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Research Article

Iterative Approximation of Fixed Point of Multivalued ρ-Quasi-Nonexpansive Mappings in Modular Function Spaces with Applications

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Recently, Khan and Abbas initiated the study of approximating fixed points of multivalued nonlinear mappings in modular function spaces. It is our purpose in this study to continue this recent trend in the study of fixed point theory of multivalued nonlinear mappings in modular function spaces. We prove some interesting theorems for ρ-quasi-nonexpansive mappings using the Picard-Krasnoselskii hybrid iterative process. We apply our results to solving certain initial value problem.

1. Introduction

Recently, Khan and Abbas [1] initiated the study of approximating fixed points of multivalued nonlinear mappings in modular function spaces. The purpose of this paper is to continue this recent trend in the study of fixed point theory of multivalued nonlinear mappings in modular function spaces. We prove some interesting theorems for ρ-quasi-nonexpansive mappings using the Picard-Krasnoselskii hybrid iterative process, recently introduced by Okeke and Abbas [2] as a modification of the Picard-Mann hybrid iterative process, introduced by Khan [3]. We also prove some stability results using this iterative process. Moreover, we apply our results in solving certain initial value problem.

For over a century now, the study of fixed point theory of multivalued nonlinear mappings has attracted many well-known mathematicians and mathematical scientists (see, e.g., Brouwer [4], Downing and Kirk [5], Geanakoplos [6], Kakutani [7], Nash [8], Nash [9], Nadler [10], Abbas and Rhoades [11], and Khan et al. [12]). The motivation for such studies stems mainly from the usefulness of fixed point theory results in real-world applications, as in *Game Theory and Market Economy* and in other areas of mathematical sciences such as in *Nonsmooth Differential Equations*.

The theory of modular spaces was initiated in 1950 by Nakano [13] in connection with the theory of ordered spaces which was further generalized by Musielak and Orlicz [14]. Modular function spaces are natural generalizations of both function and sequence variants of several important, from application perspective, spaces like Musielak-Orlicz, Orlicz, Lorentz, Orlicz-Lorentz, Kothe, Lebesgue, and Calderon-Lozanovskii spaces and several others. Interest in quasi-nonexpansive mappings in modular function spaces stems mainly in the richness of structure of modular function spaces that, besides being Banach spaces (or F-spaces in a more general settings), are equipped with modular equivalents of norm or metric notions and also equipped with almost everywhere convergence and convergence in submeasure. It is known that modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts, particularly in applications to integral operators, approximation, and fixed point results. Moreover, there are certain fixed point results that can be proved only using the apparatus of modular function spaces. Hence, fixed point theory results in modular function spaces, in this perspective, which should be considered as complementary to the fixed point theory in normed and metric spaces (see, e.g., [15, 16]).

Several authors have proved very interesting fixed points results in the framework of modular function spaces (see, e.g., [15, 17–19]). Abbas et al. [20] proved the existence and uniqueness of common fixed point of certain nonlinear mappings satisfying some contractive conditions in partially ordered spaces. Öztürk et al. [21] established some interesting fixed point results of nonlinear mappings satisfying integral type contractive conditions in the framework of modular spaces endowed with a graph. Recently, Khan and Abbas initiated the study of approximating fixed points of multivalued nonlinear mappings in the framework of modular function spaces [1]. A very recent work was given by Khan et al. [12]. They approximated the fixed points of ρ-quasi-nonexpansive multivalued mappings in modular function spaces using a three-step iterative process, where ρ satisfies the so-called Δ_2-condition. Their results improve and generalize the results of Khan and Abbas [1].

Motivated by the above results, we prove some convergence and stability results for ρ-quasi-nonexpansive mappings using the Picard-Krasnoselskii hybrid iterative process. Our results improve, extend, and generalize several known results, including the recent results of Khan et al. [12], in the sense that the restriction that ρ satisfies the so-called Δ_2-condition in [12] is removed in the present paper. Moreover, it is known (see, [2]) that the Picard-Krasnoselskii hybrid iterative process converges faster than all of Picard, Mann, Krasnoselskii, and Ishikawa iterative processes. Furthermore, we apply our results in solving certain initial value problem.

2. Preliminaries

In this study, we let Ω denote a nonempty set and let Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathscr{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $\Omega = \bigcup K_n$ (e.g., $\mathscr{P}$ can be the class of sets of finite measure in σ-finite measure space). By 1_A, we denote the characteristic function of the set A in Ω. By ε we denote the linear space of all simple functions with support from $\mathscr{P}$. By $\mathscr{M}_\infty$ we denote the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \varepsilon$, $|g_n| \leq |f|$, and $g_n(\omega) \to f(\omega)$ for each $\omega \in \Omega$.

Definition 1. Let $\rho : \mathscr{M}_\infty \to [0, \infty]$ be a nontrivial, convex, and even function. One says that ρ is a regular convex function pseudomodular if

(1) $\rho(0) = 0$;

(2) ρ is monotone, that is, $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathscr{M}_\infty$;

(3) ρ is orthogonally subadditive, that is, $\rho(f 1_{A\cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathscr{M}_\infty$;

(4) ρ has Fatou property, that is, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathscr{M}_\infty$;

(5) ρ is order continuous in ε, that is, $g_n \in \varepsilon$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

A set $A \in \Sigma$ is said to be ρ-null if $\rho(g 1_A) = 0$ for every $g \in \varepsilon$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathscr{M}(\Omega, \Sigma, \mathscr{P}, \rho) = \{f \in \mathscr{M}_\infty : |f(\omega)| < \infty \ \rho\text{-a.e.}\}, \tag{1}$$

where $f \in \mathscr{M}(\Omega, \Sigma, \mathscr{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists, we shall write $\mathscr{M}$ instead of $\mathscr{M}(\Omega, \Sigma, \mathscr{P}, \rho)$.

The following definitions were given in [1].

Definition 2. Let ρ be a regular function pseudomodular.

(a) One says that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

(b) One says that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.

It is known (see, e.g., [15]) that ρ satisfies the following properties:

(1) $\rho(0) = 0$ iff $f = 0$ ρ-a.e.

(2) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathscr{M}$.

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$, and $f, g \in \mathscr{M}$.

ρ is called a convex modular if, in addition, the following property is satisfied:

(3′) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$, and $f, g \in \mathscr{M}$.

The class of all nonzero regular convex function modulars on Ω is denoted by $\mathfrak{R}$.

Definition 3. The convex function modular ρ defines the modular function space L_ρ as

$$L_\rho = \{f \in \mathscr{M}; \rho(\lambda f) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{2}$$

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq \alpha\right\}. \tag{3}$$

In the case that ρ is convex modular,

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\} \tag{4}$$

defines a norm on the modular space L_ρ, and it is called the Luxemburg norm.

Lemma 4 (see [15]). *Let $\rho \in \mathfrak{R}$. Defining $L_\rho^0 = \{f \in L_\rho; \rho(f,\cdot)$ is order continuous$\}$ and $E_\rho = \{f \in L_\rho; \lambda f \in L_\rho^0$ for every $\lambda > 0\}$, one has the following:*

(i) $L_\rho \supset L_\rho^0 \supset E_\rho$.

(ii) *E_ρ has the Lebesgue property; that is, $\rho(\alpha f, D_k) \to 0$, for $\alpha > 0$, $f \in E_\rho$, and $D_k \downarrow \emptyset$.*

(iii) *E_ρ is the closure of ε (in the sense of $\|\cdot\|_\rho$).*

The following uniform convexity type properties of ρ can be found in [17].

Definition 5. Let ρ be a nonzero regular convex function modular defined on Ω.

(i) Let $r > 0$, $\epsilon > 0$. Define

$$D_1(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f-g) \ge \epsilon r\}. \tag{5}$$

Let

$$\delta_1(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_1(r,\epsilon)\right\} \quad \text{if } D_1(r,\epsilon) \ne \emptyset, \tag{6}$$

and $\delta_1(r,\epsilon) = 1$ if $D_1(r,\epsilon) = \emptyset$. One says that ρ satisfies (UC1) if for every $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. Observe that for every $r > 0$, $D_1(r,\epsilon) \ne \emptyset$, for $\epsilon > 0$ small enough.

(ii) One says that ρ satisfies (UUC1) if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ depending only on s and ϵ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ for any $r > s$.

(iii) Let $r > 0$, $\epsilon > 0$. Define

$$D_2(r,\epsilon) = \left\{(f,g) : f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho\left(\frac{f-g}{2}\right) \ge \epsilon r\right\}. \tag{7}$$

Let

$$\delta_2(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_2(r,\epsilon)\right\}, \quad \text{if } D_2(r,\epsilon) \ne \emptyset, \tag{8}$$

and $\delta_2(r,\epsilon) = 1$ if $D_2(r,\epsilon) = \emptyset$. one says that ρ satisfies (UC2) if for every $r > 0$, $\epsilon > 0$, $\delta_2(r,\epsilon) > 0$. Observe that for every $r > 0$, $D_2(r,\epsilon) \ne \emptyset$, for $\epsilon > 0$ small enough.

(iv) One says that ρ satisfies (UUC2) if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_2(s,\epsilon) > 0$ depending only on s and ϵ such that $\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0$ for any $r > s$.

(v) One says that ρ is strictly convex (SC), if for every $f, g \in L_\rho$ such that $\rho(f) = \rho(g)$ and $\rho((f+g)/2) = (\rho(f) + \rho(g))/2$, there holds $f = g$.

Proposition 6 (see [15]). *The following conditions characterize relationship between the above defined notions:*

(i) *(UUCi) $\Rightarrow$ (UCi) for $i = 1, 2$.*

(ii) *$\delta_1(r,\epsilon) \le \delta_2(r,\epsilon)$.*

(iii) *(UC1) $\Rightarrow$ (UC2).*

(iv) *(UUC1) $\Rightarrow$ (UUC2).*

(v) *If ρ is homogeneous (e.g., it is a norm), then all the conditions (UC1), (UC2), (UUC1), and (UUC2) are equivalent and $\delta_1(r,2\epsilon) = \delta_1(1,2\epsilon) = \delta_2(1,\epsilon) = \delta_2(r,\epsilon)$.*

Definition 7. A nonzero regular convex function modular ρ is said to satisfy the Δ_2-condition, if $\sup_{n\ge1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to $\emptyset$ and $\sup_{n\ge1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.

Definition 8. A function modular is said to satisfy the Δ_2-type condition, if there exists $K > 0$ such that, for any $f \in L_\rho$, one has $\rho(2f) \le K\rho(f)$.

In general, Δ_2-condition and Δ_2-type condition are not equivalent, even though it is easy to see that Δ_2-type condition implies Δ_2-condition on the modular space L_ρ; see [22].

Definition 9. Let L_ρ be a modular space. The sequence $\{f_n\} \subset L_\rho$ is called

(1) ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;

(2) ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as n and $m \to \infty$.

Observe that ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, one can easily show that this will happen if and only if ρ satisfies the Δ_2-condition.

Kilmer et al. [23] defined ρ-distance from an $f \in L_\rho$ to a set $D \subset L_\rho$ as follows:

$$\operatorname{dist}_\rho(f,D) = \inf\{\rho(f-h) : h \in D\}. \tag{9}$$

Definition 10. A subset $D \subset L_\rho$ is called

(1) ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D;

(2) ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of D always belongs to D;

(3) ρ-compact if every sequence in D has a ρ-convergent subsequence in D;

(4) ρ-a.e. compact if every sequence in D has a ρ-a.e. convergent subsequence in D;

(5) ρ-bounded if

$$\operatorname{diam}_\rho(D) = \sup\{\rho(f-g) : f,g \in D\} < \infty. \tag{10}$$

It is known that the norm and modular convergence are also the same when we deal with the Δ_2-type condition (see, e.g., [15]).

A set $D \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = \operatorname{dist}_\rho(f, D)$. We shall denote the family of nonempty ρ-bounded ρ-proximinal subsets of D by $P_\rho(D)$, the family of nonempty ρ-closed ρ-bounded subsets of D by $C_\rho(D)$, and the family of ρ-compact subsets of D by $K_\rho(D)$. Let $H_\rho(\cdot,\cdot)$ be the ρ-Hausdorff distance on $C_\rho(L_\rho)$; that is,

$$H_\rho(A,B) = \max\left\{\sup_{f\in A}\operatorname{dist}_\rho(f,B), \sup_{g\in B}\operatorname{dist}_\rho(g,A)\right\}, \quad A, B \in C_\rho(L_\rho). \tag{11}$$

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be

(a) ρ-contraction mapping if there exists a constant $k \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \le k\rho(f-g), \quad \forall f, g \in D, \tag{12}$$

(b) ρ-nonexpansive (see, e.g., Khan and Abbas [1]) if

$$H_\rho(Tf, Tg) \le \rho(f-g), \quad \forall f, g \in D, \tag{13}$$

(c) ρ-quasi-nonexpansive mapping if

$$H_\rho(Tf, p) \le \rho(f-p) \quad \forall f \in D,\ p \in F_\rho(T). \tag{14}$$

A sequence $\{t_n\} \subset (0, 1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \ge a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0, 1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \le b$ for every $n \in \mathbb{N}$.

Okeke and Abbas [2] introduced the Picard-Krasnoselskii hybrid iterative process. The authors proved that this new hybrid iterative process converges faster than all of Picard, Mann, Krasnoselskii, and Ishikawa iterative processes when applied to contraction mappings. We now give the analogue of the Picard-Krasnoselskii hybrid iterative process in modular function spaces as follows: let $T : D \to P_\rho(D)$ be a multivalued mapping and $\{f_n\} \subset D$ be defined by the following iteration process:

$$\begin{aligned} f_{n+1} &\in P_\rho^T(g_n) \\ g_n &= (1-\lambda) f_n + \lambda P_\rho^T(v_n), \quad n \in \mathbb{N}, \end{aligned} \tag{15}$$

where $v_n \in P_\rho^T(f_n)$ and $0 < \lambda < 1$. It is our purpose in the present paper to prove some new fixed point theorems using this iteration process in the framework of modular function spaces.

Definition 11. A sequence $\{f_n\} \subset D$ is said to be Fejér monotone with respect to subset $P_\rho(D)$ of D if $\rho(f_{n+1} - p) \le \rho(f_n - p)$, for all $p \in P_\rho^T(D)$ of D, $n \in \mathbb{N}$.

The following Lemma will be needed in this study.

Lemma 12 (see [22]). *Let ρ be a function modular and f_n and g_n be two sequences in X_ρ. Then*

$$\begin{aligned} \lim_{n\to\infty}\rho(g_n) = 0 &\Longrightarrow \limsup_{n\to\infty}\rho(f_n+g_n) \\ &= \limsup_{n\to\infty}\rho(f_n), \\ \lim_{n\to\infty}\rho(g_n) = 0 &\Longrightarrow \liminf_{n\to\infty}\rho(f_n+g_n) \\ &= \liminf_{n\to\infty}\rho(f_n). \end{aligned} \tag{16}$$

Lemma 13 (see [17]). *Let ρ satisfy $(UUC1)$ and let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1. If there exists $R > 0$ such that*

$$\begin{aligned} \limsup_{n\to\infty}\rho(f_n) &\le R, \\ \limsup_{n\to\infty}\rho(g_n) &\le R, \\ \lim_{n\to\infty}\rho(t_n f_n + (1-t_n) g_n) &= R, \end{aligned} \tag{17}$$

and then $\lim_{n\to\infty}\rho(f_n - g_n) = 0$.

The above lemma is an analogue of a famous lemma due to Schu [24] in Banach spaces.

A function $f \in L_\rho$ is called a fixed point of $T : L_\rho \to P_\rho(D)$ if $f \in Tf$. The set of all fixed points of T will be denoted by $F_\rho(T)$.

Lemma 14 (see [1]). *Let $T : D \to P_\rho(D)$ be a multivalued mapping and*

$$P_\rho^T(f) = \left\{g \in Tf : \rho(f-g) = \operatorname{dist}_\rho(f, Tf)\right\}. \tag{18}$$

Then the following are equivalent:

(1) *$f \in F_\rho(T)$, that is, $f \in Tf$.*

(2) *$P_\rho^T(f) = \{f\}$, that is, $f = g$ for each $g \in P_\rho^T(f)$.*

(3) *$f \in F(P_\rho^T(f))$, that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T(f))$, where $F(P_\rho^T(f))$ denotes the set of fixed points of $P_\rho^T(f)$.*

The following examples were presented by Razani et al. [25].

Example 15. Let $(X, \|\cdot\|)$ be a norm space; then $\|\cdot\|$ is a modular. But the converse is not true.

Example 16. Let $(X, \|\cdot\|)$ be a norm space. For any $k \ge 1$, $\|\cdot\|^k$ is a modular on X.

3. Iterative Approximation of Fixed Points in Modular Function Spaces

We begin this section with the following proposition.

Proposition 17. *Let ρ satisfy $(UUC1)$ and let D be a nonempty ρ-closed, ρ-bounded, and convex subset of L_ρ. Let $T : D \to$*

$P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-nonexpansive mapping. Then the Picard-Krasnoselskii hybrid iterative process (15) is Fejér monotone with respect to $F_\rho(T)$.

Proof. Suppose $p \in F_\rho(T)$. By Lemma 13, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$. Using (15), we have the following estimate:

$$\rho(f_{n+1} - p) \le H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \le \rho(g_n - p). \quad (19)$$

Next, we have

$$\rho(g_n - p) = \rho[(1-\lambda) f_n + \lambda P_\rho^T v_n - p]. \quad (20)$$

By convexity of ρ, we have

$$\begin{aligned}\rho(g_n - p) &\le (1-\lambda)\rho(f_n - p) \\ &\quad + \lambda H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \\ &\le (1-\lambda)\rho(f_n - p) + \lambda\rho(f_n - p) \\ &= \rho(f_n - p).\end{aligned} \quad (21)$$

Using (21) in (19), we have

$$\rho(f_{n+1} - p) \le \rho(f_n - p). \quad (22)$$

Hence, the Picard-Krasnoselskii hybrid iterative process (15) is Fejér monotone with respect to $F_\rho(T)$. This completes the proof of Proposition 17. □

Next, we prove the following proposition.

Proposition 18. *Let ρ satisfy (UUC1) and let D be a nonempty ρ-closed, ρ-bounded, and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-nonexpansive mapping. Let $\{f_n\}$ be the Picard-Krasnoselskii hybrid iterative process (15); then*

(i) *the sequence $\{f_n\}$ is bounded;*

(ii) *for each $f \in D$, $\{\rho(f_n - f)\}$ converges.*

Proof. Since $\{f_n\}$ is Fejér monotone as shown in Proposition 17, we can easily show (i) and (ii). This completes the proof of Proposition 18. □

Theorem 19. *Let ρ satisfy (UUC1) and let D be a nonempty ρ-closed, ρ-bounded, and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-nonexpansive mapping. Suppose that $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be the Picard-Krasnoselskii hybrid iterative process (15). Then $\lim_{n\to\infty}\rho(f_n - p)$ exists for all $p \in F_\rho(T)$ and $\lim_{n\to\infty}\mathrm{dist}_\rho(f_n, P_\rho^T(f_n)) = 0$.*

Proof. Suppose $p \in F_\rho(T)$. By Lemma 13, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$. Using (15), we have the following estimate:

$$\rho(f_{n+1} - p) \le H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \le \rho(g_n - p). \quad (23)$$

Next, we have

$$\rho(g_n - p) = \rho[(1-\lambda) f_n + \lambda P_\rho^T v_n - p]. \quad (24)$$

By convexity of ρ, we have

$$\begin{aligned}\rho(g_n - p) &\le (1-\lambda)\rho(f_n - p) \\ &\quad + \lambda H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \\ &\le (1-\lambda)\rho(f_n - p) + \lambda\rho(f_n - p) \\ &= \rho(f_n - p).\end{aligned} \quad (25)$$

Using (25) in (23), we have

$$\rho(f_{n+1} - p) \le \rho(f_n - p). \quad (26)$$

This shows that $\lim_{n\to\infty}\rho(f_n - p)$ exists for all $p \in F_\rho(T)$.

Suppose that

$$\lim_{n\to\infty}\rho(f_n - p) = L, \quad (27)$$

where $L \ge 0$.

We next prove that $\lim_{n\to\infty}\mathrm{dist}_\rho(f_n, P_\rho^T(f_n)) = 0$. Since $\mathrm{dist}_\rho(f_n, P_\rho^T(f_n)) \le \rho(f_n - v_n)$, it suffices to show that

$$\lim_{n\to\infty}\rho(f_n - v_n) = 0. \quad (28)$$

Now,

$$\rho(v_n - p) \le H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \le \rho(f_n - p), \quad (29)$$

and this implies that

$$\limsup_{n\to\infty}\rho(v_n - p) \le \limsup_{n\to\infty}\rho(f_n - p), \quad (30)$$

and, by (27), we have

$$\limsup_{n\to\infty}\rho(v_n - p) \le L. \quad (31)$$

Using (25), we have

$$\limsup_{n\to\infty}\rho(g_n - p) \le \limsup_{n\to\infty}\rho(f_n - p), \quad (32)$$

and, hence, we have

$$\limsup_{n\to\infty}\rho(g_n - p) \le L. \quad (33)$$

Next, we have

$$H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \le \rho(g_n - p) \le \rho(f_n - p), \quad (34)$$

and this implies that

$$\limsup_{n\to\infty}\rho(g_n - p) \le \limsup_{n\to\infty}\rho(f_n - p), \quad (35)$$

and, hence, we have

$$\limsup_{n\to\infty}\rho(g_n - p) \le L. \quad (36)$$

Using (23) and (24), we have

$$\begin{aligned}\lim_{n\to\infty}\rho\left(f_{n+1}-p\right) &= \lim_{n\to\infty}\rho\left[(1-\lambda) f_n+\lambda P_\rho^T v_n-p\right] \\ &\le \lim_{n\to\infty}\left[(1-\lambda)\rho\left(f_n-p\right)+\lambda H_\rho\left(P_\rho^T\left(f_n\right), P_\rho^T(p)\right)\right] \\ &\le \lim_{n\to\infty}\left[(1-\lambda)\rho\left(f_n-p\right)+\lambda\rho\left(f_n-p\right)\right] \\ &= \lim_{n\to\infty}\rho\left(f_n-p\right) \le L.\end{aligned} \tag{37}$$

Moreover,

$$\begin{aligned}\rho\left(f_{n+1}-p\right) &\le \rho\left[(1-\lambda) f_n+\lambda v_n-p\right] \\ &= \rho\left[\left(f_n-p\right)+\lambda\left(v_n-f_n\right)\right].\end{aligned} \tag{38}$$

Using Lemma 4 and (38), we have

$$\begin{aligned}L &= \liminf_{n\to\infty}\rho\left(f_{n+1}-p\right) \\ &= \liminf_{n\to\infty}\rho\left[\left(f_n-p\right)+\lambda\left(v_n-f_n\right)\right] \\ &= \liminf_{n\to\infty}\rho\left(f_n-p\right).\end{aligned} \tag{39}$$

This means that

$$L = \liminf_{n\to\infty}\rho\left(f_n-p\right). \tag{40}$$

Using (27) and (40), we have

$$\lim_{n\to\infty}\rho\left(f_n-p\right) = L. \tag{41}$$

Using (27), (31), (37), and Lemma 12, we have

$$\lim_{n\to\infty}\rho\left(f_n-v_n\right) = 0\ . \tag{42}$$

Hence,

$$\lim_{n\to\infty}\operatorname{dist}_\rho\left(f_n, P_\rho^T\left(f_n\right)\right) = 0. \tag{43}$$

The proof of Theorem 19 is completed. □

Next, we prove the following theorem.

Theorem 20. *Let D be a ρ-closed, ρ-bounded, and convex subset of a ρ-complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-contraction mapping and $F_\rho(T) \neq \emptyset$. Then T has a unique fixed point. Moreover, the Picard-Krasnoselskii hybrid iterative process (15) converges to this fixed point.*

Proof. Suppose $p \in F_\rho(T)$. By Lemma 13, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$. Using (15), we have the following estimate:

$$\begin{aligned}\rho\left(f_{n+1}-p\right) &\le H_\rho\left(P_\rho^T\left(g_n\right), P_\rho^T(p)\right) \le k\rho\left(g_n-p\right) \\ &\le \rho\left(g_n-p\right).\end{aligned} \tag{44}$$

Next, we have

$$\rho\left(g_n-p\right) = \rho\left[(1-\lambda) f_n+\lambda P_\rho^T\left(v_n\right)-p\right]. \tag{45}$$

By convexity of ρ, we have

$$\begin{aligned}\rho\left(g_n-p\right) &\le (1-\lambda)\rho\left(f_n-p\right) \\ &\quad + \lambda H_\rho\left(P_\rho^T\left(f_n\right), P_\rho^T(p)\right) \\ &\le (1-\lambda)\rho\left(f_n-p\right)+\lambda k\rho\left(f_n-p\right) \\ &\le (1-\lambda)\rho\left(f_n-p\right)+\lambda\rho\left(f_n-p\right) \\ &= \rho\left(f_n-p\right).\end{aligned} \tag{46}$$

Using (46) in (44), we have

$$\rho\left(f_{n+1}-p\right) \le \rho\left(f_n-p\right). \tag{47}$$

This shows that $\lim_{n\to\infty}\rho(f_n-p)$ exists for all $p \in F_\rho(T)$. Using a similar approach as in the proof of Theorem 19, we see that $\lim_{n\to\infty}\rho(f_n-p) = 0$.

Next, we show that $\{f_n\}$ is a ρ-Cauchy sequence. Since $\lim_{n\to\infty}(f_n-p) = 0$, we proceed by contradiction. Hence, there exists $\epsilon > 0$ and two sequences of natural numbers $\{m(i)\}$, $\{n(i)\}$ such that

$$\begin{gathered} n(i) > m(i) \ge i, \\ \rho\left(f_{n(i)}-f_{m(i)}\right) > \epsilon. \end{gathered} \tag{48}$$

For all integer i, let $n(i)$ be the least integer exceeding $m(i)$ which satisfy (48); then

$$\begin{gathered}\rho\left(f_{n(i)}-f_{m(i)}\right) > \epsilon, \\ \rho\left(f_{n(i)-1}-f_{m(i)}\right) \le \epsilon.\end{gathered} \tag{49}$$

So, we have

$$\begin{aligned}\epsilon < \rho\left(f_{n(i)}-f_{m(i)}\right) &\le \rho\left(\frac{f_{n(i)}-p}{2}\right)+\rho\left(\frac{p-f_{m(i)}}{2}\right) \\ &\le \frac{1}{2}\rho\left(f_{n(i)}-p\right)+\frac{1}{2}\rho\left(p-f_{m(i)}\right) \\ &\le \rho\left(f_{n(i)}-p\right)+\rho\left(p-f_{m(i)}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{50}$$

This is a contradiction. Hence, $\{f_n\}$ is a ρ-Cauchy sequence. Therefore, there exists $p \in D$ such that $f_n \to 0$ as $n \to \infty$.

Next, we have $Tp = p$. Clearly,

$$\begin{aligned}\rho\left(\frac{p-Tp}{2}\right) &\le \rho\left(\frac{p-f_n}{2}\right)+\rho\left(\frac{f_n-Tp}{2}\right) \\ &\le \frac{1}{2}\rho\left(p-f_n\right)+\frac{1}{2}\rho\left(f_n-Tp\right) \\ &\le \rho\left(p-f_n\right)+\rho\left(f_n-Tp\right) \\ &= \rho\left(p-f_n\right)+\rho\left(f_n-p\right) \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{51}$$

Hence, $\rho((p-Tp)/2) = 0$. Therefore, $p = Tp$.

Next, we prove the uniqueness of p. Suppose that q is another fixed point of T, and then we have

$$\begin{aligned}\rho\left(\frac{p-q}{2}\right) &\leq \rho\left(\frac{p-f_n}{2}\right)+\rho\left(\frac{f_n-q}{2}\right)\\ &\leq \frac{1}{2}\rho\left(p-f_n\right)+\frac{1}{2}\rho\left(f_n-q\right)\\ &\leq \rho\left(p-f_n\right)+\rho\left(f_n-q\right)\longrightarrow 0\\ &\qquad\text{as } n\longrightarrow\infty.\end{aligned} \tag{52}$$

Hence, $p=q$. The proof of Theorem 20 is completed. □

Next, we give the following example.

Example 21. Let $L_\rho=[0,\infty)$ be a vector space and ρ be an application defined as follows:

$$\begin{aligned}\rho: L_\rho &\longrightarrow L_\rho\\ t &\longrightarrow t^2.\end{aligned} \tag{53}$$

We see that ρ is not a norm. However, it is a modular since the function $t\to t^2$ is convex. Consider $D=[0,1]$ as the closed interval in $[0,\infty)$ which is ρ-closed, ρ-bounded, and ρ-complete, since ρ is continuous. Then the mapping

$$\begin{aligned}T: D &\longrightarrow P_\rho(D)\\ t &\longrightarrow \frac{t}{2}\end{aligned} \tag{54}$$

is a ρ-contraction mapping with $k=1/2$. Therefore, by Theorem 20, it has a unique fixed point in D, which is $F_\rho(T)=\{0\}$.

4. Stability Results

We begin this section by defining the concept of T-stable and almost T-stable of an iterative process in modular function spaces. Moreover, we prove some stability results for Picard-Krasnoselskii hybrid iterative process (15).

Definition 22. Let D be a nonempty convex subset of a modular function space L_ρ and $T: D\to D$ be an operator. Assume that $x_1\in D$ and $x_{n+1}=f(T,x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=1}^{\infty}\subset D$. Suppose, furthermore, that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^*\in F_\rho(T)\neq\emptyset$. Let $\{y_n\}_{n=1}^{\infty}$ be any bounded sequence in D and put $\varepsilon_n=\rho(y_{n+1}-f(T,y_n))$.

(1) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1}=f(T,x_n)$ is said to be T-stable on D if $\lim_{n\to\infty}\varepsilon_n=0$ implies that $\lim_{n\to\infty}y_n=x^*$.

(2) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1}=f(T,x_n)$ is said to be almost T-stable on D if $\sum_{n=1}^{\infty}\varepsilon_n<\infty$ implies that $\lim_{n\to\infty}y_n=x^*$.

It is easy to show that an iteration process $\{x_n\}_{n=1}^{\infty}$ which is T-stable on C is almost T-stable on D.

Next, we provide the following numerical example to show that Picard-Krasnoselskii hybrid iterative process (15) is T-stable.

Example 23. Let $L_\rho=[0,\infty)$ be a vector space and ρ be an application defined as follows

$$\begin{aligned}\rho: L_\rho &\longrightarrow L_\rho\\ t &\longrightarrow |t|.\end{aligned} \tag{55}$$

Let $D=[0,1]$ be the closed interval in $[0,\infty)$ which is ρ-closed, ρ-bounded, and ρ-complete. Let $T:[0,1]\to[0,1]$ be a multivalued mapping such that P_ρ^T is a ρ-contraction mapping satisfying contractive condition $Tx=x/2$. We now show that Picard-Krasnoselskii hybrid iterative process (15) is T-stable and hence almost T-stable with $k=1/2$ and $F_\rho(T)=\{0\}$. Suppose that $\{y_n\}=1/n$ is an arbitrary sequence in L_ρ. Take $\lambda=1/2$. Then $\lim_{n\to\infty}y_n=0$. Put

$$\begin{aligned}\varepsilon_n &= \rho\left(y_{n+1}-f\left(T,y_n\right)\right)\\ &= \operatorname{dist}_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\\ &\leq H_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right),\end{aligned} \tag{56}$$

and we have

$$\begin{aligned}\varepsilon_n &= \operatorname{dist}_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\\ &\leq H_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\leq\rho\left(y_{n+1}-g_n\right)\\ &= \left|y_{n+1}-(1-\lambda)\,y_n-\lambda y_n\right| = \left|\frac{1}{n+1}-\frac{1}{2n}-\frac{1}{2n}\right|\\ &= \left|\frac{1}{n+1}-\frac{1}{n}\right|.\end{aligned} \tag{57}$$

Hence,

$$\lim_{n\to\infty}\varepsilon_n=0. \tag{58}$$

Therefore, Picard-Krasnoselskii hybrid iterative process (15) is T-stable. Clearly, (15) is almost T-stable.

Next, we prove the following stability results.

Theorem 24. *Let D be a ρ-closed, ρ-bounded, and convex subset of a ρ-complete modular space L_ρ and $T: D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-contraction mapping and $F_\rho(T)\neq\emptyset$. Then Picard-Krasnoselskii hybrid iterative process (15) is T-stable.*

Proof. Suppose $\{y_n\}\subset L_\rho$, and define $\varepsilon_n=\rho(y_{n+1}-f(T,y_n))$. Let p be the unique fixed point of T. We want to show that $\lim_{n\to\infty}y_n=p$ if and only if $\lim_{n\to\infty}\varepsilon_n=0$. Suppose that $\{y_n\}$ converges to p. Using (15) and the convexity of ρ, we have

$$\begin{aligned}\varepsilon_n &= \operatorname{dist}_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\\ &\leq H_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\leq\rho\left(y_{n+1}-g_n\right)\end{aligned}$$

$$\leq \rho\left(y_{n+1}-(1-\lambda)\,y_n-\lambda y_n\right)=\rho\left(y_{n+1}-y_n\right)$$
$$\leq \rho\left(y_{n+1}-p\right)+\rho\left(p-y_n\right). \tag{59}$$

Hence,

$$\lim_{n\to\infty}\varepsilon_n=0. \tag{60}$$

Conversely, suppose that $\lim_{n\to\infty}\varepsilon_n=0$. Then we have

$$\begin{aligned}\varepsilon_n&=\operatorname{dist}_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\\&\leq H_\rho\left(P_\rho^T\left(y_{n+1}\right),P_\rho^T\left(g_n\right)\right)\leq\rho\left(y_{n+1}-g_n\right)\\&\leq\rho\left(y_{n+1}-(1-\lambda)\,y_n-\lambda y_n\right)=\rho\left(y_{n+1}-y_n\right)\\&\leq\rho\left(y_{n+1}-p\right)+\rho\left(p-y_n\right).\end{aligned} \tag{61}$$

Since $\lim_{n\to\infty}\varepsilon_n=0$, it follows from relation (61) that $\lim_{n\to\infty}y_n=p$. The proof of Theorem 24 is completed. □

Remark 25. Theorem 24 generalizes the results of Mbarki and Hadi [26] to multivalued mappings in modular function spaces.

5. Applications to Differential Equations

In this section, we apply our results to differential equations. The results of this section follow similar applications in [15]. Let $\rho\in\mathfrak{R}$, and we consider the following initial value problem for an unknown function $u:[0,A]\to C$, where $C\in E_\rho$.

$$\begin{aligned}u(0)&=f\\u'(t)+(I-T)\,u(t)&=0,\end{aligned} \tag{62}$$

where $f\in C$ and $A>0$ are fixed and $T:C\to C$ is such that P_ρ^T is ρ-quasi-nonexpansive mapping. The following notations will be used in this section. For $t>0$ we define

$$K(t)=1-e^{-t}=\int_0^t e^{s-t}ds. \tag{63}$$

For any function $\nu:[0,A]\to L_\rho$, where $A>0$, and any $t\in[0,A]$, we define

$$S(\nu)(t)=\int_0^t e^{s-t}\nu(s)\,ds. \tag{64}$$

We also denote

$$S_\tau(\nu)(t)=\sum_{i=0}^{n-1}\left(t_{i+1}-t_i\right)e^{t_i-t}\nu\left(t_i\right), \tag{65}$$

for any $\tau=\{t_0,\dots,t_n\}$, a subdivision of the interval $[0,A]$.

The following lemma which is needed to prove our results in this section can be found in [15].

Lemma 26. *Let $\rho\in\mathfrak{R}$ be separable. Let $x,y:[0,A]\to L_\rho$ be two Bochner-integrable $\|\cdot\|_\rho$-bounded functions, where $A>0$. Then for every $t\in[0,A]$ one has*

$$\begin{aligned}&\rho\left(e^{-t}y(t)+\int_0^t e^{s-t}x(s)\,ds\right)\\&\quad\leq e^{-t}\rho\left(y(t)\right)+K(t)\sup_{s\in[0,t]}\rho\left(x(s)\right).\end{aligned} \tag{66}$$

We now state our results for this section.

Theorem 27. *Let $\rho\in\mathfrak{R}$ be separable. Let $D\subset E_\rho$ be a nonempty, convex, ρ-bounded, ρ-closed set with the Vitali property. Let $T:D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-nonexpansive mapping. Let one fix $f\in C$ and $A>0$ and define the sequence of functions $u_n:[0,A]\to C$ by the following inductive formula:*

$$\begin{aligned}u_0(t)&=f\\u_{n+1}(t)&=e^{-t}f+\int_0^t e^{s-t}T\left(u_n(s)\right)ds.\end{aligned} \tag{67}$$

Then for every $t\in[0,A]$ there exists $u(t)\in C$ such that

$$\rho\left(u_n(t)-u(t)\right)\longrightarrow 0 \tag{68}$$

and the function $u:[0,A]\to C$ defined by (68) is a solution of initial value problem (62). Moreover,

$$\rho\left(f-u_n(t)\right)\leq K^{n+1}(A)\,\delta_\rho(C). \tag{69}$$

Proof. Since P_ρ^T is ρ-quasi-nonexpansive mapping, the proof of Theorem 27 follows the proof of ([15], Theorem 5.28). □

Next, we obtain the following corollaries as a consequence of Theorem 27.

Corollary 28. *Let $\rho\in\mathfrak{R}$ be separable. Let $D\subset E_\rho$ be a nonempty, convex, ρ-bounded, ρ-closed set with the Vitali property. Let $T:D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Let one fix $f\in C$ and $A>0$ and define the sequence of functions $u_n:[0,A]\to C$ by the following inductive formula:*

$$\begin{aligned}u_0(t)&=f\\u_{n+1}(t)&=e^{-t}f+\int_0^t e^{s-t}T\left(u_n(s)\right)ds.\end{aligned} \tag{70}$$

Then for every $t\in[0,A]$ there exists $u(t)\in C$ such that

$$\rho\left(u_n(t)-u(t)\right)\longrightarrow 0 \tag{71}$$

and the function $u:[0,A]\to C$ defined by (71) is a solution of initial value problem (62). Moreover,

$$\rho\left(f-u_n(t)\right)\leq K^{n+1}(A)\,\delta_\rho(C). \tag{72}$$

Corollary 29. *Let $\rho \in \mathfrak{R}$ be separable. Let $D \subset E_\rho$ be a nonempty, convex, ρ-bounded, ρ-closed set with the Vitali property. Let $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-contraction mapping. Let one fix $f \in C$ and $A > 0$ and define the sequence of functions $u_n : [0, A] \to C$ by the following inductive formula:*

$$u_0(t) = f$$
$$u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T(u_n(s))\, ds. \tag{73}$$

Then for every $t \in [0, A]$ there exists $u(t) \in C$ such that

$$\rho(u_n(t) - u(t)) \longrightarrow 0 \tag{74}$$

and the function $u : [0, A] \to C$ defined by (74) is a solution of initial value problem (62). Moreover,

$$\rho(f - u_n(t)) \le K^{n+1}(A)\, \delta_\rho(C). \tag{75}$$

Remark 30. Corollary 28 generalizes the results of Khamsi and Kozlowski ([15], Theorem 5.28) to a multivalued mapping.

Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this research paper. Each author read and approved the final manuscript.

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