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Research Article

Numerical Ranges of Normal Weighted Composition Operators on the Fock Space of $\mathbb{C}^N$

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Numerical ranges of normal weighted composition operators on the Fock space of $\mathbb{C}^N$ are completely characterized. The main result shows that numerical ranges of such operators are closely related to their composition symbols.

1. Introduction

Inspired by results on numerical ranges of (weighted) composition operators on Hardy space [1], in this paper we give a complete characterization of numerical ranges of normal weighted composition operators on the Fock space of $\mathbb{C}^N$ $(N \geq 1)$, the N-dimensional complex Euclidean space.

Recall that for a bounded operator T on a complex Hilbert space $\mathscr{H}$, the numerical range $W(T)$ of T is defined as

$$W(T) = \{\langle Tf, f\rangle_{\mathscr{H}} \mid f \in \mathscr{H},\ \|f\|_{\mathscr{H}} = 1\}. \tag{1}$$

The Fock space $\mathscr{F}^2$ over $\mathbb{C}^N$ is the space of analytic functions f on $\mathbb{C}^N$ with

$$\|f\|^2 = \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |f(z)|^2 \exp\left(-\frac{|z|^2}{2}\right) dm_{2N}(z), \tag{2}$$

where $|z|$ is the norm for $z \in \mathbb{C}^N$ and dm_{2N} denotes usual Lebesgue measure on $\mathbb{C}^N$. It is well known that $\mathscr{F}^2$ is a reproducing kernel Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} f(z)\overline{g(z)} \exp\left(-\frac{|z|^2}{2}\right) dm_{2N}(z), \quad f, g \in \mathscr{F}^2 \tag{3}$$

and reproducing kernel functions

$$K_w(z) = \exp\left(\frac{\langle z, w\rangle}{2}\right), \quad w, z \in \mathbb{C}^N, \tag{4}$$

where $\langle z, w\rangle$ denotes the inner product for $z, w \in \mathbb{C}^N$ and $|z|^2 = \langle z, z\rangle$. Usually, k_w denotes the normalization of K_w,

$$k_w(z) = \frac{K_w(z)}{\|K_w\|} = \exp\left(\frac{\langle z, w\rangle}{2} - \frac{|w|^2}{4}\right). \tag{5}$$

A weighted composition operator $C_{\psi,\varphi}$ on $\mathscr{F}^2$ with ψ an analytic function on $\mathbb{C}^N$ and φ an analytic self-map of $\mathbb{C}^N$ is defined as

$$C_{\psi,\varphi} f = \psi(f \circ \varphi), \quad f \in \mathscr{F}^2. \tag{6}$$

Weighted composition operators on $\mathscr{F}^2$ have been studied extensively. It is noteworthy that weighted composition operators on $\mathscr{F}^2$ of $\mathbb{C}$ are closely related to some important transforms on $L^2(\mathbb{R})$ [2].

In [3], normal weighted composition operators on $\mathscr{F}^2$ are characterized completely. We have the followingresult.

Theorem 1. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ if and only if*

$$\varphi(z) = Az + b,$$
$$\psi(z) = sK_c(z), \qquad (7)$$
$$z \in \mathbb{C}^N,$$

where A is a normal operator on $\mathbb{C}^N$ with $|A| \le 1$, $b, c \in \mathbb{C}^N$ with $(I-A)c = (I-A^*)b$, $|b| = |c|$, and s is a nonzero constant. Furthermore

$$\langle A\xi, b + Ac\rangle = 0 \qquad (8)$$

whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$.

For the presentation of our main result, we make some notations.

Let A be an operator on $\mathbb{C}^N$. $|A|$ means the norm of A. $\ker A = \{z \in \mathbb{C}^N \mid Az = 0\}$ and $\operatorname{ran} A = \{Az \mid z \in \mathbb{C}^N\}$. For $z \in \mathbb{C}^N$, $A^{-1}z = \{w \in \mathbb{C}^N \mid Aw = z\}$. I is the identity operator on $\mathbb{C}^N$.

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in \mathbb{C}^N$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{N}_0^N$ ($\mathbb{N}_0$ denotes the set of nonnegative integers), let $\lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}$.

For a set $E \subset \mathbb{C}$, let $\overline{E}$ and $\operatorname{co} E$ denote the closure of E and the convex set generated by E, respectively. For $s \in \mathbb{C}$, $sE = \{su \mid u \in E\}$. $\mathbb{D}$ is the open unit disk of $\mathbb{C}$ and $\mathbb{T}$ is the unit circle of $\mathbb{C}$. For $r > 0$, $\mathrm{D}(0, r) = \{u \in \mathbb{C} \mid |u| < r\}$.

Based on Theorem 1, our main result reads as follows.

Theorem 2. *Let $\varphi(z) = Az + b$ and $\psi(z) = K_c(z)$ such that $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$, $\lambda_1, \lambda_2, \cdots, \lambda_N$ be eigenvalues of A.*

(1) If $b \in \operatorname{ran}(I - A)$, then

$$W\left(C_{\psi,\varphi}\right) = \exp\left(\frac{\langle p, c\rangle}{2}\right)\operatorname{co}\left\{\lambda^\alpha \mid \alpha \in \mathbb{N}_0^N\right\} \qquad (9)$$

with $p \in (I - A)^{-1}b$.

(2) If $b \notin \operatorname{ran}(I - A)$ and $b = b_1 + b_2$ with $b_1 \in \operatorname{ran}(I - A)^\perp$, $b_2 \in \operatorname{ran}(I - A)$, then

$$W\left(C_{\psi,\varphi}\right) = \mathrm{D}\left(0, \left|\exp\left(\frac{\langle p, c\rangle + \langle b_1, p\rangle}{2} + \frac{|b_1|^2}{4}\right)\right|\right) \qquad (10)$$

with $p \in (I - A)^{-1}b_2$.

In [4], the spectrum of normal weighted composition operators on $\mathscr{F}^2$ is considered. Although the spectrum of a normal operator is closely related to its numerical ranges, it seems that a complete characterization of spectrum of normal weighted composition operators on $\mathscr{F}^2$ is a little more difficult. For more results on numerical ranges of (weighted) composition operators on Hardy space, see [5–8].

2. Proof of Main Result

In this section, we give the proof of our main result, Theorem 2. First, we recall some known results on numerical ranges of operators and a class of very important unitary operators on the Fock space.

Let T be a bounded operator on a complex Hilbert space $\mathscr{H}$. Denote $\sigma(T)$ with the spectrum of T and $\sigma_p(T)$ the eigenvalues of T. $u \in \sigma_p(T)$ if and only if there exists $f \in H$, $f \neq 0$ such that $Tf = uf$, and f is called an eigenvector of T corresponding to the eigenvalue u. It is well known that $\sigma_p(T) \subset W(T) \subset \overline{\mathrm{D}(0, \|T\|_{\mathscr{H}})}$. If T is a normal operator, then eigenvectors that correspond to distinct eigenvalues are orthogonal. Moreover, for a normal operator T, $\overline{W(T)} = \operatorname{co}(\sigma(T))$ [9, Theorem 1.4-4].

For $p \in \mathbb{C}^N$, denote $\varphi_p(z) = z - p$, $U_p = C_{k_p,\varphi_p}$. Then U_p is a unitary operator on $\mathscr{F}^2$ and $U_p^{-1} = U_{-p}$ [10, Proposition 2.3].

For convenience, we make some notations.

Let ψ be an analytic function on $\mathbb{C}^N$ and φ be an analytic mapping of $\mathbb{C}^N$. Denote

$$\Phi_p(z) = \left(\varphi_p \circ \varphi \circ \varphi_{-p}\right)(z) \qquad (11)$$

and

$$\Psi_p(z) = \left(k_{-p} \cdot \left(\psi \circ \varphi_{-p}\right) \cdot \left(k_p \circ \varphi \circ \varphi_{-p}\right)\right)(z). \qquad (12)$$

Then

$$U_{-p}C_{\psi,\varphi}U_p = C_{k_{-p}(\psi\circ\varphi_{-p})(k_p\circ\varphi\circ\varphi_{-p}),\varphi_p\circ\varphi\circ\varphi_{-p}} = C_{\Psi_p,\Phi_p} \qquad (13)$$

and $C_{\psi,\varphi}$ is unitarily equivalent to C_{Φ_p,Ψ_p}.

Let $\varphi(z) = Az + b$, $\psi(z) = K_c(z)$. Then

$$\Phi_p(z) = Az + \left(b - (I - A)p\right) \qquad (14)$$

and

$$\Psi_p(z) = \exp\left(\frac{\langle z, c - (I - A^*)p\rangle}{2} + \frac{\langle p, c\rangle + \langle b, p\rangle - \langle (I - A)p, p\rangle}{2}\right). \qquad (15)$$

Now we consider normal weighted composition operators on $\mathscr{F}^2$ from a different view, which will simplify the characterization of such operators' numerical ranges.

For an operator A on $\mathbb{C}^N$, let C_A denote the composition operator defined by $C_A f(z) = f(Az)$.

Theorem 3. *Let $C_{\psi,\varphi}$ be a bounded normal operator on $\mathscr{F}^2$ with $\varphi(z) = Az + b$, $\psi(z) = K_c(z)$.*

(1) If $b \in \operatorname{ran}(I - A)$, then $C_{\psi,\varphi}$ is unitarily equivalent to $\exp(\langle p, c\rangle/2)C_A$ with $p \in (I - A)^{-1}b$.

(2) If $b \notin \operatorname{ran}(I - A)$, then for any $t \in \mathbb{R}$, $C_{\psi,\varphi}$ is unitarily equivalent to $e^{it}C_{\psi,\varphi}$.

(3) If $b \notin \operatorname{ran}(I - A)$ and $b = b_1 + b_2$ with $b_1 \in \operatorname{ran}(I - A)^\perp$, $b_2 \in \operatorname{ran}(I - A)$, then $C_{\psi,\varphi}$ is unitarily equivalent to $\exp((\langle p, c\rangle + \langle b_1, p\rangle)/2)C_{\psi_1,\varphi_1}$, where $\varphi_1(z) = Az + b_1$, $\psi_1(z) = K_{-b_1}(z)$ and $p \in (I - A)^{-1}b_2$.

Proof. (1) If $b \in \operatorname{ran}(I-A)$, then there exists $p \in \mathbb{C}^N$ such that

$$(I-A)\,p = b. \tag{16}$$

Since $C_{\psi,\varphi}$ is bounded and normal, $|A| \le 1$, A is normal and $(I-A)c = (I-A^*)b$ by Theorem 1. So we have

$$\begin{aligned}(I-A)\,c &= (I-A^*)\,b = (I-A^*)(I-A)\,p \\ &= (I-A)(I-A^*)\,p.\end{aligned} \tag{17}$$

I.e., $(I-A)(c-(I-A^*)p) = 0$. It follows that

$$c-(I-A^*)\,p \in \ker(I-A). \tag{18}$$

For any $\xi \in \ker(I-A)$, $A\xi = \xi$ and by Proposition 3.1 in [11, Ch I], $A^*\xi = \xi$. Since $\operatorname{ran}(I-A)^\perp = \ker(I-A^*)$, $\xi \perp b$. By Theorem 1, $\langle A\xi, b+Ac\rangle = 0$. But

$$\begin{aligned}\langle A\xi, b+Ac\rangle &= \langle \xi, b+Ac\rangle = \langle \xi, b\rangle + \langle A^*\xi, c\rangle \\ &= \langle \xi, c\rangle.\end{aligned} \tag{19}$$

So $\langle c, \xi\rangle = 0$. Hence

$$\begin{aligned}\langle c-(I-A^*)\,p, \xi\rangle &= \langle c,\xi\rangle - \langle p,\xi\rangle + \langle p, A\xi\rangle \\ &= \langle c,\xi\rangle = 0,\end{aligned} \tag{20}$$

which implies that

$$c-(I-A^*)\,p \perp \ker(I-A). \tag{21}$$

It follows from (18) and (21) that

$$(I-A^*)\,p = c. \tag{22}$$

By formulas (14) and (15), $\Phi_p(z) = Az$ and $\Psi_p(z) = \exp(\langle p,c\rangle/2)$. So $C_{\psi,\varphi}$ is unitarily equivalent to $C_{\Psi_p,\phi_p} = \exp(\langle p,c\rangle/2)C_A$.

(2) If $b \notin \operatorname{ran}(I-A)$, then $b \neq 0$ and there exist $b_1 \in \operatorname{ran}(I-A)^\perp$ and $b_2 \in \operatorname{ran}(I-A)$ such that

$$b = b_1 + b_2. \tag{23}$$

Obviously, $b_1 \neq 0$.

For any $t \in \mathbb{R}$, let $p = itb_1$, i is the imaginary unit. Notice $\ker(I-A) = \ker(I-A^*) = \operatorname{ran}(I-A)^\perp$ by Proposition 3.1 in [11, Ch I], so we have

$$A^*p = Ap = p. \tag{24}$$

By Theorem 1, $0 = \langle Ap, Ac+b\rangle = \langle p, Ac+b\rangle = \langle A^*p, c\rangle + \langle p, b\rangle = \langle p,c\rangle + \langle p,b\rangle$. So

$$\langle p,c\rangle + \langle b,p\rangle = -\langle p,b\rangle + \langle b,p\rangle = -2it\,|b_1|^2. \tag{25}$$

By formulas (14) and (15),

$$\begin{aligned}\Phi_p(z) &= Az + b = \varphi(z), \\ \Psi_p(z) &= \exp\left(\frac{\langle z,c\rangle}{2} - it\,|b_1|^2\right) \\ &= \exp\left(-it\,|b_1|^2\right)\psi(z).\end{aligned} \tag{26}$$

So $C_{\psi,\varphi}$ is unitarily equivalent to $\exp(-it|b_1|^2)C_{\psi,\varphi}$.

(3) Since $b_2 \in \operatorname{ran}(I-A)$, there exists $p \in \mathbb{C}^N$ such that

$$(I-A)\,p = b_2. \tag{27}$$

By Theorem 1, A is normal and $(I-A)c = (I-A^*)b$. We have

$$\begin{aligned}&(I-A)\left(c-(I-A^*)\,p\right) \\ &\quad= (I-A)\,c - (I-A)(I-A^*)\,p \\ &\quad= (I-A^*)\,b - (I-A^*)(I-A)\,p \\ &\quad= (I-A^*)(b_1+b_2) - (I-A^*)\,b_2 = (I-A^*)\,b_1 \\ &\quad= 0,\end{aligned} \tag{28}$$

since $b_1 \in \operatorname{ran}(I-A)^\perp = \ker(I-A^*) = \ker(I-A)$ by Proposition 3.1 in [11, Ch I], which implies that $c-(I-A^*)p \in \ker(I-A)$. Hence

$$c-(I-A^*)\,p + b_1 \in \ker(I-A). \tag{29}$$

On the other hand, for any $\xi \in \ker(I-A)$, by Ch 1, Proposition 3.1 in [11],

$$\xi \in \ker(I-A^*) = \operatorname{ran}(I-A)^\perp. \tag{30}$$

So, by Theorem 1,

$$\begin{aligned}0 &= \langle A\xi, b+Ac\rangle = \langle \xi, b+Ac\rangle = \langle \xi, b_1+b_2\rangle + \langle \xi, c\rangle \\ &= \langle \xi, b_1\rangle + \langle \xi, c\rangle.\end{aligned} \tag{31}$$

But

$$\begin{aligned}&\langle c-(I-A^*)\,p + b_1, \xi\rangle \\ &\quad= \langle c,\xi\rangle - \langle (I-A^*)\,p, \xi\rangle + \langle b_1,\xi\rangle \\ &\quad= \langle c,\xi\rangle - \langle p, (I-A)\,\xi\rangle + \langle b_1,\xi\rangle \\ &\quad= \langle c,\xi\rangle + \langle b_1,\xi\rangle.\end{aligned} \tag{32}$$

So we have $\langle c-(I-A^*)p + b_1, \xi\rangle = 0$, which implies that

$$c-(I-A^*)\,p + b_1 \perp \ker(I-A). \tag{33}$$

It follows from (29) and (33) that

$$c-(I-A^*)\,p + b_1 = 0. \tag{34}$$

By formulas (14) and (15),

$$\begin{aligned}\Phi_p(z) &= Az + b_1, \\ \Psi_p(z) &= \exp\left(\frac{\langle z, -b_1\rangle}{2} + \frac{\langle p,c\rangle + \langle b_1,p\rangle}{2}\right) \\ &= \exp\left(\frac{\langle p,c\rangle + \langle b_1,p\rangle}{2}\right)K_{-b_1}(z).\end{aligned} \tag{35}$$

So $C_{\psi,\varphi}$ is unitarily equivalent to $\exp((\langle p,c\rangle + \langle b_1,p\rangle)/2)C_{\psi_1,\varphi_1}$. □

For an analytic function φ on $\mathbb{C}^N$, it is well known that the composition operator C_φ on $\mathscr{F}^2$ is a normal operator if and only if $\varphi(z) = Az$ for some normal operator A on $\mathbb{C}^N$ with $|A| \le 1$. In fact, such operators are diagonalizable. In [8], the numerical ranges of such operators on Hardy space of $\mathbb{D}$ are studied.

Lemma 4. *Let A be a normal operator on $\mathbb{C}^N$ with $|A| \le 1$ and $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Then on $\mathscr{F}^2$, $\|C_A\| = 1$.*

Proof. Since A is a normal operator on $\mathbb{C}^N$, there exists a unitary operator U on $\mathbb{C}^N$ such that

$$UAU^* := \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_{k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix} = D \tag{36}$$

Then $C_{U^*}C_AC_U = C_D$ and $\|C_A\| = \|C_D\|$.

For any $f \in \mathscr{F}^2$, let $f(z) = \sum_{\alpha\in\mathbb{N}_0^N} c(\alpha)z^\alpha$; then

$$(C_D f)(z) = f(Dz) = \sum_{\alpha\in\mathbb{N}_0^N} c(\alpha)\lambda^\alpha z^\alpha, \tag{37}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Since $|\lambda^\alpha| \le 1$, $\|C_D f\| \le \|f\|$. It follows that $\|C_D\| \le 1$. But $C_D 1 = 1$. So $\|C_D\| = 1$. □

Theorem 5. *Let $C_{\psi,\varphi}$ be a bounded normal operator on $\mathscr{F}^2$ with $\varphi(z) = Az + b$, $\psi(z) = K_c(z)$.*

(1) If $b \in \operatorname{ran}(I - A)$, then $\|C_{\psi,\varphi}\| = |\exp(\langle p, c\rangle/2)|$ with $p \in (I - A)^{-1}b$.

(2) If $b \notin \operatorname{ran}(I - A)$ and $b = b_1 + b_2$ with $b_1 \in \operatorname{ran}(I - A)^\perp$, $b_2 \in \operatorname{ran}(I - A)$, then $\|C_{\psi,\varphi}\| = |\exp((\langle p, c\rangle + \langle b_1, p\rangle)/2 + |b_1|^2/4)|$, where $p \in (I - A)^{-1}b_2$.

Proof. (1) The conclusion follows from Theorem 3 (1) and Lemma 4.

(2) By Theorem 3(3), $C_{\psi,\varphi}$ is unitarily equivalent to $\exp((\langle p, c\rangle + \langle b_1, p\rangle)/2)C_{\psi_1,\varphi_1}$ with

$$\begin{aligned} \varphi_1(z) &= Az + b_1, \\ \psi_1(z) &= K_{-b_1}(z) \end{aligned} \tag{38}$$

and $p \in (I - A)^{-1}b_2$.

For any $f \in \mathscr{F}^2$,

$$\begin{aligned} \left\|C_{\psi_1,\varphi_1} f\right\|^2 &= \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |\psi_1(z) f(\varphi_1(z))|^2 \\ &\quad \cdot \exp\left(-\frac{|z|^2}{2}\right) dm_{2N}(z) = \frac{1}{(2\pi)^N} \\ &\quad \cdot \int_{\mathbb{C}^N} |f(Az + b_1)|^2 \\ &\quad \cdot \exp\left(-\frac{|z|^2 + \langle z, b_1\rangle + \langle b_1, z\rangle}{2}\right) dm_{2N}(z) \\ &= \exp\left(\frac{|b_1|^2}{2}\right)\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |f(Az + b_1)|^2 \\ &\quad \cdot \exp\left(-\frac{|z + b_1|^2}{2}\right) dm_{2N}(z) \\ &= \exp\left(\frac{|b_1|^2}{2}\right)\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |f(A(z - b_1) + b_1)|^2 \\ &\quad \cdot \exp\left(-\frac{|z|^2}{2}\right) dm_{2N}(z). \end{aligned} \tag{39}$$

Since $b_1 \in \operatorname{ran}(I - A)^\perp = \ker(I - A^*) = \ker(I - A)$, $b_1 = Ab_1$. Therefore,

$$\begin{aligned} \left\|C_{\psi_1,\varphi_1} f\right\|^2 &= \exp\left(\frac{|b_1|^2}{2}\right)\frac{1}{(2\pi)^N} \\ &\quad \cdot \int_{\mathbb{C}^N} |f(Az)|^2 \exp\left(-\frac{|z|^2}{2}\right) dm_{2N}(z) \\ &= \exp\left(\frac{|b_1|^2}{2}\right)\|C_A f\|^2. \end{aligned} \tag{40}$$

It follows from Lemma 4 that $\|C_{\psi_1,\varphi_1}\| = \exp(|b_1|^2/4)$. Hence

$$\left\|C_{\psi,\varphi}\right\| = \left|\exp\left(\frac{\langle p, c\rangle + \langle b_1, p\rangle}{2} + \frac{|b_1|^2}{4}\right)\right|. \tag{41}$$

□

The following result is well known.

Lemma 6. *Let A be a normal operator on $\mathbb{C}^N$, $|A| \le 1$ and $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Then on $\mathscr{F}^2$,*

$$\begin{aligned} \sigma_p(C_A) &= \left\{\lambda^\alpha : \alpha \in \mathbb{N}_0^N\right\}, \\ \sigma(C_A) &= \overline{\left\{\lambda^\alpha : \alpha \in \mathbb{N}_0^N\right\}} = \overline{\sigma_p(C_A)}, \end{aligned} \tag{42}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$.

Theorem 7. *Let A be a normal operator on $\mathbb{C}^N$, $|A| \le 1$, and $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Then on $\mathscr{F}^2$, $W(C_A) = \operatorname{co}\sigma_p(C_A)$.*

Proof. Note that C_A is a normal operator on $\mathscr{F}^2$; we have

$$\operatorname{co}\sigma_p(C_A) \subset W(C_A) \subset \overline{W(C_A)} = \operatorname{co}\sigma(C_A). \tag{43}$$

If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) = (0, 0, \cdots, 0)$, then $A = 0$. In this case, $\sigma(C_A) = \sigma_p(C_A) = \{0, 1\}$. By formula (43),

$$W(C_A) = \operatorname{co}\sigma_p(C_A) = [0, 1]. \tag{44}$$

If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) = (1, 1, \cdots, 1)$, then $A = I$. In this case, $\sigma(C_A) = \sigma_p(C_A) = \{1\}$. By formula (43),

$$W(C_A) = \operatorname{co}\sigma_p(C_A) = \{1\}. \tag{45}$$

In the following, we assume $(0, 0, \cdots, 0) \neq (\lambda_1, \lambda_2, \cdots, \lambda_N) \neq (1, 1, \cdots, 1)$. We take the proof into several parts.

Case I. There exists $\lambda_j \in \sigma(A)$, $1 \le j \le N$, such that $|\lambda_j| = 1$; λ_j is not a root of 1.

It is well known that $\{\lambda_j^m\}_{m=0}^{\infty}$ is dense in $\mathbb{T}$. So we have

$$\begin{aligned} \mathbb{D} &\subset \operatorname{co}\left\{\lambda_j^m\right\}_{m=0}^{\infty} \subset \operatorname{co}\sigma_p(C_A) \subset W(C_A) \\ &\subset \overline{\mathrm{D}(0, \|C_A\|)} = \overline{\mathbb{D}}, \end{aligned} \tag{46}$$

the last equality follows from Lemma 4.

For any $u \in \mathbb{T}$, if $u \in W(C_A)$, then by [1, lemma 2.3], $u \in \sigma_p(C_A)$ since $|u| = 1 = \|C_A\|$. So

$$W(C_A) \subset \mathbb{D} \cup \sigma_p(C_A) \subset \operatorname{co}\sigma_p(C_A). \tag{47}$$

Therefore $W(C_A) = \operatorname{co}\sigma_p(C_A)$.

Case II. $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset (0, 1]$.

In this case, C_A is a positive operator on $\mathscr{F}^2$. Since $(\lambda_1, \lambda_2, \cdots, \lambda_N) \neq (1, 1, \cdots, 1)$, there exists λ_j, $1 \le j \le N$, $|\lambda_j| < 1$. Obviously, $1 \in \sigma_p(C_A) \subset (0, 1]$. Since $\lim_{m\longrightarrow\infty}\lambda_j^m = 0$, we have

$$\begin{aligned} (0, 1] &= \operatorname{co}\sigma_p(C_A) \subset W(C_A) \subset \overline{W(C_A)} = \operatorname{co}\sigma(C_A) \\ &= [0, 1]. \end{aligned} \tag{48}$$

If there exists $f \in \mathscr{F}^2$ such that $\langle C_A f, f\rangle = 0$, then $C_A f = 0$ and hence $f = 0$. Therefore $0 \notin W(C_A)$. So $W(C_A) = \operatorname{co}\sigma_p(C_A) = (0, 1]$.

Case III. For any j, $1 \le j \le N$, either $|\lambda_j| < 1$ or λ_j is a root of 1, and there exists $\lambda_n \in \sigma(A)$, $1 \le n \le N$, $\lambda_n \notin (0, 1]$.

Note that, in this case, $\sigma(C_A) = \overline{\sigma_p(C_A)} \subset \sigma_p(C_A) \cup \{0\}$. Since $|\lambda_n| < 1$ or λ_n is a root of 1 and $\lambda_n \notin (0, 1]$,

$$0 \in \operatorname{co}\{\lambda_n^m\}_{m=0}^{\infty} \subset \operatorname{co}\sigma_p(C_A) \tag{49}$$

So we obtain

$$\begin{aligned} \operatorname{co}\sigma_p(C_A) &\subset W(C_A) \subset \overline{W(C_A)} = \operatorname{co}(\sigma(C_A)) \\ &\subset \operatorname{co}\left\{\sigma_p(C_A) \cup \{0\}\right\} \subset \operatorname{co}\sigma_p(C_A). \end{aligned} \tag{50}$$

It follows that $W(C_A) = \operatorname{co}\sigma_p(C_A)$. □

Lastly, we give the proof of Theorem 2. Some ideas are derived from [1, 8].

Proof. (1) Since unitarily equivalent operators have the same numerical ranges, the conclusion follows from Theorems 3(1), and 7 and Lemma 6.

(2) Since $C_{\psi,\varphi}$ is normal, there exists $v \in \sigma(C_{\psi,\varphi})$ such that $|v| = \|C_{\psi,\varphi}\|$. By Theorem 3(2), for any $t \in \mathbb{R}$, $e^{it}v \in \sigma(C_{\psi,\varphi})$. It follows that

$$\left\{u \in \mathbb{C} \mid |u| = \left\|C_{\psi,\varphi}\right\|\right\} \subset \sigma\left(C_{\psi,\varphi}\right). \tag{51}$$

So we have

$$\begin{aligned} \overline{W\left(C_{\psi,\varphi}\right)} &= \operatorname{co}\sigma\left(C_{\psi,\varphi}\right) = \operatorname{co}\left\{u \in \mathbb{C} \mid |u| = \left\|C_{\psi,\varphi}\right\|\right\} \\ &= \overline{\mathrm{D}\left(0, \left\|C_{\psi,\varphi}\right\|\right)}. \end{aligned} \tag{52}$$

If there exists $v \in \{u \in \mathbb{C} \mid |u| = \|C_{\psi,\varphi}\|\}$ such that $v \in W(C_{\psi,\varphi})$, then by [1, Lemma 2.3] and Theorem 3(2), for any $t \in \mathbb{R}$, $e^{it}v \in \sigma_p(C_{\psi,\varphi})$, which implies that there is an uncountable collection of orthogonal vectors in $\mathscr{F}^2$ since $C_{\psi,\varphi}$ is normal and eigenvalues of $C_{\psi,\varphi}$ that correspond to distinct eigenvalues are orthogonal, a contradiction. So

$$W\left(C_{\psi,\varphi}\right) \subset \mathrm{D}\left(0, \left\|C_{\psi,\varphi}\right\|\right) \tag{53}$$

and $W(C_{\psi,\varphi})$ is dense in $\mathrm{D}(0, \|C_{\psi,\varphi}\|)$. Since $W(C_{\psi,\varphi})$ is convex, we must have $\mathrm{D}(0, \|C_{\psi,\varphi}\|) \subset W(C_{\psi,\varphi})$. Hence

$$\begin{aligned} W\left(C_{\psi,\varphi}\right) &= \mathrm{D}\left(0, \left\|C_{\psi,\varphi}\right\|\right) \\ &= \mathrm{D}\left(0, \left|\exp\left(\frac{\langle p, c\rangle + \langle b_1, p\rangle}{2} + \frac{|b_1|^2}{4}\right)\right|\right) \end{aligned} \tag{54}$$

by Theorem 5(2). □

Remark 8. In Theorem 2(2), we can choose $p \in \ker(I - A)^{\perp}$ such that $(I - A)p = b_2$, and then $\langle b_1, p\rangle = 0$ since $b_1 \in \operatorname{ran}(I - A)^{\perp} = \ker(I - A^*) = \ker(I - A)$. So

$$W\left(C_{\psi,\varphi}\right) = \mathrm{D}\left(0, \left|\exp\left(\frac{\langle p, c\rangle}{2} + \frac{|b_1|^2}{4}\right)\right|\right), \tag{55}$$

where $p \in (I - A)^{-1}b_2 \cap \ker(I - A)^{\perp}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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