Hindawi
Journal of Function Spaces
Volume 2018, Article ID 1571365, 7 pages
https://doi.org/10.1155/2018/1571365

Research Article

Sarason's Conjecture of Toeplitz Operators on Fock-Sobolev Type Spaces

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Received 1 October 2017; Accepted 23 November 2017; Published 2 May 2018

Academic Editor: Aurelian Gheondea

In this note, we will solve Sarason's conjecture on the Fock-Sobolev type spaces and give a well solution that if Toeplitz product $T_u T_{\overline{v}}$, with entire symbols u and v, is bounded if and only if $u = e^q$, $v = Ce^{-q}$, where q is a linear complex polynomial and C is a nonzero constant.

1. Introduction

Let $\mathbb{C}^n$ denote the complex n-space and dv be the ordinary volume measure on $\mathbb{C}^n$ that is normalized so that $\int_{C^n} e^{-|z|^2} dv(z) = 1$. If given any two points $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, we denote

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w_j}, \qquad |z| = \sqrt{z \cdot \overline{z}}. \tag{1}$$

For every $0 < p < \infty$, $\alpha \in \mathbb{R}$, we denote by $L^p_\alpha(\mathbb{C}^n)$ the space of measurable functions f such that

$$\|f\|_{L^p_\alpha} = \left(\int_{C^n} \left| f(z) e^{-(1/2)|z|^2} \right|^p \frac{dv(z)}{(1 + |z|)^\alpha} \right)^{1/p} < \infty. \tag{2}$$

Let $H(\mathbb{C}^n)$ be the set of entire functions on $\mathbb{C}^n$. Then for a given $0 < p < \infty$, the Fock-Sobolev type space F^p_α with the norm $\|\cdot\|_{F^p_\alpha} = \|\cdot\|_{L^p_\alpha}$ is defined as

$$F^p_\alpha = \left\{ f \in H(\mathbb{C}^n) \mid \|f\|_{L^p_\alpha} < \infty \right\}. \tag{3}$$

Obviously, the Fock-Sobolev type space F^2_α equipped with the natural inner product defined by

$$\langle f, g \rangle_{L^2_\alpha} = \int_{C^n} f(z) \overline{g(z)} e^{-|z|^2} \frac{dv(z)}{(1 + |z|)^\alpha} \tag{4}$$

is a reproducing kernel Hilbert space for every real α. As stated in [1], with respect to the above inner product, it is difficult to compute the reproducing kernel of F^2_α explicitly. So we use the equivalent norm with respect to a new measure $|z|^{-\alpha} dv(z)$. In more detail, for $\alpha \le 0$, we will let

$$\langle f, g \rangle_\alpha = \int_{C^n} f(z) \overline{g(z)} e^{-|z|^2} \frac{dv(z)}{|z|^\alpha}, \tag{5}$$

and for $\alpha > 0$ we let

$$\langle f, g \rangle_\alpha = \int_{C^n} f^-_{\alpha/2}(z) \overline{g^-_{\alpha/2}(z)} e^{-|z|^2} dv(z) + \int_{C^n} f^+_{\alpha/2}(z) \overline{g^+_{\alpha/2}(z)} e^{-|z|^2} \frac{dv(z)}{|z|^\alpha}, \tag{6}$$

where $f^-_{\alpha/2}$ is the Taylor expansion of f up to order $\alpha/2$ and $f^+_{\alpha/2} = f - f^-_{\alpha/2}$. Now we can bravely make sure that the inner product $\langle \cdot, \cdot \rangle_\alpha$ generates a new Hilbert space norm on F^2_α that is equivalent to the F^2_α norm $\|\cdot\|_{F^2_\alpha}$. In particular, if we define the norm $\|\cdot\|_{\widetilde{F^2_\alpha}}$ on F^2_α by, when $\alpha \le 0$,

$$\|f\|_{\widetilde{F^2_\alpha}} = \left(\int_{C^n} |f(z)|^2 e^{-|z|^2} \frac{dv(z)}{|z|^\alpha} \right)^{1/2} \tag{7}$$

and when $\alpha > 0$,

$$\|f\|_{\widetilde{F^2_\alpha}} = \left(\int_{C^n} \left|(f)^-_{\alpha/2}(z)\right|^2 e^{-|z|^2} dv(z)\right)^{1/2} + \left(\int_{C^n} \left|(f)^+_{\alpha/2}(z)\right|^2 e^{-|z|^2} \frac{dv(z)}{|z|^\alpha}\right)^{1/2}, \tag{8}$$

and then we have that both $\|\cdot\|_{\widetilde{F^2_\alpha}}$ and $\|\cdot\|_{F^2_\alpha}$ are equivalent norms.

As is well known, F^2_α is indeed a reproducing kernel Hilbert space (see Lemma 2.1 of [1] for more details). Therefore its reproducing kernel is

$$K^\alpha_z(w) = \sum_\beta \phi_\beta(w)\overline{\phi_\beta(z)}, \tag{9}$$

where $\{\phi_\beta\}$ is any orthonormal basis for F^2_α with respect to $\langle\cdot,\cdot\rangle_\alpha$. Note that polynomials form a dense subset of F^2_α (see Proposition 2.3 in [2]). Also note that monomials are mutually orthogonal, which means that $\{z^\beta/\sqrt{\langle z^\beta, z^\beta\rangle_\alpha}\}$ is an orthonormal basis for F^2_α. The arguments that are identical to the ones in the proof of Theorem 4.5 in [2] then give us that

$$K^\alpha_z(w) = \begin{cases} \mathcal{J}^{-\alpha/2}K_z(w), & \text{if } \alpha \le 0; \\ \mathcal{J}^{-\alpha/2}K_z(w) + (K_z)^-_{\alpha/2}(w), & \text{if } \alpha > 0. \end{cases} \tag{10}$$

Here $\mathcal{J}^s$ is the fractional integration operator defined as

$$\mathcal{J}^s f(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k(z), & \text{if } s \ge 0; \\ \displaystyle\sum_{k>|s|}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k(z), & \text{if } s < 0, \end{cases} \tag{11}$$

where each f_k is a polynomial of degree k. Moreover, for $s > 0$, f^+_s is the tail part of the Taylor expansion of f of degree higher than $|s|$ given by

$$f^+_s(z) = \sum_{k>|s|} f_k(z) \tag{12}$$

and we let $f^-_s = f - f^+_s$ (see [2] for more information on fractional differentiation and integration).

Now it is easy to see that if $\alpha \le 0$, $(F^2_\alpha, \|\cdot\|_{\widetilde{F^2_\alpha}})$ is a closed subspace of L^2_α with respect to $\langle\cdot,\cdot\rangle_\alpha$. In this case, let P_α denote the orthogonal projection, so that

$$P_\alpha f(z) = \left\langle f, K^\alpha_z\right\rangle_\alpha \tag{13}$$

for any $f \in L^2_\alpha$. Unfortunately, the inner product $\langle\cdot,\cdot\rangle_\alpha$ does not make sense on L^2_α when $\alpha > 0$. That means we can not define the Toeplitz operator on F^2_α in the usual way in terms of this inner product. However according to the ideas of [1], it makes sense to define the Toeplitz operator with the symbols in F^2_α by the following formula:

$$T^\alpha_\varphi f(z) = \int_{C^n} f(w)\varphi(w)\overline{K^\alpha_z(w)}e^{-|w|^2}\frac{dv(w)}{|w|^\alpha}, \tag{14}$$

if $\alpha \le 0$ and

$$T^\alpha_\varphi f(z) = \int_{C^n} \varphi(w)(f)^-_{\alpha/2}(w)\overline{(K^\alpha_z)^-_{\alpha/2}(w)}e^{-|w|^2}dv(w) + \int_{C^n} \varphi(w)(f)^+_{\alpha/2}(w)\overline{(K^\alpha_z)^+_{\alpha/2}(w)}e^{-|w|^2}\frac{dv(w)}{|w|^\alpha} \tag{15}$$

if $\alpha > 0$, for any $\varphi, f \in F^2_\alpha$. In the sequel, we can reasonably define the Berezin transform of Toeplitz operator on F^2_α by

$$\widetilde{T^\alpha_\varphi}(z) = \left\langle T^\alpha_\varphi k^\alpha_z, k^\alpha_z\right\rangle_\alpha = \int_{C^n} \varphi(w)\left|k^\alpha_z(w)\right|^2 e^{-|w|^2}\frac{dv(w)}{|w|^\alpha} \tag{16}$$

if $\alpha \le 0$, and

$$\widetilde{T^\alpha_\varphi}(z) = \left\langle T^\alpha_\varphi k^\alpha_z, k^\alpha_z\right\rangle_\alpha = \int_{C^n} \varphi(w)\left|(k^\alpha_z)^-_{\alpha/2}(w)\right|^2 e^{-|w|^2}dv(w) + \int_{C^n} \varphi(w)\left|(k^\alpha_z)^+_{\alpha/2}(w)\right|^2 e^{-|w|^2}\frac{dv(w)}{|w|^\alpha} \tag{17}$$

if $\alpha > 0$ for any $\varphi \in F^2_\alpha$, where $k^\alpha_z(w)$ is the normalization of the kernel $K^\alpha_z(w)$, that is, $k^\alpha_z(w) = K^\alpha_z(w)/\sqrt{K^\alpha_z(z)}$.

The original product problem, owed to Sarason firstly in [3], describes the pairs of outer functions g and h in the Hardy space such that the operator $T_g T_{\overline{h}}$ is bounded on the Hardy space. Sequentially, this problem was partially researched for the Hardy space in [4] and for the Bergman space in [5–8]. Unluckily it turns out that the Sarason's conjecture is not true for both Hardy space and Bergman space of unit disk. See [9, 10] for counterexamples.

We will, in this note, give the equivalent conditions about the Sarason's conjecture of Toeplitz product on Fock-Sobolev type spaces F^2_α. Our main result will be the following.

Main Theorem. *Suppose that u and v are two nonzero functions in Fock-Sobolev type spaces F^2_α. Then the following conditions are equivalent:*

(1) *The Toeplitz product $T^\alpha_u T^\alpha_{\overline{v}}$ is bounded on Fock-Sobolev type spaces F^2_α.*

(2) *There exists a complex linear polynomial $q(z)$ on $\mathbb{C}^n$ such that $u = e^q$ and $v = Ce^{-q}$, where C is a nonzero complex constant.*

(3) *The product $\widetilde{|u|^2}\widetilde{|v|^2}$ is a bounded function on the complex space $\mathbb{C}^n$.*

In 2014, Cho et al. studied the products of Toeplitz operators on the classical Fock space (see [11]). In the case of the Fock-Sobolev space, Chen et al. (see [12]) had already proven the same topics and obtained the similar results. What is more, they claimed that if f and g are two nonzero functions in the Fock(-Sobolev) space, then the Toeplitz product $T_f T_{\overline{g}}$ is bounded if and only if $f = e^q$ and $g = Ce^{-q}$, where C is a nonzero constant and q is a linear polynomial. More properties about Toeplitz operators on Fock-Sobolev spaces are referred to in [13]. Sequentially, Bommier-Hato et al. in [14] continued to research Cho's results on the general Fock-type space with the weight functions $\exp(-|\cdot|^{2m})$. They took full advantage of the exact form of the reproducing kernel of the general Fock-type space and concluded that if u and v are two nonzero functions, then the Toeplitz product $T_u T_{\overline{v}}$ is bounded if and only if $u = e^g$ and $v = Ce^{-g}$, where C is a nonzero constant and g is a polynomial of degree at most m. The similar techniques are founded in [15, 16]. However, the translations appearing to the classical Fock spaces are not suitable to the generalized Fock space. To tackle the main theorem, we have to use the main ideas of [14], that is, making good use of the explicit properties of the reproducing kernel K_z^α in Fock-Sobolev type spaces F_α^2 instead of the Weyl operators defined by translations on the complex plane.

At last, it is remarked that, as stated in [1], the Fock-Sobolev type spaces F_α^2 are in fact very natural generalization of the Fock-Sobolev spaces and the Fock-Sobolev spaces of fractional order. For example, when $\alpha = 0$, F_0^2 is the classical Fock space F^2. Thus in this paper, we always omit discussing the case of $\alpha = 0$ and the similar result of this case is obtained in [11, 14].

Throughout this paper we write $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y whenever there is a constant $C > 0$ independent of X and Y such that $X \leq CY$. Similarly we write $X \simeq Y$ if $X \lesssim Y$ and $Y \lesssim X$.

2. Proof of the Main Result

We begin with some properties of the Fock-Sobolev spaces F_α^2. See [1] for more information.

Lemma 1. *Suppose that f belongs to the Fock-Sobolev type space F_α^p for any real α. Then for any $z, w \in \mathbb{C}^n$, we have*

$$\left|f(z)\right|^p e^{-(p/2)|z|^2} \frac{1}{(1+|z|)^\alpha} \lesssim \|f\|_{F_\alpha^p}^p, \tag{18}$$

and when $\alpha < 0$,

$$\begin{aligned} &\left|K_z^\alpha(w)\right| \\ &\lesssim (1+|z||w|)^{\alpha/2} \exp\left(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2\right), \end{aligned} \tag{19}$$

when $\alpha > 0$,

$$\begin{aligned} \left|K_z^\alpha(w)\right| &\lesssim (1+|w\cdot\overline{z}|)^{\alpha/2} \\ &\quad \cdot \exp\left(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2\right). \end{aligned} \tag{20}$$

More specifically,

$$\left|K_z^\alpha(z)\right| \simeq (1+|z|)^\alpha e^{|z|^2} \tag{21}$$

for any $z \in \mathbb{C}^n$ and there is a $r > 0$ such that

$$\left|K_z^\alpha(w)\right| \gtrsim (1+|z|)^\alpha \exp\left(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2\right), \tag{22}$$

for any $z \in B(w, r)$.

A consequence of the first estimate in Lemma 1 is that, for any function $u \in F_\alpha^2$, the Toeplitz operators T_u and $T_{\overline{u}}$ are both densely defined on F_α^2.

Lemma 2. *If the function u belongs to the Fock-Sobolev type space F_α^2, we then have $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$.*

Proof. In views of the Lemma 3.4 in [1], we can calculate that, for any polynomial f and $g \in F^\infty$ (see [1] for the definition of F^∞),

$$\begin{aligned} \left\langle (T_u^\alpha)^* f, g\right\rangle_\alpha &= \int_{C^n} \overline{u}(z) f(z) \overline{g(z)} \frac{e^{-|z|^2}}{|z|^\alpha} dv(z) \\ &= \left\langle T_{\overline{u}}^\alpha f, g\right\rangle_\alpha, \end{aligned} \tag{23}$$

if $\alpha \leq 0$, and if $\alpha > 0$,

$$\begin{aligned} \left\langle (T_u^\alpha)^* f, g\right\rangle_\alpha &= \int_{C^n} \overline{u}(z) f_{\alpha/2}^-(z) \overline{g_{\alpha/2}^-(z)} e^{-|z|^2} dv(z) \\ &\quad + \int_{C^n} \overline{u}(z) f_{\alpha/2}^+(z) \overline{g_{\alpha/2}^+(z)} \frac{e^{-|z|^2}}{|z|^\alpha} dv. \end{aligned} \tag{24}$$

Lastly the fact that the set of all holomorphic polynomials is dense in F_α^2 completes the proof. ☐

Lemma 3. *For given $u, v \in F_\alpha^2$, if $T_u^\alpha T_{\overline{v}}^\alpha$ is bounded on F_α^2, then $T_u^\alpha T_{\overline{v}}^\alpha K_z^\alpha(w) = u(w)\overline{v}(z) K_z^\alpha(w)$ for any $z, w \in \mathbb{C}^n$.*

Proof. When $\alpha \leq 0$, in view of reproducing properties of $K_z^\alpha(w)$, Lemma 3.4, the claim $(d) \Rightarrow (a)$ of Lemma 3.10 in [1], and Lemma 2, we see that

$$\begin{aligned} T_u^\alpha T_{\overline{v}}^\alpha K_z^\alpha(w) &= \left\langle T_u^\alpha T_{\overline{v}}^\alpha K_z^\alpha, K_w^\alpha\right\rangle_\alpha \\ &= \left\langle \overline{v}(z) K_z^\alpha, \overline{u}(w) K_w^\alpha\right\rangle_\alpha \\ &= u(w)\overline{v}(z) K_z^\alpha(w). \end{aligned} \tag{25}$$

On the other side, when $\alpha > 0$, we have to use the Lemma 3.4, the claim $(d) \Rightarrow (a)$ of Lemma 3.10 in [1] to achieve that, if $u, v \in F_\alpha^2$, $T_u^\alpha T_{\overline{v}}^\alpha$ is bounded,

$$\begin{aligned} &\left\langle T_u^\alpha T_{\overline{v}}^\alpha K_z^\alpha, K_w^\alpha\right\rangle_\alpha \\ &= \int_{C^n} u(\lambda) \left(T_{\overline{v}}^\alpha K_z^\alpha\right)_{\alpha/2}^-(\lambda) \overline{\left(K_w^\alpha\right)_{\alpha/2}^-(\lambda)} e^{-|\lambda|^2} dv(\lambda) \\ &+ \int_{C^n} u(\lambda) \left(T_{\overline{v}}^\alpha K_z^\alpha\right)_{\alpha/2}^+(\lambda) \overline{\left(K_w^\alpha\right)_{\alpha/2}^+(\lambda)} \frac{e^{-|\lambda|^2}}{|\lambda|^\alpha} dv(\lambda). \end{aligned} \tag{26}$$

Together with (3.5) in [1], Fubini's theorem, and the reproducing property, we can see that

$$\begin{aligned}
&\int_{C^n} u(\lambda)\left(T_{\overline{v}}^{\alpha}K_z^{\alpha}\right)_{\alpha/2}^{+}(\lambda)\,\overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{+}(\lambda)}\frac{e^{-|\lambda|^2}}{|\lambda|^{\alpha}}dv(\lambda) \\
&= \int_{C^n}\int_{C^n} u(\lambda)\,\overline{v}(\xi)\left(K_z^{\alpha}\right)_{\alpha/2}^{+}(\xi)\left(K_{\xi}^{\alpha}\right)_{\alpha/2}^{+}(\lambda) \\
&\cdot \overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{+}(\lambda)}\frac{e^{-|\lambda|^2}}{|\lambda|^{\alpha}}\frac{e^{-|\xi|^2}}{|\xi|^{\alpha}}dv(\xi)\,dv(\lambda) \\
&= \int_{C^n} u(\lambda)\left(K_{\xi}^{\alpha}\right)_{\alpha/2}^{+}(\lambda)\,\overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{+}(\lambda)}\frac{e^{-|\lambda|^2}}{|\lambda|^{\alpha}}dv(\lambda) \\
&\cdot \int_{C^n}\overline{v}(\xi)\left(K_z^{\alpha}\right)_{\alpha/2}^{+}(\xi)\frac{e^{-|\xi|^2}}{|\xi|^{\alpha}}dv(\xi) = u(w)\int_{C^n}\overline{v}(\xi) \\
&\cdot \left(K_{\xi}^{\alpha}\right)_{\alpha/2}^{+}(w)\left(K_z^{\alpha}\right)_{\alpha/2}^{+}(\xi)\frac{e^{-|\xi|^2}}{|\xi|^{\alpha}}dv(\xi) = u(w)\,\overline{v}(z) \\
&\cdot \left(K_z^{\alpha}\right)_{\alpha/2}^{+}(w).
\end{aligned} \tag{27}$$

Similarly, we can achieve that

$$\begin{aligned}
&\int_{C^n} u(\lambda)\left(T_{\overline{v}}^{\alpha}K_z^{\alpha}\right)_{\alpha/2}^{-}(\lambda)\,\overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{-}(\lambda)}e^{-|\lambda|^2}dv(\lambda) \\
&\quad = u(w)\,\overline{v}(z)\left(K_z^{\alpha}\right)_{\alpha/2}^{-}(w).
\end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned}
T_u^{\alpha}T_{\overline{v}}^{\alpha}K_z^{\alpha}(w) &= \left\langle T_u^{\alpha}T_{\overline{v}}^{\alpha}K_z^{\alpha}, K_w^{\alpha}\right\rangle_{\alpha} \\
&= u(w)\,\overline{v}(z)\,K_z^{\alpha}(w).
\end{aligned} \tag{29}$$

□

Theorem 4 ((1) ⇒ (2)). *If we give that u and v are two nonzero functions in the Fock-Sobolev type space F_{α}^2 such that Toeplitz product $T_u^{\alpha}T_{\overline{v}}^{\alpha}$ is bounded on F_{α}^2; then there is a complex linear polynomial $q(z)$ on $\mathbb{C}^n$ such that $u = e^q$ and $v = Ce^{-q}$, where C is a nonzero complex constant.*

Proof. This proof is similar to Theorem 2.4 in [12] and here we only give its brief illustration.

If the condition holds that the Toeplitz product $T_u^{\alpha}T_{\overline{v}}^{\alpha}$ is bounded on F_{α}^2, by Lemmas 2 and 3 and the Cauchy-Schwarz inequality, we can see that

$$|u(z)\,v(z)| = \left|\widetilde{T_u^{\alpha}T_{\overline{v}}^{\alpha}}\right| \le \left\|T_u^{\alpha}T_{\overline{v}}^{\alpha}\right\| < \infty. \tag{30}$$

Sequentially, the local property of reproducing kernel K_z shows us that the module of function

$$T(z,w) = \frac{\left\langle T_u^{\alpha}T_{\overline{v}}^{\alpha}K_w^{\alpha}, K_z^{\alpha}\right\rangle_{\alpha}}{\sqrt{K_z^{\alpha}(z)}\sqrt{K_w^{\alpha}(w)}} \tag{31}$$

is equivalent to $|e^{q(z)-\overline{q(w)}}|$ when $|z-w| < \epsilon_0$. It implies that $|T(z,w)|$ is bounded in that situation.

On the other side, we give the representation of quadratic polynomial in the case of real inner product as follows: $q_2(z) = \langle Az, z\rangle$, where $q = q_1 + q_2$, q_1 is linear, q_2 is a homogeneous polynomial of degree 2, and $A = A_{n\times n}$ is a complex matrix symmetric in the real sense. After we choose $w = r\xi$ and $z = r\xi + (\epsilon_0/2)\eta$, where r is any real positive number, we achieve that

$$\left|e^{q(z)-\overline{q(w)}}\right| = M\exp\left(r\epsilon_0\langle A\xi, \eta\rangle\right) \tag{32}$$

is not bounded as $r \to \infty$. This contradiction finishes the proof. □

Theorem 5 ((2) ⇒ (1)). *If $u = e^q$ and $v = e^{-q}$ where q is a complex linear polynomial on $\mathbb{C}^n$, then $T_u^{\alpha}T_{\overline{v}}^{\alpha}$ is bounded on the Fock-Sobolev type space F_{α}^2.*

Proof. To prove the boundedness of $T_u^{\alpha}T_{\overline{v}}^{\alpha}$, we will sufficiently obtain that $\|T_u^{\alpha}T_{\overline{v}}^{\alpha}f\|_{\widetilde{F_{\alpha}^2}}$ is bounded by means of the idea of [1]. In fact we only discuss the case of $\alpha > 0$ because the other case is the same as the proof of Theorem 2.5 in [12]. Using the similar ways, our goal is to obtain that

$$\begin{aligned}
\|T_u^{\alpha}T_{\overline{v}}^{\alpha}f\|_{\widetilde{F_{\alpha}^2}} &= \left(\int_{C^n}\left|\left(T_u^{\alpha}T_{\overline{v}}^{\alpha}f\right)_{\alpha/2}^{-}(z)\right.\right. \\
&\quad\left.\left.\cdot\, e^{-(1/2)|z|^2}\right|^2 dv(z)\right)^{1/2} \\
&\quad + \left(\int_{C^n}\left|\left(T_u^{\alpha}T_{\overline{v}}^{\alpha}f\right)_{\alpha/2}^{+}(z)\right.\right. \\
&\quad\left.\left.\cdot\, e^{-(1/2)|z|^2}|z|^{-\alpha/2}\right|^2 dv(z)\right)^{1/2}
\end{aligned} \tag{33}$$

is bounded for any $f \in F_{\alpha}^2$ in view of the definition of the norm $\|\cdot\|_{\widetilde{F_{\alpha}^2}}$. To the end, we focus our attention on the integrands in it. By formulae (3.4) (3.5) in [1] and the definition of Toeplitz operator, the integrands in the norm are

$$\begin{aligned}
\left(T_u^{\alpha}T_{\overline{v}}^{\alpha}f\right)_{\alpha/2}^{-}(z) &= \int_{C^n}\overline{v}(\eta)\,(f)_{\alpha/2}^{-}(\eta)\left(\int_{C^n}u(w)\right. \\
&\quad\left.\cdot\,\overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{-}(\eta)\left(K_z^{\alpha}\right)_{\alpha/2}^{-}(w)}e^{-|w|^2}dv(w)\right)e^{-|\eta|^2}dv(\eta), \\
\left(T_u^{\alpha}T_{\overline{v}}^{\alpha}f\right)_{\alpha/2}^{+}(z) &= \int_{C^n}\overline{v}(\eta)\,(f)_{\alpha/2}^{+}(\eta)\left(\int_{C^n}u(w)\right. \\
&\quad\left.\cdot\,\overline{\left(K_w^{\alpha}\right)_{\alpha/2}^{+}(\eta)\left(K_z^{\alpha}\right)_{\alpha/2}^{+}(w)}\frac{e^{-|w|^2}}{|w|^{\alpha}}dv(w)\right) \\
&\quad\cdot\,\frac{e^{-|\eta|^2}}{|\eta|^{\alpha}}dv(\eta).
\end{aligned} \tag{34}$$

By the reproducing property, the estimations of their module are, respectively, coming from

$$\begin{aligned}
\left|\left(T_u^{\alpha}T_{\overline{v}}^{\alpha}f\right)_{\alpha/2}^{-}(z)\right| &\le \int_{C^n}|u(z)|\,|\overline{v}(\eta)|\,\left|(f)_{\alpha/2}^{-}(\eta)\right| \\
&\quad\cdot\left|\left(K_z^{\alpha}\right)_{\alpha/2}^{-}(\eta)\right|e^{-|\eta|^2}dv(\eta),
\end{aligned} \tag{35}$$

and then, similarly,

$$\left|(T_u^\alpha T_{\overline{v}}^\alpha f)_{\alpha/2}^+(z)\right| \lesssim \int_{C^n} |u(z)|\left|\overline{v}(\eta)\right|\left|(f)_{\alpha/2}^+(\eta)\right| \cdot \left|(K_z^\alpha)_{\alpha/2}^+(\eta)\right| \frac{e^{-|\eta|^2}}{|\eta|^\alpha} dv(\eta). \tag{36}$$

Therefore, using the Cauchy-Schwarz inequality, we have the estimation of the first term of the norm as follows:

$$\begin{aligned}\int_{C^n} &\left|(T_u^\alpha T_{\overline{v}}^\alpha f)_{\alpha/2}^-(z) e^{-(1/2)|z|^2}\right|^2 dv(z) \\ &\lesssim \int_{C^n}\left(\int_{C^n} H_\alpha^1(w,z)\,dv(w)\int_{C^n} H_\alpha^1(w,z) \cdot \left|(f)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w)\right) dv(z),\end{aligned} \tag{37}$$

where $H_\alpha^1(w,z) = e^{-(1/2)|z|^2}|(K_z^\alpha)_{\alpha/2}^-(w)|e^{-(1/2)|w|^2}\exp(\mathrm{Re}(q(z) - \overline{q(w)}))$. Similarly the estimation of the second norm has been achieved that

$$\begin{aligned}\int_{C^n} &\left|(T_u^\alpha T_{\overline{v}}^\alpha f)_{\alpha/2}^+(z) e^{-(1/2)|z|^2}\,|z|^{-\alpha/2}\right|^2 dv(z) \\ &\lesssim \int_{C^n}\left(\int_{C^n} H_\alpha^2(w,z)\,dv(w)\int_{C^n} H_\alpha^2(w,z) \cdot \left|(f)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w)\right) dv(z),\end{aligned} \tag{38}$$

where $H_\alpha^2(w,z) = e^{-(1/2)|z|^2}|z|^{-\alpha/2}|(K_z^\alpha)_{\alpha/2}^+(w)|e^{-(1/2)|w|^2} \cdot |w|^{-\alpha/2}\exp(\mathrm{Re}(q(z) - \overline{q(w)}))$.

If we can affirm both

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\int_{C^n} H_\alpha^1(z,w)\,dv(w) < \infty, \\ &\sup_{z\in\mathbb{C}^n}\int_{C^n} H_\alpha^2(z,w)\,dv(w) < \infty,\end{aligned} \tag{39}$$

we would finish the proof because

$$\left\|T_u^\alpha T_{\overline{v}}^\alpha f\right\|_{F_\alpha^2}^2 \lesssim \int_{C^n}\left|(f)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w) + \int_{C^n}\left|(f)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w). \tag{40}$$

To the finish, in terms of Lemma 1 and transformation, we can assert that

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\int_{C^n} H_\alpha^1(z,w)\,dv(w) \\ &\lesssim \sup_{z\in\mathbb{C}^n}\int_{C^n}(1+|w\cdot\overline{z}|)^{\alpha/2} e^{-(1/8)|z-w|^2}\left|e^{q(z-w)}\right| dv(w) \\ &< \infty,\end{aligned}$$

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\int_{C^n} H_\alpha^2(z,w)\,dv(w) \\ &\lesssim \sup_{z\in\mathbb{C}^n}\int_{C^n}\frac{|z|^{-\alpha/2}|w|^{-\alpha/2}e^{-(1/8)|z-w|^2}}{(1+|z\cdot w|)^{-\alpha/2}}\left|e^{q(z-w)}\right| dv(w) \\ &< \infty.\end{aligned} \tag{41}$$

This implies that $\|T_uT_{\overline{v}}f\|_{\widetilde{F_\alpha^2}}$ is bounded and completes the proof. □

Theorem 6 ((1) ⇒ (3)). *If u and v are two functions in the Fock-Sobolev type space F_α^2, not identically zero, such that the operator $T_u^\alpha T_{\overline{v}}^\alpha$ is bounded on F_α^2, then $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is a bounded function on the complex space.*

Proof. We omit the proof here for it is analogous to Theorem 2.6 in [12]. □

Theorem 7 ((3) ⇒ (2)). *Suppose that u and v are two functions in the Fock-Sobolev type space F_α^2, not identically zero, such that $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on $\mathbb{C}^n$. Then there is a complex linear polynomial $q(z)$ on $\mathbb{C}^n$ satisfying $u = e^q$ and $v = Ce^{-q}$, where C is a nonzero complex constant.*

Proof. Now we only consider the case of $\alpha > 0$ while the other case would be referred to in Theorem 2.7 in [12].

It is easy to see that, for any $u \in F_\alpha^2$, $\widetilde{u}(z) = \langle T_u^\alpha k_z^\alpha, k_z^\alpha\rangle_\alpha = u(z)$. When $\alpha > 0$, we use the triangle inequality and Hölder's inequality to calculate

$$\begin{aligned}|u(z)|^2 \lesssim &\int_{C^n}\left|u(w)(k_z^\alpha)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w) \cdot \int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w) \\ &+ \int_{C^n}\left|u(w)(k_z^\alpha)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w) \cdot \int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w).\end{aligned} \tag{42}$$

Because k_z^α is a unit element, that is,

$$\left\|k_z^\alpha\right\|_{\widetilde{F_\alpha^2}} = \left(\int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w)\right)^{1/2} + \left(\int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w)\right)^{1/2}, \tag{43}$$

we can see that

$$\begin{aligned}&\int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^-(w)\right|^2 e^{-|w|^2} dv(w) \lesssim 1, \\ &\int_{C^n}\left|(k_z^\alpha)_{\alpha/2}^+(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w) \lesssim 1.\end{aligned} \tag{44}$$

From the above inequations, the estimate of $|u(z)|^2$ turns into

$$|u(z)|^2 \lesssim \int_{C^n} |u(w)|^2 \left|(k_z^\alpha)^-_{\alpha/2}(w)\right|^2 e^{-|w|^2} dv(w) + \int_{C^n} |u(w)|^2 \left|(k_z^\alpha)^+_{\alpha/2}(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w) = \widetilde{|u|^2}(z). \tag{45}$$

If $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is a bounded function on $\mathbb{C}^n$, $\widetilde{|v|^2}(z)|u|^2(z)$ and $|u|^2(z)|v|^2(z)$ are both bounded on $\mathbb{C}^n$. By Liouville's theorem, the boundedness of $|u|^2(z)|v|^2(z)$ implies that there exists a constant C such that $uv = C$. Since neither u nor v is identically zero, we have $C \neq 0$. That is, both u and v are nonvanishing. By Lemma 1, there exists a complex polynomial $q(z)$ on $\mathbb{C}^n$ with $\deg(q) \leq 2$ such that $u = e^q$ and $v = Ce^{-q}$.

On the other side, by the definition of Berezin transformation in this case,

$$\widetilde{|u|^2}(z)|v|^2(z) = \int_{C^n} \left|u(w)\bar{v}(z)(k_z^\alpha)^-_{\alpha/2}(w)\right|^2 e^{-|w|^2} dv(w) + \int_{C^n} \left|u(w)\bar{v}(z)(k_z^\alpha)^+_{\alpha/2}(w)\right|^2 \frac{e^{-|w|^2}}{|w|^\alpha} dv(w). \tag{46}$$

Now giving a sufficiently small $\delta > 0$, we further obtain that

$$\begin{aligned}\widetilde{|u|^2}(z)|v|^2(z) &\gtrsim \int_{|w|>\delta} \left|u(w)\bar{v}(z)(k_z^\alpha)^-_{\alpha/2}(w)\right|^2 \cdot e^{-|w|^2} dv(w) + \int_{|w|>\delta} \left|u(w)\bar{v}(z)(k_z^\alpha)^+_{\alpha/2}(w)\right|^2 \cdot \frac{e^{-|w|^2}}{|w|^\alpha} dv(w) \gtrsim \int_{|w|>\delta} \left|u(w)\bar{v}(z)k_z^\alpha(w)\right|^2 \cdot \frac{e^{-|w|^2}}{|w|^\alpha} dv(w) \\ &\gtrsim \int_{|w|>\delta} \left|(1+|z|)^{-\alpha/2} e^{-(1/2)|z|^2} K_z^\alpha(w) e^{-(1/2)|w|^2}\right|^2 \cdot \left|\frac{e^{q(w)-q(z)}}{|w|^{\alpha/2}}\right|^2 dv(w).\end{aligned} \tag{47}$$

When choosing a constant $\varepsilon > 0$ satisfying Lemma 1, we can see that

$$\begin{aligned}\widetilde{|u|^2}(z)|v|^2(z) &\gtrsim \int_{\substack{|w|>\delta \\ |z-w|<\varepsilon}} \left|(1+|z|)^{-\alpha/2} \cdot e^{-(1/2)|z|^2} K_z^\alpha(w) e^{-(1/2)|w|^2}\right|^2 \left|\frac{e^{q(w)-q(z)}}{|w|^{\alpha/2}}\right|^2 dv(w) \\ &+ \int_{\substack{|w|>\delta \\ |z-w|\geq\varepsilon}} \left|(1+|z|)^{-\alpha/2} e^{-(1/2)|z|^2} K_z^\alpha(w) e^{-(1/2)|w|^2}\right|^2 \cdot \left|\frac{e^{q(w)-q(z)}}{|w|^{\alpha/2}}\right|^2 dv(w).\end{aligned} \tag{48}$$

Using the similar method like the case $\alpha < 0$, we can get the desired results and the proof is finished at this moment. □

3. Conclusions

In this content, we deal with the Sarason's problem on the Fock-Sobolev type spaces and have a complete solution that $u = e^q, v = Ce^{-q}$, where q is a linear complex polynomial and C is a nonzero constant. As stated in [1], we know that the Fock-Sobolev type space F_α^2 clearly does not fall under the class of weighted Fock spaces F_φ^2. Therefore the Sarason's problem of weighted Fock spaces F_φ^2 is still open. We will focus on this open problem in the future study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally. All authors read and approved the final manuscript.

Acknowledgments

This paper is supported by National Natural Science Foundation of China (Grants nos. 11471084 and 11301101), Young Innovative Talent Project of Department of Education of Guangdong Province (no. 2017KQNCX220), and the Natural Research Project of Zhaoqing University (nos. 201732 and 221622).

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