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Research Article

Hermite-Hadamard Type Integral Inequalities for Functions Whose Second-Order Mixed Derivatives Are Coordinated (s, m)-P-Convex

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We establish some new Hermite-Hadamard type integral inequalities for functions whose second-order mixed derivatives are coordinated (s, m)-P-convex. An expression form of Hermite-Hadamard type integral inequalities via the beta function and the hypergeometric function is also presented. Our results provide a significant complement to the work of Wu et al. involving the Hermite-Hadamard type inequalities for coordinated (s, m)-P-convex functions in an earlier article.

1. Introduction

Let $f : I \to \mathbb{R}$ be a convex mapping. Then for any $a, b \in I$ with $a < b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \quad (1)$$

This celebrated inequality is known in the literature as the Hermite-Hadamard inequality. As we all know, some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Indeed, Hermite-Hadamard's inequality (1) has already found many applications in mathematical analysis and optimization (see, for example, [1–9]).

In recent years, the applications of various properties of extended convex functions in establishing and improving Hermite-Hadamard type inequalities have attracted the attention of many researchers (see [10–15] and references cited therein).

In [16], Wu et al. established some Hermite-Hadamard type inequalities under the assumption that the function f is a coordinated (s, m)-P-convex function. Motivated by the ideas of work [16], in this paper we study Hermite-Hadamard type inequalities related to the convexity of second-order mixed derivatives of f. More precisely, we focus on establishing some new Hermite-Hadamard type inequalities for functions whose second-order mixed derivatives are coordinated (s, m)-P-convex. For convenience of our discussions in subsequent sections, we begin with recalling some relevant definitions.

Definition 1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad (2)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2 (see [5]). We say that a map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ satisfies the following inequality:

$$f(tx + (1-t)y) \le f(x) + f(y). \quad (3)$$

In [17], the concept of m-convex functions was introduced as follows.

Definition 3 (see [17]). For $f : [0, b] \subseteq \mathbb{R}_0 = [0, +\infty) \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) \quad (4)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is a m-convex function on $[0, b]$.

In [18], the concept of s-convex functions was presented as follows.

Definition 4 (see [18]). Let $s \in (0, 1]$. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex (in the second sense) if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y) \quad (5)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 5 (see [19]). For $(s, m) \in (0, 1] \times (0, 1]$, a function $f : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex if

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y) \quad (6)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 6 (see [20]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y) \quad (7)$$

is valid for all $x, y \in I$ and $t \in (0, 1)$.

Dragomir [21] and Dragomir and Pearce [22] considered the convexity of a function on the coordinates and put forward the following definition.

Definition 7 (see [21, 22]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the coordinates on Δ with $a < b$ and $c < d$ if the partial functions

$$\begin{aligned} f_y &: [a, b] \longrightarrow \mathbb{R}, \\ &f_y(u) = f(u, y), \\ f_x &: [c, d] \longrightarrow \mathbb{R}, \\ &f_x(v) = f(x, v), \end{aligned} \quad (8)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

It should be noted that a formal definition for coordinated convex functions is stated as follows.

Definition 8 (see [21, 22]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the coordinates on Δ with $a < b$ and $c < d$ if the partial function

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\quad \le t\lambda f(x, y) + t(1-\lambda) f(x, w) + (1-t)\lambda f(z, y) \\ &\qquad + (1-t)(1-\lambda) f(z, w) \end{aligned} \quad (9)$$

holds for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in \Delta$.

Definition 9 (see [16]). For some $m \in (0, 1]$ and $s \in [-1, 1]$, a function $f : [0, b] \times [c, d] \to \mathbb{R}$ is said to be coordinated (s, m)-P-convex on $[0, b] \times [c, d]$ with $0 < b$ and $c < d$, if

$$\begin{aligned} &f(tx + m(1-t)z, \lambda y + (1-\lambda)w) \\ &\quad \le t^s [f(x, y) + f(x, w)] \\ &\qquad + m(1-t)^s [f(z, y) + f(z, w)] \end{aligned} \quad (10)$$

holds for all $t \in (0, 1)$, $\lambda \in [0, 1]$ and $(x, y), (z, w) \in [0, b] \times [c, d]$.

Dragomir [21] and Dragomir and Pearce [22] established the following result.

Theorem 10 (see [21, 22]). *Let $f : \Delta = [a, b] \times [c, d]$ be convex on the coordinates on $\Delta = [a, b] \times [c, d]$ with $a < b$ and $c < d$. Then, one has the inequalities:*

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \le \frac{1}{(b-a)(d-c)} \\ &\quad \cdot \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &\quad \le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x, c)\, dx + \int_a^b f(x, d)\, dx\right) \right. \\ &\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a, y)\, dy + \int_c^d f(b, y)\, dy\right)\right] \\ &\quad \le \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned} \quad (11)$$

In this paper, we shall establish some new integral inequalities of Hermite-Hadamard type for coordinated (s, m)-P-convex functions.

2. Lemma

Lemma 11 (see [23]). *If $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives and $\partial^2 f / \partial x \partial y \in L_1(\Delta)$ with $a < b$ and $c < d$, then*

$$\begin{aligned} P(a, b, c, d) &\triangleq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &\quad + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx \end{aligned}$$

$$-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy=(b-a)(d-c)\cdot\int_0^1\int_0^1 K(t,\lambda)\frac{\partial^2}{\partial x\partial y}\cdot f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,dt\,d\lambda, \tag{12}$$

where

$$K(t,\lambda)=\begin{cases} t\lambda, & (t,\lambda)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right],\\ t(\lambda-1), & (t,\lambda)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right],\\ (t-1)\lambda, & (t,\lambda)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right],\\ (t-1)(\lambda-1), & (t,\lambda)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right].\end{cases} \tag{13}$$

3. Main Results

In this section, we establish some Hermite-Hadamard type integral inequalities for functions whose second-order mixed derivatives are coordinated (s,m)-P-convex on the plane $\mathbb{R}_0\times\mathbb{R}$.

Theorem 12. *Suppose that the function $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ has continuous partial derivatives of the second-order and $\partial^2 f/\partial x\partial y\in L_1([0,b^*/m]\times[c,d])$ with $0\le a<b\le b^*$, $c<d$, for some $m\in(0,1]$ and $s\in[-1,1]$. If $|\partial^2 f/\partial x\partial y|^q$ is coordinated (s,m)-P-convex functions on $[0,b^*/m]\times[c,d]$ for $q\ge 1$, then*

(1) *if $s\in(-1,1]$, we have*

$$|P(a,b,c,d)|\le\frac{(b-a)(d-c)}{2^3\times 2^{(s+1)/q}}\left\{\left[\frac{1}{s+2}\Delta_1(q)+m\frac{2^{s+2}-s-3}{(s+1)(s+2)}\Delta_2(m,q)\right]^{1/q}+\left[\frac{2^{s+2}-s-3}{(s+1)(s+2)}\Delta_1(q)+\frac{m}{s+2}\Delta_2(m,q)\right]^{1/q}\right\}, \tag{14}$$

(2) *if $s=-1$, we have*

$$|P(a,b,c,d)|\le\frac{(b-a)(d-c)}{8}\times\left\{[\Delta_1(q)+m(2\ln 2-1)\Delta_2(m,q)]^{1/q}+[(2\ln 2-1)\Delta_1(q)+m\Delta_2(m,q)]^{1/q}\right\}, \tag{15}$$

where

$$\Delta_1(q)=\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q,\quad \Delta_2(m,q)=\left|\frac{\partial^2 f(b/m,c)}{\partial x\partial y}\right|^q+\left|\frac{\partial^2 f(b/m,d)}{\partial x\partial y}\right|^q. \tag{16}$$

Proof. By Lemma 11 and Hölder's integral inequality, we have

$$\begin{aligned}|P(a,b,c,d)|\le{}&(b-a)(d-c)\left(\int_0^1\int_0^1|K(t,\lambda)|\,dt\,d\lambda\right)^{1-1/q}\\&\times\left\{\left[\int_0^{1/2}\int_0^{1/2}t\lambda\left|\frac{\partial^2}{\partial x\partial y}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q}\right.\\&+\left[\int_{1/2}^1\int_0^{1/2}t(1-\lambda)\left|\frac{\partial^2}{\partial x\partial y}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q}\\&+\left[\int_0^{1/2}\int_{1/2}^1(1-t)\lambda\left|\frac{\partial^2}{\partial x\partial y}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q}\\&\left.+\left[\int_{1/2}^1\int_{1/2}^1(1-t)(1-\lambda)\left|\frac{\partial^2}{\partial x\partial y}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q}\right\}.\end{aligned} \tag{17}$$

A straightforward computation gives

$$\begin{aligned}
&\int_0^{1/2} \lambda\, d\lambda = \int_{1/2}^1 (1-\lambda)\, d\lambda = \frac{1}{8},\\
&\int_0^{1/2} t^{s+1} dt = \int_{1/2}^1 (1-t)^{s+1}\, dt = \frac{1}{2^{s+5}(s+2)}, \quad \text{for } s \in (-1,1],\\
&\int_0^{1/2} t(1-t)^s\, dt = \int_{1/2}^1 (1-t)\, t^s dt = \frac{2^{s+2}-s-3}{2^{s+5}(s+1)(s+2)}, \quad \text{for } s \in (-1,1],\\
&\int_0^{1/2} t^{-1+1} dt = \int_{1/2}^1 (1-t)^{-1+1}\, dt = \frac{1}{2},\\
&\int_0^{1/2} t(1-t)^{-1}\, dt = \int_{1/2}^1 (1-t)\, t^{-1} dt = \ln 2 - \frac{1}{2},\\
&\int_0^1 \int_0^1 |K(t,\lambda)|\, dt\, d\lambda = \frac{1}{16}.
\end{aligned} \tag{18}$$

Now, by using the coordinated (s,m)-P-convexity of $|\partial^2 f/\partial x\partial y|^q$, it follows that if $-1 < s \le 1$, we have

$$\begin{aligned}
&\int_0^{1/2}\int_0^{1/2} t\lambda \left|\frac{\partial^2}{\partial x\partial y} \cdot f(ta+(1-t)b, \lambda c+(1-\lambda)d)\right|^q dt\, d\lambda\\
&\le \int_0^{1/2}\int_0^{1/2} t\lambda \left[t^s \Delta_1(q) + m(1-t)^s \cdot \Delta_2(m,q)\right] dt\, d\lambda = \frac{1}{2^{s+5}(s+2)}\Delta_1(q)\\
&+ \frac{m\left(2^{s+2}-s-3\right)}{2^{s+5}(s+1)(s+2)}\Delta_2(m,q),
\end{aligned} \tag{19}$$

and if $s=-1$, we have

$$\begin{aligned}
&\int_0^{1/2}\int_0^{1/2} t\lambda \left|\frac{\partial^2}{\partial x\partial y} f(ta+(1-t)b, \lambda c+(1-\lambda)d)\right|^q dt\, d\lambda\\
&\le \int_0^{1/2}\int_0^{1/2} t\lambda \left[t^{-1}\Delta_1(q) + m(1-t)^{-1}\Delta_2(m,q)\right] dt\, d\lambda\\
&= \frac{1}{16}\left[\Delta_1(q) + m(2\ln 2 - 1)\Delta_2(m,q)\right].
\end{aligned} \tag{20}$$

By a similar argument, we obtain

$$\begin{aligned}
&\int_{1/2}^1\int_0^{1/2} t(1-\lambda) \cdot \left|\frac{\partial^2}{\partial x\partial y} f(ta+(1-t)b, \lambda c+(1-\lambda)d)\right|^q dt\, d\lambda\\
&\le \frac{1}{2^{s+5}} \times \begin{cases} \dfrac{1}{s+2}\Delta_1(q) + m\dfrac{2^{s+2}-s-3}{(s+1)(s+2)}\Delta_2(m,q), & -1 < s \le 1,\\ \Delta_1(q) + m(2\ln 2 - 1)\Delta_2(m,q), & s=-1, \end{cases}\\
&\int_0^{1/2}\int_{1/2}^1 (1-t) \cdot \lambda \left|\frac{\partial^2}{\partial x\partial y} f(ta+(1-t)b, \lambda c+(1-\lambda)d)\right|^q dt\, d\lambda\\
&\le \frac{1}{2^{s+5}} \times \begin{cases} \dfrac{2^{s+2}-s-3}{(s+1)(s+2)}\Delta_1(q) + \dfrac{m}{s+2}\Delta_2(m,q), & -1 < s \le 1,\\ (2\ln 2 - 1)\Delta_1(q) + m\Delta_2(m,q), & s=-1, \end{cases}\\
&\int_{1/2}^1\int_{1/2}^1 (1-t)(1-\lambda) \cdot \left|\frac{\partial^2}{\partial x\partial y} f(ta+(1-t)b, \lambda c+(1-\lambda)d)\right|^q dt\, d\lambda\\
&\le \frac{1}{2^{s+5}} \times \begin{cases} \dfrac{2^{s+2}-s-3}{(s+1)(s+2)}\Delta_1(q) + \dfrac{m}{s+2}\Delta_2(m,q), & -1 < s \le 1,\\ (2\ln 2 - 1)\Delta_1(q) + m\Delta_2(m,q), & s=-1. \end{cases}
\end{aligned} \tag{21}$$

Applying (18) and inequalities (19)–(21) into inequality (17), we get (14) and (15). This completes the proof of Theorem 12. □

Corollary 13. *Under the assumptions of Theorem 12, if $q=1$, then*

(1) *if $s \in (-1,1]$, then*

$$|P(a,b,c,d)| \le \frac{(b-a)(d-c)\left(2^{s+1}-1\right)}{2^{s+3}(s+1)(s+2)}\left[\Delta_1(1) + m\Delta_2(m,1)\right], \tag{22}$$

(2) *if $s=-1$, then*

$$|P(a,b,c,d)| \le \frac{(b-a)(d-c)\ln 2}{4}\left[\Delta_1(1) + m\Delta_2(m,1)\right]. \tag{23}$$

Corollary 14. *Under the assumptions of Theorem 12, if $q = m = 1$, then*

(1) *if $s \in (-1,1]$, then*

$$\begin{aligned}
|P(a,b,c,d)| \le{}& \frac{(b-a)(d-c)\left(2^{s+1}-1\right)}{2^{s+3}(s+1)(s+2)}\\
&\times \left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right| + \left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right| + \left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right| + \left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|\right],
\end{aligned} \tag{24}$$

(2) *if $s=-1$, then*

$$\begin{aligned}
|P(a,b,c,d)| \le{}& \frac{(b-a)(d-c)\ln 2}{4} \times \left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|\right.\\
&\left. + \left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right| + \left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right| + \left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|\right].
\end{aligned} \tag{25}$$

Furthermore, if $q = m = 1$, $s = 0$, *then*

$$|P(a,b,c,d)| \le \frac{(b-a)(d-c)}{16} \times \left[\left|\frac{\partial^2 f(a,c)}{\partial x \partial y}\right| + \left|\frac{\partial^2 f(a,d)}{\partial x \partial y}\right| + \left|\frac{\partial^2 f(b,c)}{\partial x \partial y}\right| + \left|\frac{\partial^2 f(b,d)}{\partial x \partial y}\right|\right]. \tag{26}$$

Theorem 15. *Suppose that the function* $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ *has continuous partial derivatives of the second-order and* $\partial^2 f/\partial x \partial y \in L_1([0, b^*/m] \times [c,d])$ *with* $0 \le a < b \le b^*$, $c < d$, *and* $0 \le r \le q$, $-1 < \ell \le q$. *If* $|\partial^2 f/\partial x \partial y|^q$ *is coordinated* (s,m)*-P-convex functions on* $[0, b^*/m] \times [c,d]$ *for some* $m \in (0,1]$, $s \in (-1,1]$, *and* $q > 1$, *then*

$$\begin{aligned}|P(a,b,c,d)| \le{}& \frac{(b-a)(d-c)}{[2^{\ell}(\ell+1)]^{1/q}} \left(\frac{(q-1)^2}{(2q-r-1)(2q-\ell-1)} \left(\frac{1}{2}\right)^{(4q-r-\ell-2)/(q-1)}\right)^{1-1/q} \\ &\times \left\{ \left[\frac{\Delta_1(q)}{2^{r+s+1}(r+s+1)} + \frac{2^{-s} m \Delta_2(m,q)}{r+1} {}_2F_1\left(-s, r+1, r+2, 2^{-1}\right)\right]^{1/q} \right. \\ &\left. + \left[\left[B(r+1,s+1) - 2^{-s} {}_2F_1\left(-r, s+1, s+2, 2^{-1}\right)\right] \frac{\Delta_1(q)}{s+1} + \frac{m\Delta_2(m,q)}{2^{r+s+1}(r+s+1)}\right]^{1/q} \right\},\end{aligned} \tag{27}$$

where $\Delta_1(q)$ *and* $\Delta_2(m,q)$ *are defined as in (16), and* $B(\alpha,\beta)$ *is the beta function defined by*

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \quad \alpha,\beta > 0, \tag{28}$$

and ${}_2F_1(c,d,e,z)$ *is the hypergeometric function defined by*

$${}_2F_1(c,d,e,z) = \frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)} \int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,dt \tag{29}$$

for $e > d > 0$, $|z| < 1$, $c \in \mathbb{R}$, $u > 0$.

Proof. Using Lemma 11 and Hölder's integral inequality, we obtain

$$\begin{aligned}|P(a,b,c,d)| \le{}& (b-a)(d-c) \left\{ \left(\int_0^{1/2}\int_0^{1/2} t^{(q-r)/(q-1)} \lambda^{(q-\ell)/(q-1)} dt\,d\lambda\right)^{1-1/q} \right. \\ &\times \left[\int_0^{1/2}\int_0^{1/2} t^r \lambda^{\ell} \left|\frac{\partial^2}{\partial x \partial y} f(ta+(1-t)b, \lambda c + (1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q} \\ &+ \left(\int_{1/2}^{1}\int_0^{1/2} t^{(q-r)/(q-1)} (1-\lambda)^{(q-\ell)/(q-1)} dt\,d\lambda\right)^{1-1/q} \\ &\times \left[\int_{1/2}^{1}\int_0^{1/2} t^r (1-\lambda)^{\ell} \left|\frac{\partial^2}{\partial x \partial y} f(ta+(1-t)b, \lambda c + (1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q} \\ &+ \left(\int_0^{1/2}\int_{1/2}^{1} (1-t)^{(q-r)/(q-1)} \lambda^{(q-\ell)/(q-1)} dt\,d\lambda\right)^{1-1/q} \\ &\times \left[\int_0^{1/2}\int_{1/2}^{1} (1-t)^r \lambda^{\ell} \left|\frac{\partial^2}{\partial x \partial y} f(ta+(1-t)b, \lambda c + (1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q} \\ &+ \left(\int_{1/2}^{1}\int_{1/2}^{1} (1-t)^{(q-r)/(q-1)} (1-\lambda)^{(q-\ell)/(q-1)} dt\,d\lambda\right)^{1-1/q} \\ &\left. \times \left[\int_{1/2}^{1}\int_{1/2}^{1} (1-t)^r (1-\lambda)^{\ell} \left|\frac{\partial^2}{\partial x \partial y} f(ta+(1-t)b, \lambda c + (1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q} \right\}.\end{aligned} \tag{30}$$

After some calculations, it follows that

$$\begin{aligned}
&\int_0^{1/2}\int_0^{1/2} t^{(q-r)/(q-1)}\lambda^{(q-\ell)/(q-1)}dt\,d\lambda \\
&= \int_{1/2}^{1}\int_0^{1/2} t^{(q-r)/(q-1)}(1-\lambda)^{(q-\ell)/(q-1)}dt\,d\lambda \\
&= \int_0^{1/2}\int_{1/2}^{1} (1-t)^{(q-r)/(q-1)}\lambda^{(q-\ell)/(q-1)}dt\,d\lambda \\
&= \int_{1/2}^{1}\int_{1/2}^{1} (1-t)^{(q-r)/(q-1)}(1-\lambda)^{(q-\ell)/(q-1)}dt\,d\lambda \\
&= \frac{(q-1)^2}{(2q-r-1)(2q-\ell-1)} \times 2^{-(4q-r-\ell-2)/(q-1)}.
\end{aligned} \tag{31}$$

From the coordinated (s,m)-P-convexity of $|\partial^2 f/\partial x\partial y|^q$, we deduce that

$$\begin{aligned}
&\int_0^{1/2}\int_0^{1/2} t^r\lambda^\ell \left|\frac{\partial^2}{\partial x\partial y}\cdot f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda \\
&\le \frac{2^{-(\ell+1)}}{\ell+1}\int_0^{1/2} t^r\left[t^s\Delta_1(q)+m(1-t)^s\cdot\Delta_2(m,q)\right]dt = \frac{2^{-(\ell+1)}}{\ell+1}\left[\frac{\Delta_1(q)}{2^{r+s+1}(r+s+1)}\right. \\
&\left.+\frac{2^{-s}m\Delta_2(m,q)}{r+1}\,{}_2F_1\left(-s,r+1,r+2,2^{-1}\right)\right], \\
&\int_{1/2}^{1}\int_0^{1/2} t^r(1-\lambda)^\ell \left|\frac{\partial^2}{\partial x\partial y}\cdot f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda \\
&\le \frac{2^{-(\ell+1)}}{\ell+1}\left[\frac{\Delta_1(q)}{2^{r+s+1}(r+s+1)}+\frac{2^{-s}m\Delta_2(m,q)}{r+1}\cdot{}_2F_1\left(-s,r+1,r+2,2^{-1}\right)\right], \\
&\int_0^{1/2}\int_{1/2}^{1} (1-t)^r\lambda^\ell \left|\frac{\partial^2}{\partial x\partial y}\cdot f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda \\
&\le \frac{2^{-(\ell+1)}}{\ell+1}\left[\left[B(r+1,s+1)-2^{-s}{}_2F_1\left(-r,s+1,s+2,2^{-1}\right)\right]\times\frac{\Delta_1(q)}{s+1}+\frac{m\Delta_2(m,q)}{2^{r+s+1}(r+s+1)}\right], \\
&\int_{1/2}^{1}\int_{1/2}^{1} (1-t)^r(1-\lambda)^\ell \left|\frac{\partial^2}{\partial x\partial y}\cdot f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda \\
&\le \frac{2^{-(\ell+1)}}{\ell+1}\left[\left[B(r+1,s+1)-2^{-s}{}_2F_1\left(-r,s+1,s+2,2^{-1}\right)\right]\times\frac{\Delta_1(q)}{s+1}+\frac{m\Delta_2(m,q)}{2^{r+s+1}(r+s+1)}\right].
\end{aligned} \tag{32}$$

Applying (31) and inequalities (32) into inequality (30), we get inequality (27). The proof of Theorem 15 is complete. □

Corollary 16. *Under the assumptions of Theorem 15, if $r=0$, then*

$$\begin{aligned}
|P(a,b,c,d)| &\le \frac{(b-a)(d-c)}{\left[2^{s+\ell+1}(s+1)(\ell+1)\right]^{1/q}}\left(\frac{(q-1)^2}{(2q-1)(2q-\ell-1)}\left(\frac{1}{2}\right)^{(4q-\ell-2)/(q-1)}\right)^{1-1/q} \\
&\quad\times\left\{\left[\Delta_1(q)+m\left(2^{s+1}-1\right)\Delta_2(m,q)\right]^{1/q}+\left[\left(2^{s+1}-1\right)\Delta_1(q)+m\Delta_2(m,q)\right]^{1/q}\right\}.
\end{aligned} \tag{33}$$

In particular, if $r=\ell=0$, then

$$\begin{aligned}
&|P(a,b,c,d)| \\
&\le \frac{(b-a)(d-c)}{\left[2^{s+1}(s+1)\right]^{1/q}}\left(\frac{q-1}{2q-1}\left(\frac{1}{2}\right)^{(2q-1)/(q-1)}\right)^{2(1-1/q)} \\
&\quad\times\left\{\left[\Delta_1(q)+m\left(2^{s+1}-1\right)\Delta_2(m,q)\right]^{1/q}\right. \\
&\quad\left.+\left[\left(2^{s+1}-1\right)\Delta_1(q)+m\Delta_2(m,q)\right]^{1/q}\right\}.
\end{aligned} \tag{34}$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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