3},x_{n-2},x_{n-2}\right)\right)\right]\\ &\le \cdots \le \phi^n\left[\theta\left(G\left(x_0,x_1,x_1\right)\right)\right].\end{aligned} \tag{11}$$

By the definition of θ and (Φ_2), we have

$$\lim_{n\to\infty}\phi^n\left[\theta\left(G\left(x_0,x_1,x_1\right)\right)\right] = 1. \tag{12}$$

By (Θ_2), we obtain

$$\lim_{n\to\infty}G\left(x_n,x_{n+1},x_{n+1}\right) = 0. \tag{13}$$

Thus, $\{x_n\}$ is an asymptotically regular sequence.

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

Suppose, on the contrary, that, by Lemma 8, there exist a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in N$,

$$\begin{gathered}k \le n(k) < m(k) < n(k+1),\\ G\left(x_{n(k)},x_{m(k)-1},x_{m(k)-1}\right) \le \varepsilon_0\\ < G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right),\end{gathered} \tag{14}$$

and also, for all given $p_1 = p_2 = p_3 \in Z$,

$$\begin{aligned}&\lim_{k\to\infty}G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\\ &\quad= \lim_{k\to\infty}G\left(x_{n(k)+p_1},x_{m(k)+p_2},x_{m(k)+p_3}\right) = \varepsilon_0.\end{aligned} \tag{15}$$

Pick k large enough, by (13), (15), and Lemma 9,

$$\begin{aligned}N\left(x_{n(k)},x_{m(k)},x_{m(k)}\right) &= \max\Big\{G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right),\\ &G\left(x_{n(k)},x_{n(k)+1},x_{n(k)+1}\right),G\left(x_{m(k)},x_{m(k)+1},x_{m(k)+1}\right),\\ &\frac{1}{2}G\left(x_{n(k)},x_{n(k)+2},x_{n(k)+2}\right),\frac{1}{2}\cdot G\left(x_{m(k)},x_{m(k)+1},x_{m(k)+1}\right),\frac{1}{2}\cdot G\left(x_{m(k)},x_{n(k)+1},x_{n(k)+1}\right),\\ &\frac{1}{3}\left(G\left(x_{n(k)},x_{n(k)+1},x_{n(k)+1}\right)+G\left(x_{m(k)},x_{m(k)+1},x_{m(k)+1}\right)+G\left(x_{m(k)},x_{n(k)+1},x_{n(k)+1}\right)\right)\Big\}\\ &= \max\Big\{G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right),\frac{1}{2}\cdot G\left(x_{m(k)},x_{n(k)+1},x_{n(k)+1}\right)\Big\}\\ &\le \max\left\{G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right),G\left(x_{n(k)+1},x_{n(k)+1},x_{m(k)}\right)\right\} \longrightarrow\\ &\varepsilon_0, \quad (\text{as } k \longrightarrow \infty).\end{aligned} \tag{16}$$

Using the contractivity condition (4),

$$\begin{aligned}&\theta\left(G\left(x_{n(k)+1},x_{m(k)+1},x_{m(k)+1}\right)\right)\\ &\quad= \theta\left(G\left(Tx_{n(k)},Tx_{m(k)},Tx_{m(k)}\right)\right)\\ &\quad\le \phi\left[\theta\left(N\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\right)\right].\end{aligned} \tag{17}$$

Passing to limit as $k \to \infty$, then we get $\theta(\varepsilon_0) \le \phi[\theta(\varepsilon_0)]$. By Lemma 1, $\phi[\theta(\varepsilon_0)] < \theta(\varepsilon_0)$, then $\theta(\varepsilon_0) \le \phi[\theta(\varepsilon_0)] < \theta(\varepsilon_0)$, which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X.

Taking into account the fact that (X, G) is complete, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. In particular,

$$\lim_{n\to\infty} G(x_n, x^*, x^*) = 0. \quad (18)$$

Using the fact that G is continuous on each variable,

$$G(x^*, Tx^*, Tx^*) = \lim_{n\to\infty} G(x_{n+1}, Tx^*, Tx^*). \quad (19)$$

We claim that x^* is a fixed point of T. Suppose, on the contrary, if $x^* \neq Tx^*$, then by (18), (19),

$$\begin{aligned} N(x_n, x^*, x^*) = \max \Big\{ & G(x_n, x^*, x^*), \\ & G(x_n, x_{n+1}, x_{n+1}), G(x^*, Tx^*, Tx^*), \\ & G(x^*, Tx^*, Tx^*), \frac{1}{2}(G(x_n, Tx^*, Tx^*)), \frac{1}{2} \\ & \cdot G(x^*, Tx^*, Tx^*), \frac{1}{2} G(x^*, x_{n+1}, x_{n+1}), \\ & \frac{1}{3}(G(x_n, Tx^*, Tx^*) + G(x^*, Tx^*, Tx^*) \\ & + G(x^*, x_{n+2}, x_{n+2}))\Big\} \longrightarrow \\ & G(x^*, Tx^*, Tx^*), \quad (\text{as } n \to \infty). \end{aligned} \quad (20)$$

Using the contractivity condition (4),

$$\begin{aligned} \theta(G(x_{n+1}, Tx^*, Tx^*)) &= \theta(G(Tx_n, Tx^*, Tx^*)) \\ &\leq \phi[\theta(N(x_n, x^*, x^*))]. \end{aligned} \quad (21)$$

Passing to limit as $n \to \infty$, then we have

$$\theta(G(x^*, Tx^*, Tx^*)) \leq \phi[\theta(G(x^*, Tx^*, Tx^*))]. \quad (22)$$

By Lemma 1, $\phi[\theta(G(x^*, Tx^*, Tx^*))] < \theta(G(x^*, Tx^*, Tx^*))$. Then

$$\begin{aligned} \theta(G(x^*, Tx^*, Tx^*)) &\leq \phi[\theta(G(x^*, Tx^*, Tx^*))] \\ &< \theta(G(x^*, Tx^*, Tx^*)), \end{aligned} \quad (23)$$

which is a contradiction. As a consequence, we conclude that $Tx^* = x^*$.

Now, we will prove that T has at most one fixed point. Suppose, on the contrary, that there exists another distinct fixed point y^* of T such that $Tx^* = x^* \neq Ty^* = y^*$. Therefore, $G(Tx^*, Ty^*, Ty^*) = G(x^*, y^*, y^*) > 0$, and $N(x^*, y^*, y^*) = G(x^*, y^*, y^*)$, and then by (4)

$$\begin{aligned} \theta(G(x^*, y^*, y^*)) &= \theta(G(Tx^*, Ty^*, Ty^*)) \\ &\leq \phi[\theta(N(x^*, y^*, y^*))] \\ &= \phi[\theta(G(x^*, y^*, y^*))], \end{aligned} \quad (24)$$

and by Lemma 1, $\theta(G(x^*, y^*, y^*)) \leq \phi[\theta(G(x^*, y^*, y^*))] < \theta(G(x^*, y^*, x^*))$, which is a contradiction. Therefore, the fixed point of T is unique. □

Theorem 12. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$\begin{aligned} & G(Tx, Ty, Ty) \neq 0 \Longrightarrow \\ & \theta(G(Tx, Ty, Ty)) \leq \phi[\theta(N(x, y, y))], \end{aligned} \quad (25)$$

where

$$\begin{aligned} N(x, y, y) = \max \Big\{ & G(x, y, y), G(x, Tx, Tx), \\ & G(y, Ty, Ty), \frac{1}{2} G(y, Tx, Tx), \frac{1}{3}(G(x, Ty, Ty) \\ & + G(y, Ty, Ty) + G(y, Tx, Tx))\Big\}. \end{aligned} \quad (26)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

The following Theorem 13 is the main result of [5].

Theorem 13 (see [5]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping which satisfies the following condition, for all $x, y \in X$,*

$$\begin{aligned} G(Tx, Ty, Ty) \leq \max \{ & aG(x, y, y), \\ & b(G(x, Tx, Tx) + 2G(y, Ty, Ty)), \\ & b(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx))\}, \end{aligned} \quad (27)$$

where $0 \leq a < 1$ and $0 \leq b < 1/3$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $\lambda = \max\{a, 3b\}$; then $0 \leq \lambda < 1$. And let $\theta(t) = e^t$, $\phi(t) = t^\lambda$; then $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned} \max \{ & aG(x, y, y), b(G(x, Tx, Tx) + 2G(y, Ty, Ty)), \\ & b(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx))\} \\ \leq & \lambda \max \Big\{ G(x, y, y), \\ & \frac{1}{3}(G(x, Tx, Tx) + 2G(y, Ty, Ty)), \\ & \frac{1}{3}(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx))\Big\} \\ \leq & \lambda \max \Big\{ G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), \\ & \frac{1}{3}(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx))\Big\} \\ \leq & \lambda N(x, y, y). \end{aligned} \quad (28)$$

Therefore,

$$\begin{aligned}\theta(G(Tx,Ty,Ty)) &= e^{G(Tx,Ty,Ty)} \leq e^{\lambda N(x,y,y)} \\ &= \left(e^{N(x,y,y)}\right)^{\lambda} \\ &= \phi(\theta(N(x,y,y))).\end{aligned} \tag{29}$$

From Theorem 12, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. □

The following Theorem 14 is the main result of [6].

Theorem 14 (see [6]). *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping which satisfies the following condition, for all $x, y \in X$,*

$$\begin{aligned}G(Tx,Ty,Tz) \leq k \max\{&G(x,y,z), G(x,Tx,Tx), \\ &G(y,Ty,Ty), G(z,Tz,Tz), G(x,Ty,Ty), \\ &G(y,Tz,Tz), G(z,Tx,Tx)\},\end{aligned} \tag{30}$$

where $0 \leq k < 1/2$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $\lambda = 2k$; then $0 \leq \lambda < 1$. And let $\theta(t) = e^t$, $\phi(t) = t^{\lambda}$; then $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned}&k \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), \\ &\quad G(z,Tz,Tz), G(x,Ty,Ty), G(y,Tz,Tz), \\ &\quad G(z,Tx,Tx)\} = \lambda \max\left\{\frac{1}{2}G(x,y,z), \frac{1}{2}\right. \\ &\quad \cdot G(x,Tx,Tx), \frac{1}{2}G(y,Ty,Ty), \frac{1}{2}G(z,Tz,Tz), \frac{1}{2} \\ &\quad \left.\cdot G(x,Ty,Ty), \frac{1}{2}G(y,Tz,Tz), \frac{1}{2}G(z,Tx,Tx)\right\} \\ &\quad \leq \lambda N(x,y,z),\end{aligned} \tag{31}$$

therefore,

$$\begin{aligned}\theta(G(Tx,Ty,Tz)) &= e^{G(Tx,Ty,Tz)} \leq e^{\lambda N(x,y,z)} \\ &= \left(e^{N(x,y,z)}\right)^{\lambda} \\ &= \phi(\theta(N(x,y,z))).\end{aligned} \tag{32}$$

From Theorem 11, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. □

Theorem 15. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$,*

$$\begin{aligned}&G(Tx,Ty,Tz) \neq 0 \Longrightarrow \\ &\theta(G(Tx,Ty,Tz)) \leq \phi[\theta(G(x,y,z))].\end{aligned} \tag{33}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Theorem 16 (see [4]). *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping such that there exists $\lambda \in [0,1)$ satisfying, for any $x, y, z \in X$,*

$$G(Tx,Ty,Tz) \leq \lambda G(x,y,z). \tag{34}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $\theta(t) = e^t$, $\phi(t) = t^{\lambda}$; then $\theta \in \Theta$ and $\phi \in \Phi$.

$G(Tx,Ty,Tz) \leq \lambda G(x,y,z)$ is equivalent to $e^{G(Tx,Ty,Ty)} \leq e^{\lambda G(x,y,z)} = (e^{G(x,y,z)})^{\lambda}$; that is, $\theta(G(Tx,Ty,Tz)) \leq \phi(\theta(G(x,y,z)))$.

From Theorem 15, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. □

Corollary 17. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$\begin{aligned}&G(Tx,Ty,Ty) \neq 0 \Longrightarrow \\ &\theta(G(Tx,Ty,Ty)) \leq \phi[\theta(G(x,y,y))].\end{aligned} \tag{35}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 18. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping such that there exists $\lambda \in [0,1)$ satisfying, for any $x, y \in X$,*

$$G(Tx,Ty,Ty) \leq \lambda G(x,y,y). \tag{36}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 19. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$,*

$$\begin{aligned}&G(Tx,Ty,Ty) \neq 0 \Longrightarrow \\ &\theta(G(Tx,Ty,Tz)) \\ &\quad \leq \phi\left[\theta\left(\frac{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)}{3}\right)\right].\end{aligned} \tag{37}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 20. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a self-mapping such that there exists $\lambda \in [0,1)$ satisfying, for any $x, y, z \in X$,*

$$\begin{aligned}&G(Tx,Ty,Tz) \\ &\quad \leq \lambda\left(\frac{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)}{3}\right).\end{aligned} \tag{38}$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 21. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$, $G(Tx, Ty, Ty) \neq 0$, we have*

$$\theta(G(Tx, Ty, Ty)) \leq \phi\left[\theta\left(\max\left\{\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}, \frac{G(y, Ty, Ty) + G(z, Tz, Tz)}{2}, \frac{G(x, Tx, Tx) + G(z, Tz, Tz)}{2}\right\}\right)\right]. \quad (39)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 22. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping which satisfies the following condition, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq k \max\{G(x, Tx, Tx) + G(y, Ty, Ty), G(y, Ty, Ty) + G(z, Tz, Tz), G(x, Tx, Tx) + G(z, Tz, Tz)\}, \quad (40)$$

where $0 \leq k < 1/2$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 23. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$G(Tx, Ty, Ty) \neq 0 \Longrightarrow \theta(G(Tx, Ty, Ty)) \leq \phi\left[\theta\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right)\right]. \quad (41)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 24. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping such that there exists $\lambda \in [0, 1)$ satisfying, for any $x, y \in X$,*

$$G(Tx, Ty, Ty) \leq \lambda\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right). \quad (42)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Theorem 25 (see [9]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping such that, for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z)), \quad (43)$$

where $\varphi : [0, \infty) \to [0, \infty)$ is an increasing continuous function such that $\lim_{n\to\infty}\varphi^n(t) = 0$ for $t > 0$.

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. Let $\theta(t) = e^t$ for all $t \in [0, +\infty)$, and $\phi(t) = e^{\varphi(\ln t)}$ for all $t \in [1, +\infty)$.

Obviously, $\theta \in \Theta$, $\phi \in \Phi$.

By the definition of ϕ, we have $\phi(e^t) = e^{\varphi(t)}$.

$$\theta(G(Tx, Ty, Tz)) = e^{G(Tx,Ty,Tz)} \leq e^{\varphi(G(x,y,z))} = \phi\left[e^{G(x,y,z)}\right] = \phi[\theta(G(x, y, z))]. \quad (44)$$

Therefore, from Theorem 15, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*. □

Remark 26. In [9], the function φ is not required to be continuous. But due to Theorem 1 of [16] and item 4.2.3 of [11], we can suppose that φ is continuous.

$\mathcal{F}_{alt} = \{\varphi : [0, \infty) \to [0, \infty) \mid \varphi$ is continuous and nondecreasing, $\varphi(t) = 0 \Leftrightarrow t = 0\}$, and

$\mathcal{F}'_{alt} = \{\omega : [0, \infty) \to [0, \infty) \mid \omega$ is lower semicontinuous, $\omega(t) = 0 \Leftrightarrow t = 0\}$.

Theorem 27 (see [11]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a self-mapping. Assume that there exist $\varphi \in \mathcal{F}_{alt}$ and $\omega \in \mathcal{F}'_{alt}$ such that, for any $x, y \in X$,*

$$\varphi(G(Tx, Ty, Ty)) \leq \varphi(G(x, y, y)) - \omega(G(x, y, y)). \quad (45)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Due to Theorem 1 of [16] and item 4.2.3 of [11], the condition where there exist $\varphi \in \mathcal{F}_{alt}$ and $\omega \in \mathcal{F}'_{alt}$ such that, for any $x, y \in X$, $\varphi(G(Tx, Ty, Ty)) \leq \varphi(G(x, y, y)) - \omega(G(x, y, y))$ is equivalent to the condition where there exist $a \in [0, 1)$ and $\psi \in \mathcal{F}_{alt}$ such that, for any $x, y \in X$, $\psi(G(Tx, Ty, Ty)) \leq a\psi(G(x, y, y))$. Let $\theta(t) = e^{\psi(t)}$, $\phi(t) = t^a$; then $\theta \in \Theta$ and $\phi \in \Phi$. $\psi(G(Tx, Ty, Ty)) \leq a\psi(G(x, y, y))$ is equivalent to $e^{\psi(G(Tx,Ty,Ty))} \leq e^{a\psi(G(x,y,y))}$; that is, $\theta(G(Tx, Ty, Ty)) \leq \phi[\theta(G(x, y, y))]$.

From Theorem 15, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*. □

Remark 28. According to fixed point theory of metric spaces, we divide contractions into different type in the setting of G-metrics. Then Theorem 16 and Corollary 18 belong to

Banach type, Corollaries 19–24 Kannan type [17], Theorem 25 Browder type [18], and Theorem 27 Choudhury type. To some extent, our results unify them.

3. Example

In this section, we give an example to illustrate our results.

Example 29. Let $X = \{0, \pm 1, \pm 2, \ldots\}$ be endowed with the G-metric $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all x, y, $z \in X$. Then (X, G) is a complete G-metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0; \\ -(n-1), & \text{if } x = n; \\ n-1, & \text{if } x = -n. \end{cases} \quad (46)$$

At first, we observe that Theorems 16, 25, and 27 cannot be applied since for all $x = n > y = z = m > 2$, $G(Tx, Ty, Tz) = G(Tn, Tm, Tm) = 2n - 2m = G(n, m, m)$.

And Theorem 13 cannot be applied too. In fact, let $x = n$, $y = z = 0$; then $G(Tx, Ty, Tz) = G(Tn, T0, T0) = 2n - 2$, while

$$\begin{aligned} &\max\{aG(x, y, y), b(G(x, Tx, Tx) + 2G(y, Ty, Ty)), \\ &\quad b(G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx))\} \\ &\quad = \max\{aG(n, 0, 0), b(G(n, -(n-1), -(n-1)) \\ &\quad + 0), b(G(n, 0, 0) + G(0, T0, T0) \\ &\quad + G(0, -(n-1), -(n-1)))\} = \max\{2an, b(4n \\ &\quad - 2)\} \le \max\left\{2an, \frac{4n-2}{3}\right\}, \\ &G(Tx, Ty, Ty) \le \max\{aG(x, y, y), b(G(x, Tx, Tx) \\ &\quad + 2G(y, Ty, Ty)), b(G(x, Ty, Ty) + G(y, Ty, Ty) \\ &\quad + G(y, Tx, Tx))\}, \end{aligned} \quad (47)$$

is equivalent to $2n - 2 \le \max\{2an, (4n - 2)/3\}$.

Since $2n-2 \le \max\{2an, (4n-2)/3\}$ for all $n \in N$, we have $a = 1$, which yields a contradiction since $a < 1$.

By the same way, we can see that Theorem 14 cannot be applied.

Now, let the function $\theta : (0, \infty) \to (1, \infty)$ be defined by

$$\theta(t) = 5^t. \quad (48)$$

And define $\phi : [1, \infty) \to [1, \infty)$ by

$$\phi(t) = \begin{cases} 1, & \text{if } 1 \le t \le 2; \\ t - 1, & \text{if } t \ge 2. \end{cases} \quad (49)$$

Obviously, $\theta \in \Theta$, $\phi \in \Phi$.

In what follows, we prove that T is some θ–ϕ Kannan-type contraction; that is, T satisfies the condition of Corollary 23.

We consider three cases.

Case 1 ($x = n \ge 1$, $y = 0$ or $x = -n$ $(n \ge 1)$, $y = 0$). In this case, we have

$$\begin{aligned} &G(Tx, Ty, Ty) = 2(n-1), \\ &G(x, Tx, Tx) = 2(2n-1), \\ &G(y, Ty, Ty) = 0, \\ &\theta(G(Tx, Ty, Ty)) = \theta(2n-2) = 5^{2n-2}, \\ &\phi\left(\theta\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right)\right) \\ &\quad = \phi\left(\theta\left(\frac{G(x, Tx, Tx)}{2}\right)\right) = \phi(\theta(2n-1)) \\ &\quad = \phi\left(5^{2n-1}\right) = 5^{2n-1} - 1 = 5 \times 5^{2(n-1)} - 1 \ge 5^{2(n-1)} \\ &\quad = \theta(G(Tx, Ty, Ty)). \end{aligned} \quad (50)$$

Case 2 ($x = n > y = m \ge 1$ or $x = -n < y = -m \le -1$). In this case, we have

$$\begin{aligned} &G(Tx, Ty, Ty) = 2n - 2m, \\ &G(x, Tx, Tx) = 4n - 2, \\ &G(y, Ty, Ty) = 4m - 2, \\ &\phi\left(\theta\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right)\right) \\ &\quad = \phi(\theta(2n + 2m - 2)) = \phi\left(5^{2n+2m-2}\right) \\ &\quad = 5^{4m-2} \times 5^{2n-2m} - 1 \ge 5^{2n-2m} \\ &\quad = \theta(G(Tx, Ty, Ty)). \end{aligned} \quad (51)$$

Case 3 ($x = n$, $y = -m$, $n > m \ge 1$ or $x = -n$, $y = m$, $n > m \ge 1$). In this case, we have

$$\begin{aligned} &G(Tx, Ty, Ty) = 2n + 2m - 4, \\ &G(x, Tx, TX) = 4n - 2, \\ &G(y, Ty, Ty) = 4m - 2, \\ &\phi\left(\theta\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right)\right) \\ &\quad = \phi(\theta(2n + 2m - 2)) = \phi\left(5^{2n+2m-2}\right) \\ &\quad = 25 \times 5^{2n+2m-4} - 1 \ge 5^{2n+2m-4} = \theta(G(Tx, Ty)). \end{aligned} \quad (52)$$

Therefore, we have for all $x, y \in X$

$$\begin{aligned} &\theta(G(Tx, Ty, Ty)) \\ &\quad \le \phi\left[\theta\left(\frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}\right)\right]. \end{aligned} \quad (53)$$

Thus, T is a $\theta - \phi$ Kannan-type contraction.

So all the hypotheses of Corollary 23 are satisfied, thus T has a fixed point. In this example $x = 0$ is the fixed point.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research is supported by National Natural Science Foundation of China (nos. 11461002 and 11461003) and Guangxi Natural Science Foundation (2016GXNSFAA380003, 2016GXNSFAA380317, and 2017GXNSFAA198100).

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