Research Article

Existence of Uniqueness and Nonexistence Results of Positive Solution for Fractional Differential Equations Integral Boundary Value Problems

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In this paper, we consider a class of fractional differential equations with conjugate type integral conditions. Both the existence of uniqueness and nonexistence of positive solution are obtained by means of the iterative technique. The interesting point lies in that the assumption on nonlinearity is closely associated with the spectral radius corresponding to the relevant linear operator.

1. Introduction

In this paper, we consider the existence of uniqueness and nonexistence of positive solution for the following fractional differential equations:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad (1)$$

$$D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t) dt,$$

where $n-1 < \alpha \leq n$, $n \geq 3$, $0 < \beta < 1$, $\Gamma(\alpha - \beta) \int_0^{\eta} a(t) t^{\alpha-\gamma-1} dt < \Gamma(\alpha - \gamma)$, $\eta \in (0, 1]$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f : (0, 1) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, and $a(t) \in L^1[0, 1]$ is nonnegative.

In the recent years, many results were obtained to deal with the existence of solutions for nonlinear differential equations by using nonlinear analysis methods; see [1–16] and references therein. The fractional nonlocal boundary value problems have particularly attracted a great deal of attention (see [17–27]). While there are a lot of works dealing with the existence and multiplicity of solutions for nonlinear fractional differential equations, the results dealing with the uniqueness of solution are relatively scarce (see [28–35]). The main tool used in most of the papers dealing with the uniqueness of solution is the Banach contraction map

principle provided that the nonlinearity f is a Lipschitz continuous function. When $1 \leq \beta < \alpha - 1$, and f is continuous on $[0, 1] \times (-\infty, +\infty)$, Zhang and Zhong [34] established the uniqueness results of solution to problem (1) by using the Banach contraction map principle. It is worth mentioning that only positive solutions are meaningful in most practical problems. As far as we know, the nonexistence of positive solution has seldom been considered up to now.

Motivated by the above work, the aim of this paper is to establish the existence of uniqueness and nonexistence of positive solution to problem (1). Our analysis relies on the iterative technique on the cone derived from the properties of the Green function. This article provides some new insights. Firstly, the uniqueness results are obtained under some conditions concerning the spectral radius with respect to the relevant linear operator. In addition, the error estimation of the iterative sequences is given. Secondly, we impose weaker positivity conditions on f ; that is, the Lipschitz constant is generalized to a function and $f(t, x)$ may be singular at $t = 0, 1$. Finally, the nonexistence results of positive solution are obtained under conditions concerning the spectral radius of the relevant linear operator.

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory and lemmas.

Definition 1. The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \quad (2)$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad (3)$$

where $n = [\alpha] + 1$; $[\alpha]$ denotes the integer part of number α , provided that the right-hand side is point-wise defined on $(0, +\infty)$.

For convenience, we here list the assumptions to be used throughout the paper.

(A₁) $a(t) \in L^1[0, 1]$ is nonnegative, and

$$\Delta := \Gamma(\alpha - \gamma) - \Gamma(\alpha - \beta) \int_0^{\eta} a(t) t^{\alpha-\gamma-1} dt > 0; \quad (4)$$

(A₂) $f : (0, 1) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous.

Lemma 3 ([34]). For any $\gamma \in L[0, 1] \cap C(0, 1)$, the unique solution of the boundary value problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) + \gamma(t) &= 0, \quad 0 < t < 1, \\ u(0) &= u'(0) = \dots = u^{(n-2)}(0) = 0, \end{aligned} \quad (5)$$

$$D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t),$$

is

$$u(t) = \int_0^1 G(t, s) \gamma(s) ds, \quad (6)$$

where

$$G(t, s) = G_1(t, s) + h(s) t^{\alpha-1},$$

$$\begin{aligned} G_1(t, s) &= \frac{1}{\Gamma(\alpha)} \\ &\cdot \begin{cases} t^{\alpha-1} (1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\ t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \end{cases} \\ G_2(t, s) &= \frac{1}{\Gamma(\alpha)} \\ &\cdot \begin{cases} t^{\alpha-\gamma-1} (1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\ t^{\alpha-\gamma-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1, \end{cases} \end{aligned} \quad (7)$$

$$h(s) = \frac{\Gamma(\alpha - \gamma)}{\Delta} \int_0^{\eta} a(t) G_2(t, s) dt.$$

Lemma 4. The function $G_1(t, s)$ has the following properties:

- (1) $G_1(t, s) > 0, \forall t, s \in (0, 1)$;
- (2) $\Gamma(\alpha) G_1(t, s) \leq t^{\alpha-1} (1-s)^{\alpha-\beta-1}, \forall t, s \in [0, 1]$;
- (3) $\beta s (1-s)^{\alpha-\beta-1} t^{\alpha-1} \leq \Gamma(\alpha) G_1(t, s) \leq s (1-s)^{\alpha-\beta-1}, \forall t, s \in [0, 1]$.

Proof. It is obvious that (1) and (2) hold. In the following, we will prove (3).

Case (i) ($0 < s \leq t < 1$). Noticing $\alpha > 2$, we have

$$\begin{aligned} &\frac{\partial}{\partial t} \left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}} \right] \\ &= (\alpha-1) t^{\alpha-2} \left[1 - \left(\frac{t-s}{t(1-s)} \right)^{\alpha-2} \right] \geq 0, \end{aligned} \quad (8)$$

which implies

$$t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}} \leq 1 - (1-s) = s. \quad (9)$$

Noticing $0 < \beta < 1$, we have

$$\begin{aligned} &t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &= (1-s)^{\alpha-\beta-1} \left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-\beta-1}} \right] \\ &\leq (1-s)^{\alpha-\beta-1} \left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}} \right] \\ &\leq s (1-s)^{\alpha-\beta-1}. \end{aligned} \quad (10)$$

On the other hand, it follows from

$$\frac{\partial}{\partial s} [\beta s + (1-s)^{\beta}] \leq 0, \quad \forall s \in [0, 1] \quad (11)$$

that

$$\beta s + (1-s)^{\beta} \leq 1, \quad \forall s \in [0, 1]. \quad (12)$$

Therefore,

$$\begin{aligned} &t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &\geq t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\beta} (t-ts)^{\alpha-\beta-1} \\ &= t^{\alpha-1} \left[1 - \left(1 - \frac{s}{t} \right)^{\beta} \right] (1-s)^{\alpha-\beta-1} \\ &\geq t^{\alpha-1} [1 - (1-s)^{\beta}] (1-s)^{\alpha-\beta-1} \\ &\geq \beta s (1-s)^{\alpha-\beta-1} t^{\alpha-1}. \end{aligned} \quad (13)$$

Case (ii) ($0 \leq t \leq s \leq 1$). It is easy to see that

$$t^{\alpha-1} (1-s)^{\alpha-\beta-1} \leq s^{\alpha-1} (1-s)^{\alpha-\beta-1} \leq s (1-s)^{\alpha-\beta-1}. \quad (14)$$

On the other hand, we have

$$\begin{aligned} t^{\alpha-1} (1-s)^{\alpha-\beta-1} &\geq s t^{\alpha-1} (1-s)^{\alpha-\beta-1} \\ &\geq \beta s (1-s)^{\alpha-\beta-1} t^{\alpha-1}. \end{aligned} \quad (15)$$

It follows from (10), (13), (14), and (15) that (3) holds. $\square$

Lemma 5. *The function $G(t, s)$ has the following properties:*

- (1) $G(t, s) > 0$, $\forall t, s \in (0, 1)$;
- (2) $G(t, s) \leq t^{\alpha-1} \Phi_1(s)$, $\forall t, s \in [0, 1]$;
- (3) $\beta t^{\alpha-1} \Phi_2(s) \leq G(t, s) \leq \Phi_2(s)$, $\forall t, s \in [0, 1]$,

where

$$\begin{aligned} \Phi_1(s) &= \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s), \\ \Phi_2(s) &= \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s). \end{aligned} \quad (16)$$

Proof. It can be directly deduced from Lemma 4 and the definition of $G(t, s)$, so we omit the proof. $\square$

Let $E = C[0, 1]$ be endowed with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$, $B_r = \{u \in E : \|u\| < r\}$. Define cones P , Q by

$$\begin{aligned} P &= \{u \in E : u(t) \geq 0\}, \\ Q &= \{u \in P : \text{there exists } l_u > 0 \text{ such that } \beta \|u\| t^{\alpha-1} \\ &\leq u(t) \leq l_u t^{\alpha-1}\}. \end{aligned} \quad (17)$$

It is clear that Q is nonempty set since $t^{\alpha-1} \in Q$.

For convenience, we list here one more assumption to be used later:

(A₃) There exists $\lambda \in C(0, 1) \cap L[0, 1]$ such that

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq \lambda(t) |x - y|, \\ t &\in [0, 1], \quad x, y \in [0, \infty). \end{aligned} \quad (18)$$

Moreover,

$$\begin{aligned} 0 &< \int_0^1 \Phi_1(s) \lambda(s) ds < +\infty; \\ 0 &< \int_0^1 \Phi_1(s) f(s, 0) ds < +\infty. \end{aligned} \quad (19)$$

Define operators A and T as follows:

$$\begin{aligned} Au(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \\ T_\lambda u(t) &= \int_0^1 G(t, s) \lambda(s) u(s) ds. \end{aligned} \quad (20)$$

Lemma 6. *Assume that (A₃) holds; then $A : P \rightarrow Q$.*

Proof. It is clear that $f(t, x) \leq \lambda(t)x + f(t, 0)$, $\forall x \geq 0$. For any $u \in P$, we have

$$\begin{aligned} |Au(t)| &= \int_0^1 G(t, s) f(s, u(s)) ds \\ &\leq \int_0^1 \Phi_1(s) \lambda(s) \|u\| ds \\ &\quad + \int_0^1 \Phi_1(s) f(s, 0) ds. \end{aligned} \quad (21)$$

Then A is well-defined on P . It follows from Lemma 5 that $A : P \rightarrow Q$. $\square$

By virtue of the Krein-Rutmann theorem and Lemma 5, we have the following lemma.

Lemma 7. *Assume that (A₃) holds. Then $T_\lambda : P \rightarrow Q$ is a completely continuous linear operator. Moreover, the spectral radius $r(T_\lambda) > 0$ and T_λ has a positive eigenfunction φ_1 corresponding to its first eigenvalue $(r(T_\lambda))^{-1}$; that is, $T_\lambda \varphi_1 = r(T_\lambda) \varphi_1$.*

3. Main Results

Theorem 8. *Assume that (A₃) holds. Then (1) has a unique positive solution if the spectral radius $r(T_\lambda) \in (0, 1)$.*

Proof. It follows from $0 < \int_0^1 \Phi_1(s) f(s, 0) ds < +\infty$ that θ is not a fixed point of A . Then we only need to prove that A has a unique fixed point in Q .

Firstly, we will prove A has a fixed point in Q .

For any $u \in Q$, let

$$l_u = \int_0^1 \Phi_1(s) \lambda(s) u(s) ds. \quad (22)$$

Then

$$\beta \|u\| t^{\alpha-1} \leq (T_\lambda u)(t) \leq l_u t^{\alpha-1}. \quad (23)$$

By Lemma 7, we have that $T_\lambda \varphi_1 = r(T_\lambda) \varphi_1$. It is easy to see that

$$\frac{\beta \|\varphi_1\|}{r(T_\lambda)} t^{\alpha-1} \leq \varphi_1(t) \leq \frac{l_{\varphi_1}}{r(T_\lambda)} t^{\alpha-1}. \quad (24)$$

For any $u_0 \in Q$, set

$$u_n = A(u_{n-1}), \quad n = 1, 2, \dots \quad (25)$$

We may suppose that $u_1 - u_0 \neq \theta$ (otherwise, the proof is finished). It follows from (23) and (24) that

$$T_\lambda(|u_1 - u_0|) \leq \frac{r(T_\lambda) l_{|u_1 - u_0|}}{\beta \|\varphi_1\|} \varphi_1. \quad (26)$$

Then, we have

$$\begin{aligned} |u_2 - u_1| &= \left| \int_0^1 G(t, s) [f(s, u_1(s)) - f(s, u_0(s))] ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 G(t, s) \lambda(s) |u_1(s) - u_0(s)| ds \\
&\leq \frac{r(T_\lambda) l_{|u_1 - u_0|}}{\beta \|\varphi_1\|} \varphi_1.
\end{aligned} \tag{27}$$

By induction, we can get

$$|u_{n+1} - u_n| \leq \frac{[r(T_\lambda)]^n l_{|u_1 - u_0|}}{\beta \|\varphi_1\|} \varphi_1, \quad n = 1, 2, \dots \tag{28}$$

Then, for any $n, m \in \mathbb{N}$, one has

$$\begin{aligned}
|u_{n+m} - u_n| &\leq |u_{n+m} - u_{n+m-1}| + \dots + |u_{n+1} - u_n| \\
&\leq ([r(T_\lambda)]^{n+m-1} + \dots + [r(T_\lambda)]^n) \frac{l_{|u_1 - u_0|}}{\beta \|\varphi_1\|} \varphi_1 \\
&\leq \frac{[r(T_\lambda)]^n l_{|u_1 - u_0|}}{[1 - r(T_\lambda)] \beta \|\varphi_1\|} \varphi_1.
\end{aligned} \tag{29}$$

It follows from $r(T_\lambda) < 1$ that

$$\|u_{n+m} - u_m\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{30}$$

which implies $\{u_n\}$ is a Cauchy sequence. Therefore, there exists $u^* \in Q$ such that $\{u_n\}$ converges to u^* . Clearly u^* is a fixed point of A .

In the following, we will prove the fixed point of A is unique.

Suppose $v \neq u^*$ is a positive fixed point of A . Then there exists $l_{|u^* - v|} > 0$ such that

$$T_\lambda(|u^* - v|) \leq \frac{r(T_\lambda) l_{|u^* - v|}}{\beta \|\varphi_1\|} \varphi_1. \tag{31}$$

Therefore,

$$\begin{aligned}
|Au^* - Av| &= \left| \int_0^1 G(t, s) [f(s, u^*(s)) - f(s, v(s))] ds \right| \\
&\leq \int_0^1 G(t, s) \lambda(s) |u^*(s) - v(s)| ds \\
&\leq \frac{r(T_\lambda) l_{|u^* - v|}}{\beta \|\varphi_1\|} \varphi_1.
\end{aligned} \tag{32}$$

By induction, we can get

$$|A^n u^* - A^n v| \leq \frac{[r(T_\lambda)]^n l_{|u^* - v|}}{\beta \|\varphi_1\|} \varphi_1. \tag{33}$$

It follows from $r(T_\lambda) < 1$ that

$$\|u^* - v\| = \|A^n u^* - A^n v\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{34}$$

which implies the positive fixed point of A is unique. $\square$

Remark 9. The unique positive solution u^* of (1) can be approximated by the iterative schemes: for any $u_0 \in Q$, let

$$u_n = A(u_{n-1}), \quad n = 1, 2, \dots, \tag{35}$$

and then $u_n \longrightarrow u^*$. Furthermore, we have error estimation

$$|u_n - u^*| \leq \frac{[r(T_\lambda)]^n l_{|u_1 - u_0|}}{[1 - r(T_\lambda)] \beta \|\varphi_1\|} \varphi_1, \tag{36}$$

and with the rate of convergence

$$\|u_n - u^*\| = O([r(T_\lambda)]^n). \tag{37}$$

Remark 10. The spectral radius satisfies $r(T_\lambda) = \lim_{n \rightarrow \infty} \|T_\lambda^n\|^{1/n}$ and $r(T_\lambda) \leq \|T_\lambda\|^{1/n}$. Particularly,

$$r(T_\lambda) \leq \|T_\lambda\| = \sup_{0 \leq t \leq 1} \int_0^1 G(t, s) \lambda(s) ds. \tag{38}$$

Theorem 11. Assume that the following condition holds:

(A₄) There exists $\lambda_1 \in C(0, 1) \cap L[0, 1]$ satisfying

$$0 < \int_0^1 \Phi_1(s) \lambda_1(s) ds < +\infty, \tag{39}$$

such that

$$f(t, x) \leq \lambda_1(t) x, \quad t \in [0, 1], \quad x \in [0, \infty). \tag{40}$$

Then (1) has no positive solution if the spectral radius $r(T_{\lambda_1}) \in (0, 1)$, where

$$T_{\lambda_1} u(t) = \int_0^1 G(t, s) \lambda_1(s) u(s) ds. \tag{41}$$

Proof. We only need to prove that A has no fixed point in $Q \setminus \{\theta\}$. Otherwise, there exists $v \in Q \setminus \{\theta\}$, such that $Av = v$.

By Lemma 7, we have that the spectral radius $r(T_{\lambda_1}) > 0$ and T_{λ_1} has a positive eigenfunction ψ_1 satisfying

$$T_{\lambda_1} \psi_1 = r(T_{\lambda_1}) \psi_1. \tag{42}$$

It is clear that $\psi_1 \in Q \setminus \{\theta\}$. Therefore, there exists $c_1 > 0$ such that

$$v \leq c_1 \psi_1. \tag{43}$$

It follows from $f(t, x) \leq b_1(t)x$ that $v = Av \leq T_{\lambda_1} v$. It is obvious that T_{λ_1} is increasing on Q . By induction, we can get $v \leq T_{\lambda_1}^n v, \forall n = 1, 2, 3, \dots$. Thus,

$$v \leq T_{\lambda_1}^n v \leq T_{\lambda_1}^n c_1 \psi_1 = c_1 [r(T_{\lambda_1})]^n \psi_1, \tag{44}$$

$$\forall n = 1, 2, 3, \dots$$

Noticing $r(T_{\lambda_1}) < 1$, we have $v = \theta$, which contradicts with $v \in Q \setminus \{\theta\}$. $\square$

Theorem 12. Assume that the following condition holds:

(A₅) There exists $\lambda_2 \in C(0, 1) \cap L[0, 1]$ satisfying

$$0 < \int_0^1 \Phi_1(s) \lambda_2(s) ds < +\infty, \tag{45}$$

such that

$$f(t, x) \geq \lambda_2(t)x, \quad t \in [0, 1], \quad x \in [0, \infty). \quad (46)$$

Then (1) has no positive solution if the spectral radius $r(T_{\lambda_2}) > 1$, where

$$T_{\lambda_2} u(t) = \int_0^1 G(t, s) \lambda_2(s) u(s) ds. \quad (47)$$

Proof. Suppose that there exists $v \in Q \setminus \{\theta\}$, such that $Av = v$.

By Lemma 7, we have that the spectral radius $r(T_{\lambda_2}) > 0$ and T_{λ_2} has a positive eigenfunction ψ_2 satisfying

$$T_{\lambda_2} \psi_2 = r(T_{\lambda_2}) \psi_2. \quad (48)$$

It is clear that $\psi_2 \in Q \setminus \{\theta\}$. Therefore, there exists $c_2 > 0$ such that

$$v \geq c_2 \psi_1. \quad (49)$$

It follows from $f(t, x) \geq \lambda_2(t)x$ that $v = Av \geq T_{\lambda_2} v$. Noticing that T_{λ_2} is increasing on Q , by induction, we have $v \geq T_{\lambda_2}^n v$, $\forall n = 1, 2, 3, \dots$. Thus,

$$v \geq T_{\lambda_2}^n v \geq T_{\lambda_2}^n c_2 \psi_2 = c_2 [r(T_{\lambda_2})]^n \psi_2, \quad (50)$$

$$\forall n = 1, 2, 3, \dots$$

It follows from $r(T_{\lambda_1}) > 1$ that $\|v\| = \infty$, which contradicts with $v \in Q$. $\square$

4. Example

Example 1. Consider the following integral boundary value problem:

$$\begin{cases} D_{0+}^{5/2} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & D_{0+}^{1/2} u(1) = \int_0^1 \frac{1}{\sqrt{t}} D_{0+}^{1/2} u(t) dt, \end{cases} \quad (51)$$

with

$$f(t, x) = \frac{1 + x + |\sin x|}{2C\sqrt{t}}, \quad (52)$$

where

$$C = \frac{8}{3\sqrt{\pi}} - \frac{\sqrt{\pi}}{2}. \quad (53)$$

By direct calculations, we have

$$\Delta = \Gamma(2) - \Gamma(2) \int_0^1 \frac{t}{\sqrt{t}} dt = \frac{1}{3}. \quad (54)$$

It is clear that (A_1) and (A_2) hold. Clearly, we have

$$G_1(t, s) = \frac{4}{3\sqrt{\pi}} \begin{cases} t^{3/2}(1-s), & 0 \leq t \leq s \leq 1, \\ t^{3/2}(1-s) - (t-s)^{3/2}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t, s) = \frac{4}{3\sqrt{\pi}} \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$h(s) = \frac{16}{3\sqrt{\pi}} s(1-\sqrt{s}),$$

$$\Phi_1(s) = \frac{4(1-s)}{3\sqrt{\pi}} + h(s),$$

$$G(t, s) = G_1(t, s) + h(s)t^{\alpha-1}. \quad (55)$$

Let $\lambda(t) = 1/C\sqrt{t}$; then we have

$$|f(t, x) - f(t, y)| \leq \lambda(t)|x - y|, \quad (56)$$

$$t \in [0, 1], \quad x, y \in [0, \infty).$$

It is easy to get that (A_3) holds.

Denote

$$e(t) \equiv 1, \quad t \in [0, 1]. \quad (57)$$

By direct calculations, we have

$$(T_\lambda e)(t) = \int_0^1 G(t, s) \lambda(s) ds = \frac{8t^{3/2}}{3\sqrt{\pi}C} - \frac{\sqrt{\pi}t^2}{2C}, \quad (58)$$

$$\begin{aligned} (T_\lambda^2 e)(t) &= \int_0^1 G(t, s) \lambda(s) (T_\lambda e)(s) ds = \left(\frac{4}{3\sqrt{\pi}C} \right)^2 \\ &\cdot \left[\left(\frac{5}{7} - \frac{27\pi}{280} \right) t^{3/2} - \frac{8}{35} t^{7/2} + \frac{9\pi^2}{1024} t^4 \right] \\ &\leq \left(\frac{4}{3\sqrt{\pi}C} \right)^2 \end{aligned} \quad (59)$$

$$\begin{aligned} &\cdot \left[\left(\frac{5}{7} - \frac{27\pi}{280} \right) t^{3/2} - \frac{8}{35} t^{7/2} + \frac{9\pi^2}{1024} t^{7/2} \right] \\ &\approx \left(\frac{4}{3\sqrt{\pi}C} \right)^2 (0.41135t^{3/2} - 0.14183t^{7/2}). \end{aligned}$$

$\forall t \in [0, 1]$, we have

$$\begin{aligned} &\left(\frac{8t^{3/2}}{3\sqrt{\pi}C} - \frac{\sqrt{\pi}t^2}{2C} \right)' = \frac{4t^{1/2} - \pi t}{\sqrt{\pi}C} \geq 0, \\ &(0.41135t^{3/2} - 0.14183t^{7/2})' \\ &= 0.617025t^{1/2} - 0.496405t^{5/2} \geq 0. \end{aligned} \quad (60)$$

Therefore,

$$\begin{aligned} \max_{0 \leq t \leq 1} (T_\lambda e)(t) &= (T_\lambda e)(1) = 1, \\ \max_{0 \leq t \leq 1} (T_\lambda^2 e)(t) &\leq \left(\frac{4}{3\sqrt{\pi}C} \right)^2 (0.41135 - 0.14183) \\ &\approx 0.39898. \end{aligned} \quad (61)$$

It is obvious that

$$\|L_\lambda\| = \max_{0 \leq t \leq 1} (T_\lambda e)(t) = 1, \quad (62)$$

and

$$\|L_\lambda^2\| = \max_{0 \leq t \leq 1} (T_\lambda^2 e)(t) < 1. \quad (63)$$

which implies that

$$\|L_\lambda^2\|^{1/2} < 1, \quad (64)$$

Lemma 7 and Remark 10 can guarantee that

$$0 < r(L_\lambda) \leq \|L_\lambda^2\|^{1/2} < 1 < \|L_\lambda\| = 1. \quad (65)$$

So all of the assumptions of Theorem 8 are satisfied. As a result, BVP (51) has a unique positive solution.

Example 2. Consider BVP (51) with

$$f(t, x) = \frac{x + |\sin x|}{2C\sqrt{t}}. \quad (66)$$

Clearly, we have

$$f(t, x) \leq \lambda(t)x, \quad t \in [0, 1], \quad x \in [0, \infty). \quad (67)$$

It is not difficult to check that (A_4) holds.

It follows from Example 1 that

$$0 < r(L_\lambda) < 1 < \|L_\lambda\| = 1. \quad (68)$$

So all of the assumptions of Theorem 11 are satisfied. As a result, BVP (51) has no positive solution.

5. Conclusions

In this paper, we consider the existence of positive solution for fractional differential equations with conjugate type integral conditions. Both the existence of uniqueness and nonexistence of positive solution are established under conditions closely associated with the spectral radius with respect to the relevant linear operator. In addition, the unique positive solution can be approximated by an iterative scheme, and the error estimation of the iterative sequences is also given.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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