Hindawi
Journal of Function Spaces
Volume 2018, Article ID 1949747, 9 pages
https://doi.org/10.1155/2018/1949747

Research Article

The Higher Integrability of Commutators of Calderón-Zygmund Singular Integral Operators on Differential Forms

Jinling Niu and Yuming Xing

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to Yuming Xing; xyuming@hit.edu.cn

Received 18 August 2018; Accepted 30 September 2018; Published 6 November 2018

Academic Editor: Gestur Ólafsson

In this article, the higher integrability of commutators of Calderón-Zygmund singular integral operators on differential forms is derived. Also, the higher order Poincaré-type inequalities for the commutators acting on the solutions of Dirac-harmonic equations are obtained. Finally, some applications of the main results are demonstrated by examples.

1. Introduction

Differential forms as the key tools are widely used in many fields including quasiconformal analysis, nonlinear elasticity, and differential geometry, due to their advantage of being coordinate system independent; see [1–5]. The integrability of various operators and the upper bound estimates for the norms of operators are very important and core topics while studying the L^p-theory of differential forms and investigating the qualitative and quantitative properties of the solutions of partial differential equations. In last few decades, a lot of related research has been done and many results on estimates for the L^p-norms of various operators applied to the differential form u in terms of the L^p-norms of u have been obtained; see [6–12]. In this paper, we define the commutators of Calderón-Zygmund singular integral operators $[b, T_\Omega]$ on differential forms and give the strong type estimates for the commutators, which allows one to estimate $\|[b, T_\Omega]u\|_p$ in terms of the norm $\|u\|_p$ with $p > 1$. Meanwhile, we make a contribution to the estimates of the commutators $\|[b, T_\Omega]u\|_s$ in terms of the norm $\|u\|_p$, where $s > p$, that is the higher integrability of commutators of Calderón-Zygmund singular integral operators on differential forms. Then, we will establish the higher order Poincaré-type inequalities for the commutators applied to the solutions of Dirac-harmonic equations. The higher integrability and higher order inequalities in this paper can be used to study the regularity properties of the related operators. More results on the problem of higher order estimates and their applications in potential theory, quantum mechanics, and partial differential equations can be found in [13–17].

This paper is organised as follows. Section 2 contains, in addition to definitions and other preliminary material, the main lemmas. In Section 3, Theorems 12 and 13 show the local higher integrability of commutators of Calderón-Zygmund singular integral operators on differential forms. Based on these local results, the global higher integrability is presented in Theorems 14 and 15 by the well-known covering lemma. Especially, when the differential forms satisfy the Dirac-harmonic equations (in [18]), we establish the higher order Poincaré-type inequalities for the commutators in Section 4. The local higher order Poincaré-type inequalities are given in Theorems 16 and 17 and the global higher order Poincaré-type inequalities are obtained in Theorems 18 and 19. Finally, we demonstrate some applications of the main results by examples in Section 5. These results obtained in this paper will provide a further insight into the L^p-theory and regularity theory of the related operators and differential forms.

2. Preliminary

Before specifying the main results precisely, we introduce some notation. Let $M \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$, B and σB be the balls with the same center, and $diam(\sigma B) = \sigma diam(B)$ for any real number $\sigma > 1$. We denote by $\mu(M)$ the

n-dimensional Lebesgue measure of a set $M \subseteq \mathbb{R}^n$. The set of all l-forms, denoted by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$, is a l-vector, spanned by exterior products $e_I = e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_l}$, for all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < \cdots < i_l \le n$. The l-form $u(x) = \sum_I u_I(x) dx_I$ is called a differential l-form if its coefficients u_I are differentiable functions. We shall denote by $D'(M, \Lambda^l)$ the differential l-forms space on M and denote by $L^p(M, \Lambda^l)$ the space of differential l-forms on M satisfying $\int_M |u_I|^p < \infty$ and with norm $\|u\|_{p,M} = (\int_M (\sum_I |u_I(x)|^2)^{p/2} dx)^{1/p}$. In particular, we know that a 0-form is a function. A differential l-form $u \in D'(M, \Lambda^l)$ is called a closed form if $du = 0$ in M. Similarly, a differential $(l+1)$-form $v \in D'(M, \Lambda^{l+1})$ is called a coclosed form if $d^\star v = 0$. From the Poincaré lemma, $ddu = 0$, we know that du is a closed form. The operator $\star : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^{n-l}(\mathbb{R}^n)$ is the Hodge-star operator which is an isometric isomorphism and the linear operator $d : D'(M, \Lambda^l) \longrightarrow D'(M, \Lambda^{l+1})$, $0 \le l \le n-1$, is called the exterior differential. The Hodge codifferential operator $d^\star : D'(M, \Lambda^{l+1}) \longrightarrow D'(M, \Lambda^l)$, the formal adjoint of d, is defined by $d^\star = (-1)^{nl+1} \star d \star$; see [19] for more introduction. The following Dirac-harmonic equation for differential forms was initially introduced by S. Ding and B. Liu in [18]:

$$d^\star A(x, Du) = 0, \tag{1}$$

for almost every $x \in M$, where $D = d + d^\star$ is the Dirac operator, $M \subset \mathbb{R}^n$ is a domain, and $A : M \times \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^l(\mathbb{R}^n)$ satisfies the following conditions:

$$\begin{gathered} |A(x, \xi)| \le a |\xi|^{p-1}, \\ \langle A(x, \xi), \xi \rangle \ge |\xi|^p \end{gathered} \tag{2}$$

for almost every $x \in M$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here, $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1).

We should point out that the Dirac-harmonic equation is a kind of general equation which includes many existing harmonic equations as special cases, such as the A-harmonic equation; see [18, 20, 21] for more information.

The Calderón-Zygmund singular integral operator T_Ω on differential forms is defined by

$$T_\Omega u(x) = \sum_I \left(\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} u_I(y)\, dy \right) dx_I, \tag{3}$$

where $\Omega(x)$ is defined on S^{n-1}, has mean 0, and is sufficiently smooth.

If $b \in BMO(\mathbb{R}^n)$, the commutator of Calderón-Zygmund singular integral operator on differential forms is of the form

$$\begin{aligned} [b, T_\Omega] u(x) &= b(x) T_\Omega u(x) - T_\Omega(bu)(x) \\ &= \sum_I \left(\int_{\mathbb{R}^n} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} u_I(y)\, dy \right) dx_I. \end{aligned} \tag{4}$$

When taking $u(x)$ as a 0-form, the commutator in (4) reduces to the corresponding operator on function space as follows:

$$[b, T_\Omega] f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy. \tag{5}$$

For the degenerated operator and the related applications in partial differential equations, see [22–24].

In order to prove our conclusions, we need several lemmas. The following L^p-boundedness result for commutator $[b, T_\Omega]$ on function spaces was proved in [25].

Lemma 1. *Let w be a weight satisfying A_p condition: for all cubes Q, if there is a constant C such that*

$$\left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\, dx \right)^{p-1} \le C < \infty, \tag{6}$$

where $1 < p < \infty$ and $1/p + 1/p' = 1$. T_Ω is any Calderón-Zygmund singular integral operator. Then, given any function $b \in BMO(\mathbb{R}^n)$, $[b, T_\Omega]$ satisfies the following inequality:

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left| [b, T_\Omega] f(x) \right|^p w(x)\, dx \right)^{1/p} \\ &\quad \le C \|b\|_{BMO} \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p}. \end{aligned} \tag{7}$$

The following lemma was given by S. Ding and B. Liu in [18].

Lemma 2. *Let $u \in D'(M, \Lambda^l)$ be a solution of Dirac-harmonic equation (1) in M; $\sigma > 1$ and $0 < s, t < \infty$ are constants. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C \mu(B)^{(t-s)/st} \|u\|_{t,\sigma B} \tag{8}$$

for all cubes or balls B with $\sigma B \subset M$.

In [26], T. Iwaniec and A. Lutoborski gave the following three lemmas which will be used repeatedly in this paper.

Lemma 3. *Let $u \in L^p_{loc}(M, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be a differential form and $T : L^p(M, \Lambda^l) \longrightarrow W^{1,p}(M, \Lambda^{l-1})$ be the homotopy operator. Then, we have the following decomposition:*

$$u = d(Tu) + T(du) \tag{9}$$

and the inequality

$$\|Tu\|_{p,M} \le C(n, p)\, \mu(M)\, diam(M)\, \|u\|_{p,M} \tag{10}$$

holds for any bounded domain $M \subset \mathbb{R}^n$, where $C(n, p)$ is a constant, independent of u.

From [26], the l-form $u_M \in L^p(M, \Lambda^l)$ is defined by $u_M = (1/\mu(M)) \int_M u(y) dy$ if $l = 0$ and $u_M = d(Tu)$ if $l = 1, \ldots, n$, $1 \le p \le \infty$.

Lemma 4. *Let $u \in L^p(M, \Lambda^l)$, $l = 1, 2, \ldots, n$. Then, $u_M \in L^p(M, \Lambda^l)$ and*

$$\|u_M\|_{p,M} \le C(n,p)\,\mu(M)\,\|u\|_{p,M}, \tag{11}$$

where $C(n,p)$ is a constant, independent of u and $1 < p < \infty$.

Lemma 5. *Let $u \in D'(M, \Lambda^l)$ and $du \in L^p(M, \Lambda^{l+1})$. Then $u - u_M$ is in $L^{np/(n-p)}(M, \Lambda^l)$ and*

$$\begin{aligned}\left(\int_M |u - u_M|^{np/(n-p)}\,dx\right)^{(n-p)/np} \\ \le C_p(n)\left(\int_M |du|^p\,dx\right)^{1/p},\end{aligned} \tag{12}$$

where $l = 1, 2, \ldots, n$, $1 < p < n$, and $C_p(n)$ is a constant independent of u.

The following lemma appears in [27].

Lemma 6. *Let ψ defined on $[0, +\infty)$ be a strictly increasing convex function, $\psi(0) = 0$, and $M \subset \mathbb{R}^n$ be a domain. Assume that $u(x) \in D'(M, \Lambda^l)$ satisfies $\psi(k(|u| + |u_M|)) \in L^1(M, \mu)$ for any real number $k > 0$, and $\mu(x \in M : |\mu - \mu_M| > 0) > 0$ where μ is a Radon measure defined by $d\mu(x) = \omega(x)dx$ with a weight $\omega(x)$, then for any $a > 0$, we obtain*

$$\begin{aligned}\int_M \psi\left(\frac{1}{2}|u - u_M|\right)d\mu &\le C_1\int_M \psi(a|u|)\,d\mu \\ &\le C_2\int_M \psi(2a|u - u_M|)\,d\mu,\end{aligned} \tag{13}$$

where $C_1 > 0$ and $C_2 > 0$ are constants.

Choose $\psi(t) = t^p$, $p > 1$, and $w(x) = 1$ and let M be a ball B in Lemma 6; we find that the norms $\|u\|_{p,B}$ and $\|u - u_B\|_{p,B}$ are comparable; that is,

$$\|u - u_B\|_{p,B} \le C_1\|u\|_{p,B} \le C_2\|u - u_B\|_{p,B} \tag{14}$$

for any ball B with $|\{x \in B : |u - u_B| > 0\}| > 0$.

The covering lemma below belongs to [19].

Lemma 7. *Each domain M has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\begin{gathered}\cup_i Q_i = M, \\ \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{5/4}Q_i} \le N\chi_\Omega\end{gathered} \tag{15}$$

and some $N > 1$, and if $Q_i \cap Q_j \ne \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if M is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

3. Higher Integrability

In this section, we show the higher integrability of commutators of Calderón-Zygmund singular integral operators on differential forms. We first concentrate on the local higher integrability of the commutators. We need the following lemma appearing in [28].

Lemma 8. *Let*

$$A(f)(x) = \lim_{\epsilon \to 0}\int_{|x-y|>\epsilon} k(x-y)\,f(y)\,dy, \tag{16}$$

where x, y are points in n-dimensional Euclidean space $\mathbb{R}^n$ and $k(x)$ is a homogeneous function of degree $-n$ with mean value zero on $|x| = 1$, and let $B(f) = b(x)f(x)$. If k and b are sufficiently smooth and b is bounded, then,

$$\left\|\frac{\partial}{\partial x_j}(AB - BA)\,f\right\|_p \le c\,\|f\|_p \tag{17}$$

for $1 < p < \infty$, where c is a constant, independent of f.

Lemma 9. *Let $u \in L^p(\mathbb{R}^n, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, T_Ω be the Calderón-Zygmund singular integral operator on differential forms, and b be sufficiently smooth and bounded. Then, there exists a constant C, independent of u, such that*

$$\|d[b, T_\Omega]\,u\|_{p,\mathbb{R}^n} \le C\,\|u\|_{p,\mathbb{R}^n}. \tag{18}$$

Proof. For any differential l-form $u(x)$, by the definition of commutator of Calderón-Zygmund singular integral operator on differential forms, we have

$$\begin{aligned}&[b, T_\Omega](u)(x) = b(x)\,T_\Omega u(x) - T_\Omega(bu)(x) \\ &= \sum_I\left(\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}(b(x) - b(y))\,u_I(y)\,dy\right)dx_I.\end{aligned} \tag{19}$$

Let $K(x-y) = \Omega(x-y)/|x-y|^n$ and then according to the definition of the exterior differential operator, we obtain

$$\begin{aligned}&d[b, T_\Omega](u)(x) \\ &\quad = \sum_I\sum_{k=1}^n\frac{\partial\int_{\mathbb{R}^n}(b(x) - b(y))\,K(x-y)\,u_I(y)\,dy}{\partial x_k}dx_k \\ &\quad \wedge dx_I.\end{aligned} \tag{20}$$

Using the elementary inequality $|\sum_{i=1}^N t_i|^s \le N^{s-1}\sum_{i=1}^N |t_i|^s$, for constants $N, s > 0$, we deduce

$$\begin{aligned}&\|d[b, T_\Omega]\,u(x)\|_{p,\mathbb{R}^n}^p \\ &\le \int_{\mathbb{R}^n}\left(\sum_I\sum_{k=1}^n\left|\frac{\partial\int_{\mathbb{R}^n}(b(x) - b(y))\,K(x-y)\,u_I(y)\,dy}{\partial x_k}\right|^2\right)^{p/2}dx\end{aligned}$$

$$\begin{aligned}
&\leq C_1 \int_{\mathbb{R}^n} \sum_I \sum_{k=1}^n \left| \frac{\partial \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) u_I(y)\, dy}{\partial x_k} \right|^p dx \\
&= C_1 \sum_I \sum_{k=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) u_I(y)\, dy}{\partial x_k} \right|^p dx \\
&= C_1 \sum_I \sum_{k=1}^n \left\| \frac{\partial \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) u_I(y)\, dy}{\partial x_k} \right\|_{p,\mathbb{R}^n}^p .
\end{aligned} \tag{21}$$

By Lemma 8, we obtain

$$\begin{aligned}
&\left\| \frac{\partial \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) u_I(y)\, dy}{\partial x_k} \right\|_{p,\mathbb{R}^n} \\
&\quad \leq C_2 \| u_I(x) \|_{p,\mathbb{R}^n} .
\end{aligned} \tag{22}$$

Substituting (22) into (21) and applying the fundamental inequality $\|a\|^s + \|b\|^s \leq (\|a\| + \|b\|)^s$, $s \geq 1$ yields that

$$\begin{aligned}
\| d[b, T_\Omega] u(x) \|_{p,\mathbb{R}^n}^p &\leq C_3 \sum_I \| u_I(x) \|_{p,\mathbb{R}^n}^p \\
&= C_3 \sum_I \int_{\mathbb{R}^n} |u_I(x)|^p dx \\
&= C_3 \int_{\mathbb{R}^n} \sum_I |u_I(x)|^p dx \\
&\leq C_4 \int_{\mathbb{R}^n} \left(\sum_I |u_I(x)| \right)^p dx.
\end{aligned} \tag{23}$$

Using the inequality $|\sum_{i=1}^N t_i|^s \leq N^{s-1} \sum_{i=1}^N |t_i|^s$ again, we easily have

$$\left(\sum_I |u_I(x)| \right)^2 \leq C_5 \sum_I |u_I(x)|^2 , \tag{24}$$

that is,

$$\left(\sum_I |u_I(x)| \right) \leq \left(C_5 \sum_I |u_I(x)|^2 \right)^{1/2} . \tag{25}$$

Substituting (25) into (23) gives

$$\begin{aligned}
\| d[b, T_\Omega] u \|_{p,\mathbb{R}^n}^p &\leq C_6 \int_{\mathbb{R}^n} \left(\sum_I |u_I(x)|^2 \right)^{p/2} dx \\
&= C_6 \int_{\mathbb{R}^n} |u(x)|^p dx = C_6 \|u\|_{p,\mathbb{R}^n}^p .
\end{aligned} \tag{26}$$

This completes the proof of Lemma 9. □

In Lemma 9, let $u = u$ in bounded domain M and $u = 0$ in $\mathbb{R}^n \setminus M$; we can easily obtain the following lemma.

Lemma 10. *Let $u \in L^p(M, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, T_Ω be the Calderón-Zygmund singular integral operator on differential forms, and b be sufficiently smooth and bounded. Then, there exists a constant C, independent of u, such that*

$$\| d[b, T_\Omega] u \|_{p,M} \leq C \|u\|_{p,M} , \tag{27}$$

where $M \subset \mathbb{R}^n$ is any bounded domain.

Using Lemma 1 and the analogous method developed in Lemma 9, we have the following estimate for $[b, T_\Omega]$.

Lemma 11. *Let $u \in L^p(M, \Lambda^l)$, $l = 0, 1, \ldots, n$, $1 < p < \infty$, and T_Ω be a Calderón-Zygmund singular integral operator on differential forms. Then, given any function $b \in BMO(\mathbb{R}^n)$, $[b, T_\Omega]$ satisfies the strong (p, p) inequality*

$$\| [b, T_\Omega] u \|_{p,M} \leq C \|b\|_{BMO} \|u\|_{p,M} \tag{28}$$

for any bounded domain $M \subset \mathbb{R}^n$, where C is a constant independent of u.

We now present the local higher integrability of commutators of Calderón-Zygmund singular integral operators on differential forms.

Theorem 12. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p_{loc}(M, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < p < n$, then $[b, T_\Omega] u \in L^s_{loc}(M, \Lambda^l)$ for any $0 < s < np/(n-p)$. Moreover, there exists a constant C, independent of u, such that*

$$\| [b, T_\Omega] u \|_{s,B} \leq C \mu(B)^{1/s + 1/n - 1/p} \|u\|_{p,\sigma B} \tag{29}$$

for all balls B with $\sigma B \subset M$ for some $\sigma > 1$.

Proof. For any ball B with $\sigma B \subset M$ for some $\sigma > 1$, in the case that the measure $\mu\{x \in B : |[b, T_\Omega] u - ([b, T_\Omega] u)_B| > 0\} = 0$. We have obviously that $[b, T_\Omega] u = ([b, T_\Omega] u)_B$ almost everywhere in B, which implies $[b, T] u$ is a closed form and a solution of the Dirac-harmonic equation (1) in M. Applying Lemmas 4 and 2 to $[b, T_\Omega] u$ with any $s, p > 0$, we have

$$\begin{aligned}
\| [b, T_\Omega] u \|_{s,B} &= \| ([b, T_\Omega] u)_B \|_{s,B} \\
&\leq C_1 \mu(B) \| [b, T_\Omega] u \|_{s,B} \\
&\leq C_2 \mu(B)^{1 + 1/s - 1/p} \| [b, T_\Omega] u \|_{p,\sigma B}
\end{aligned} \tag{30}$$

for all balls B with $\sigma B \subset M$ for some $\sigma > 1$. We note that b is bounded; thus $b \in BMO(\mathbb{R}^n)$. By Lemma 11, it follows that

$$\begin{aligned}
\| [b, T_\Omega] u \|_{s,B} &\leq C_2 \mu(B)^{1 + 1/s - 1/p} \| [b, T_\Omega] u \|_{p,\sigma B} \\
&\leq C_3 \mu(B)^{1 + 1/s - 1/p} \|b\|_{BMO} \|u\|_{p,\sigma B} \\
&\leq C_4 \mu(B)^{1 + 1/s - 1/p} \|u\|_{p,\sigma B} .
\end{aligned} \tag{31}$$

On the other hand, if the measure $\mu\{x \in B : |[b,T_\Omega]u - ([b,T_\Omega]u)_B| > 0\} > 0$. Applying Lemma 5 to $[b,T_\Omega]u$, we get

$$\left(\int_B \left|[b,T_\Omega]u - ([b,T_\Omega]u)_B\right|^{np/(n-p)} dx\right)^{(n-p)/np} \le C_5(n,p)\left(\int_B \left|d[b,T_\Omega]u\right|^p dx\right)^{1/p}. \tag{32}$$

Then by Lemma 10, it follows that

$$\left(\int_B \left|[b,T_\Omega]u - ([b,T_\Omega]u)_B\right|^{np/(n-p)} dx\right)^{(n-p)/np} \le C_6(n,p)\left(\int_B |u|^p dx\right)^{1/p}. \tag{33}$$

Noticing that the measure $|\{x \in B : |[b,T_\Omega]u - ([b,T_\Omega]u)_B| > 0\}| > 0$, we could use Lemma 6. Taking $\psi(t) = t^{np/(n-p)}$ in Lemma 6, we have

$$\left(\int_B |v|^{np/(n-p)} dx\right)^{(n-p)/np} \le C_7\left(\int_B \left|v - v_B\right|^{np/(n-p)} dx\right)^{(n-p)/np} \tag{34}$$

for any differential form v and any ball B with $|\{x \in B : |v - v_B| > 0\}| > 0$. Replacing v by $[b,T_\Omega]u$ in (34), we obtain

$$\left(\int_B \left|[b,T_\Omega]u\right|^{np/(n-p)} dx\right)^{(n-p)/np} \le C_8\left(\int_B \left|[b,T_\Omega]u - ([b,T_\Omega]u)_B\right|^{np/(n-p)} dx\right)^{(n-p)/np}. \tag{35}$$

Combining (33) and (35) yields that

$$\left(\int_B \left|[b,T_\Omega]u\right|^{np/(n-p)} dx\right)^{(n-p)/np} \le C_9\left(\int_B |u|^p dx\right)^{1/p}. \tag{36}$$

In view of the monotonic property of the L^p-space with $0 < s < np/(n-p)$, we have

$$\left(\frac{1}{\mu(B)}\int_B \left|[b,T_\Omega]u\right|^s dx\right)^{1/s} \le \left(\frac{1}{\mu(B)}\int_B \left|[b,T_\Omega]u\right|^{np/(n-p)} dx\right)^{(n-p)/np}. \tag{37}$$

Substituting (36) and (37) gives

$$\left(\int_B \left|[b,T_\Omega]u\right|^s dx\right)^{1/s} \le C_{10}\mu(B)^{1/s-1/p+1/n}\left(\int_B |u|^p dx\right)^{1/p}. \tag{38}$$

Therefore, inequality (29) follows in both cases, which indicates that if $u \in L^p_{loc}(M,\Lambda^l)$, then $[b,T]u \in L^s_{loc}(M,\Lambda^l)$. We have completed the proof of Theorem 12. □

Moreover, inequality (38) can be written as the following integral average inequality:

$$\left(\frac{1}{\mu(B)}\int_B \left|[b,T_\Omega]u\right|^s dx\right)^{1/s} \le C_{11}\mu(B)^{1/n}\left(\frac{1}{\mu(B)}\int_{\sigma B} |u|^p dx\right)^{1/p}. \tag{39}$$

Clearly, with p close to n, the integral exponent s on the left hand side could be much larger than the integral exponent p on the right hand side since the condition $0 < s < np/(n-p)$. Hence the higher integrability of operator $[b,T_\Omega]$ for the case that $1 < p < n$ is obtained.

Next, we consider the higher integrability of $[b,T_\Omega]$ for the case $p \ge n$.

Theorem 13. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p_{loc}(M,\Lambda^l)$, $l = 1,2,\ldots,n$, $p \ge n$, then $[b,T_\Omega]u \in L^s_{loc}(M,\Lambda^l)$ for any $s > 0$. Moreover, there exists a constant C, independent of u, such that*

$$\left\|[b,T_\Omega]u\right\|_{s,B} \le C\mu(B)^{1/s+1/n-1/p}\|u\|_{p,\sigma B} \tag{40}$$

for all balls B with $\sigma B \subset M$ for some $\sigma > 1$.

Proof. It is immediate by the same method developed in the proof of Theorem 12 that (40) holds for $\mu\{x \in B : |[b,T_\Omega]u - ([b,T_\Omega]u)_B| > 0\} = 0$. Let us consider the remain case that $\mu\{x \in B : |[b,T_\Omega]u - ([b,T_\Omega]u)_B| > 0\} > 0$. Select $\alpha = \max\{1, s/p\}$ and $q = \alpha np/(n+\alpha p)$. Then it is easy to check that

$$q - p = \frac{p(\alpha(n-p)) - n}{n+\alpha p} < 0, \tag{41}$$

since $n - p \le 0$. Notice that $1 < q = \alpha np/(n+\alpha p) < n$. Using Lemma 5 to $[b,T_\Omega]u$ and combining Lemma 10 and the monotonic property of the L^p-space, we have

$$\begin{aligned}\left(\int_B \left|[b,T_\Omega]u - ([b,T_\Omega]u)_B\right|^{nq/(n-q)} dx\right)^{(n-q)/nq} &\le C_1\left(\int_B \left|d[b,T_\Omega]u\right|^q dx\right)^{1/q} \\ &\le C_2\left(\int_B |u|^q dx\right)^{1/q} \\ &\le C_3\mu(B)^{1/q-1/p}\left(\int_B |u|^p dx\right)^{1/p}.\end{aligned} \tag{42}$$

Notice that the measure $|\{x \in B : |[b,T_\Omega]u - ([b,T_\Omega]u)_B| > 0\}| > 0$. Therefore, we can select $\psi(t) = t^{nq/(n-q)}$ in Lemma 6. Then for any differential form w, we have

$$\left(\int_B |w|^{nq/(n-q)} dx\right)^{(n-q)/nq} \le C_4\left(\int_B \left|w - w_B\right|^{nq/(n-q)} dx\right)^{(n-q)/nq}. \tag{43}$$

Replacing w by $[b,T_\Omega]u$ in (43), we get

$$\left(\int_B \left|[b,T_\Omega]u\right|^{nq/(n-q)}dx\right)^{(n-q)/nq} \le C_5\left(\int_B \left|[b,T_\Omega]u - ([b,T_\Omega]u)_B\right|^{nq/(n-q)}dx\right)^{(n-q)/nq}. \tag{44}$$

Taking into account the fact that $nq/(n-q)=\alpha p > s$, we see from the monotonic property of the L^p-space, (42), and (44) that

$$\begin{aligned}\left(\int_B \left|[b,T_\Omega]u\right|^{s}dx\right)^{1/s} &\le C_6\mu(B)^{1/s-1/\alpha p}\cdot\left(\int_B \left|[b,T_\Omega]u\right|^{nq/(n-q)}dx\right)^{(n-q)/nq}\\ &\le C_7\mu(B)^{1/s-1/\alpha p}\cdot\left(\int_B \left|[b,T_\Omega]u-([b,T_\Omega]u)_B\right|^{nq/(n-q)}dx\right)^{(n-q)/nq}\\ &\le C_8\mu(B)^{1/q-1/p+1/s-1/\alpha p}\left(\int_B |u|^p dx\right)^{1/p}\\ &= C_8\mu(B)^{1/n+1/s-1/p}\left(\int_B |u|^p dx\right)^{1/p},\end{aligned} \tag{45}$$

which means that if $u \in L^p_{loc}(M,\Lambda^l)$, then $[b,T_\Omega]u \in L^s_{loc}(M,\Lambda^l)$. We have completed the proof of Theorem 13. □

Now, we are ready to assert the global higher integrability of the commutator of Calderón-Zygmund singular integral operator on differential forms.

Theorem 14. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p(M,\Lambda^l)$, $l=1,2,\ldots,n$, $1<p<n$, then $[b,T_\Omega]u \in L^s(M,\Lambda^l)$ for any $0<s<np/(n-p)$. Moreover, there exists a constant C, independent of u, such that*

$$\left\|[b,T_\Omega]u\right\|_{s,M} \le C\mu(M)^{1/s+1/n-1/p}\|u\|_{p,M} \tag{46}$$

for any bounded domain $M\subset\mathbb{R}^n$.

Proof. Notice that $1/s+1/n-1/p>0$ since $0<s<np/(n-p)$. By Lemma 7 and Theorem 12, we have

$$\begin{aligned}\|[b,T]u\|_{s,M} &\le \sum_{B\in\nu}\|[b,T]u\|_{s,B}\\ &\le \sum_{B\in\nu}\left(C_1\mu(B)^{1/s+1/n-1/p}\|u\|_{p,\sigma B}\right)\\ &\le \sum_{B\in\nu}\left(C_1\mu(M)^{1/s+1/n-1/p}\|u\|_{p,\sigma B}\right)\\ &\le C_2\mu(M)^{1/s+1/n-1/p}N\|u\|_{p,M}\\ &\le C_3\|u\|_{p,M},\end{aligned} \tag{47}$$

which finishes the proof of Theorem 14. □

Using the similar method as we did in Theorem 14 and combining with Theorem 13, we can deduce the following global result for the case $p\ge n$.

Theorem 15. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u\in L^p(M,\Lambda^l)$, $l=1,2,\ldots,n$, $p\ge n$, then $[b,T_\Omega]u\in L^s(M,\Lambda^l)$ for any $s>0$. Moreover, there exists a constant C, independent of u, such that*

$$\left\|[b,T_\Omega]u\right\|_{s,M} \le C\mu(M)^{1/s+1/n-1/p}\|u\|_{p,M} \tag{48}$$

for any bounded domain $M\subset\mathbb{R}^n$.

4. Higher Order Poincaré-Type Inequalities

In this section, we shall state the higher order Poincaré-type inequalities for commutator of Calderón-Zygmund singular integral operator acting on the solutions of the Dirac-harmonic equations.

Theorem 16. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u\in L^p_{loc}(M,\Lambda^l)$ is a solution of the Dirac-harmonic equation (1), $l=1,2,\ldots,n$, $1<p<n$. Then, for any $0<s<np/(n-p)$, there exists a constant $C>0$, independent of u, such that*

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\ &\quad\le C\mu(B)^{1+1/s-1/p+1/n}\|u\|_{p,\sigma B}\end{aligned} \tag{49}$$

for all balls B with $\sigma B\subset M$ for some $\sigma>1$.

Proof. For any $u\in L^p_{loc}(M,\Lambda^l)$, using the L^p-decomposition formula (9) to $[b,T_\Omega]u$, we obtain

$$[b,T_\Omega]u = dT([b,T_\Omega]u)+Td([b,T_\Omega]u). \tag{50}$$

Noticing that $dT([b,T_\Omega]u)=([b,T_\Omega]u)_B$ for any differential form u, by (50), (10) and Lemma 10, we get

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{np/(n-p),B}\\ &\quad=\left\|Td([b,T_\Omega]u)\right\|_{np/(n-p),B}\\ &\quad\le C_1(n,p)\mu(B)\,diam(B)\left\|d([b,T_\Omega]u)\right\|_{np/(n-p),B}\\ &\quad\le C_2(n,p)\mu(B)^{1+1/n}\|u\|_{np/(n-p),B}.\end{aligned} \tag{51}$$

Since u is a solution of the Dirac-harmonic equation (1), by Lemma 2, we have

$$\|u\|_{np/(n-p),B}\le\mu(B)^{-1/n}\|u\|_{p,\sigma B}, \tag{52}$$

where $\sigma>1$ is a constant. Substituting (52) into (51) gives that

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{np/n-p,B}\\ &\quad\le C_3(n,p)\mu(B)\|u\|_{p,\sigma B}.\end{aligned} \tag{53}$$

Moreover, for any $0 < s < np/(n-p)$, applying the monotonic properties of L^p-space, it follows that

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\&\quad\leq \mu(B)^{1/s-1/p+1/n}\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{np/n-p,B}.\end{aligned}\tag{54}$$

Combined with (53), the above inequality becomes

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\&\quad\leq C_4(n,p)\,\mu(B)^{1+1/s-1/p+1/n}\|u\|_{p,\sigma B},\end{aligned}\tag{55}$$

which finishes the proof of Theorem 16. □

Next, we will prove the higher order Poincaré-type inequality still holds for the case that $p \geq n$.

Theorem 17. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p_{loc}(M,\Lambda^l)$ is a solution of the Dirac-harmonic equation (1), $l = 1,2,\ldots,n$, $p \geq n$. Then, for any $s > 0$, there exists a constant C, independent of u, such that*

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\&\quad\leq C\mu(B)^{1+1/s-1/p+1/n}\|u\|_{p,\sigma B}\end{aligned}\tag{56}$$

for all balls B with $\sigma B \subset M$ for some $\sigma > 1$.

Proof. Select $\alpha = \max\{1, s/p\}$ and $q = \alpha np/(n+\alpha p)$. Then, we easily obtain

$$q-p=\frac{p(\alpha(n-p))-n}{n+\alpha p}<0,\tag{57}$$

that is, $q < p$, since $n-p \leq 0$. We can also find $1 < q = \alpha np/(n+\alpha p) < n$. Using the same technique as in the proof of Theorem 16, we have

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{nq/(n-q),B}\\&\quad=\left\|Td([b,T_\Omega]u)\right\|_{nq/(n-q),B}\\&\quad\leq C_1\mu(B)\,diam(B)\left\|d([b,T_\Omega]u)\right\|_{nq/(n-q),B}\\&\quad\leq C_2\mu(B)^{1+1/n}\|u\|_{nq/(n-q),B}\\&\quad\leq C_3\mu(B)^{1+1/n}\mu(B)^{1/q-1/n-1/p}\|u\|_{p,\sigma B}\\&\quad\leq C_4\mu(B)^{1+1/q-1/p}\|u\|_{p,\sigma B}.\end{aligned}\tag{58}$$

Noticing that $nq/(n-q) = \alpha p > s$ and combining the monotonic property of the L^p-space, we obtain

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\&\quad\leq \mu(B)^{1/s-1/q+1/n}\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{nq/n-q,B}.\end{aligned}\tag{59}$$

Combining (58) and (59), we finally have

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_B\right\|_{s,B}\\&\quad\leq C_5\mu(B)^{1+1/s-1/p+1/n}\|u\|_{p,\sigma B}.\end{aligned}\tag{60}$$

The proof of Theorem 17 is completed. □

Based on Theorems 16 and 17, we can obtain the following global higher order Poincaré-type inequalities for the commutator $[b,T_\Omega]$ using the analogous method developed in Theorem 14.

Theorem 18. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p(M,\Lambda^l)$ is a solution of the Dirac-harmonic equation (1), $l = 1,2,\ldots,n$, $1 < p < n$. Then, for any $0 < s < np/(n-p)$, there exists a constant $C > 0$, independent of u, such that*

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_M\right\|_{s,M}\\&\quad\leq C\mu(M)^{1/n+1/s-1/p}\|u\|_{p,M}\end{aligned}\tag{61}$$

for any bounded domain $M \subset \mathbb{R}^n$.

Theorem 19. *Let T_Ω be the Calderón-Zygmund singular integral operator on differential forms and b be sufficiently smooth and bounded. If $u \in L^p(M,\Lambda^l)$ is a solution of the Dirac-harmonic equation (1), $l = 1,2,\ldots,n$, $p \geq n$. Then, for any $s > 0$, there exists a constant C, independent of u, such that*

$$\begin{aligned}&\left\|[b,T_\Omega]u-([b,T_\Omega]u)_M\right\|_{s,M}\\&\quad\leq C\mu(M)^{1/n+1/s-1/p}\|u\|_{p,M}\end{aligned}\tag{62}$$

for any bounded domain $M \subset \mathbb{R}^n$.

5. Applications

In this section, we demonstrate some applications of our main results established in the previous sections.

Example 20. Let $r > 0$ and $k > 0$ be any constants and $M_1 = \{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2 \leq r^2\} \subset \mathbb{R}^3$. Consider the 1-form

$$\begin{aligned}u(x_1,x_2,x_3)&=\frac{x_1}{k+x_1^2+x_2^2+x_3^2}dx_1\\&\quad+\frac{x_2}{k+x_1^2+x_2^2+x_3^2}dx_2\\&\quad+\frac{x_3}{k+x_1^2+x_2^2+x_3^2}dx_3\end{aligned}\tag{63}$$

defined in M such that $M_1 \subset M$. It is easy to check that $du = 0$. Hence, u is a solution of the Dirac-harmonic equation (1) for any operators A satisfying (2). Also, it can be calculated that

$$|u| = \left(\left(\frac{x_1}{k + x_1^2 + x_2^2 + x_3^2}\right)^2 + \left(\frac{x_2}{k + x_1^2 + x_2^2 + x_3^2}\right)^2 + \left(\frac{x_3}{k + x_1^2 + x_2^2 + x_3^2}\right)^2\right)^{1/2} = \left(\frac{x_1^2 + x_2^2 + x_3^2}{\left(k + x_1^2 + x_2^2 + x_3^2\right)^2}\right)^{1/2} < 1 \quad (64)$$

for any $x = (x_1, x_2, x_3) \in M_1$. Thus,

$$\|u\|_{p,M_1} = \left(\int_{M_1} |u|^p \, dx\right)^{1/p} = \left(\int_{M_1} \left|\frac{x_1^2 + x_2^2 + x_3^2}{\left|k + x_1^2 + x_2^2 + x_3^2\right|^2}\right|^{p/2} dx\right)^{1/p} \leq \left(\int_{M_1} 1 dx\right)^{1/p} = \left(\frac{4\pi}{3} r^3\right)^{1/p}. \quad (65)$$

Applying Theorem 14 for $1 < p < 3$, we have $[b, T_\Omega]u \in L^s(M, \Lambda^l)$ for any $0 < s < 3p/(3 - p)$. Analogously, we have $[b, T_\Omega]u \in L^s(M, \Lambda^l)$ for any $s > 0$ when $p \geq 3$ by Theorem 15. In the meantime, we have the higher estimate for $[b, T_\Omega]u$; that is,

$$\left\|[b, T_\Omega] u\right\|_{s,M_1} \leq C\left(\frac{4\pi}{3} r^3\right)^{1/p}. \quad (66)$$

In addition, we can evaluate the following integrals by Theorem 18 and Theorem 19 that

$$\left\|[b, T_\Omega] u - \left([b, T_\Omega] u\right)_{M_1}\right\|_{s,M_1} \leq C\left(\frac{4\pi}{3} r^3\right)^{1/p}. \quad (67)$$

We should notice that the above example can be extended to the case of $\mathbb{R}^n$ as follows.

Example 21. We can check that the 1-form defined in $\mathbb{R}^n$

$$u\left(x_1, \ldots, x_n\right) = \sum_{i=1}^{n} \frac{x_i}{k + x_1^2 + \cdots + x_n^2} dx_i, \quad k > 0, \quad (68)$$

is a solution of the Dirac-harmonic equation (1) for any operators A satisfying (2). Hence, Theorems 14, 15, 18, and 19 are also applicable to $u(x_1, \ldots, x_n)$.

Remark. It is worth pointing out that the global results obtained in this paper can be extended to larger classes of domains, such as L^p-averaging domains and $L^\varphi(\mu)$-averaging domains; see [1, 27]. Also, the techniques developed in this paper provide an effective method to study the higher integrability of bilinear commutators of singular integrals on differential forms, which are defined in [25]. We leave the statements and proofs to the interested readers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (Grant no. 11601105).

References

[1] R. P. Agarwal, S. Ding, and C. Nolder, *Inequalities for Differential Forms*, Springer, New York, NY, USA, 2009.

[2] Y. Xing and C. Wu, "Global weighted inequalities for operators and harmonic forms on manifolds," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 294–309, 2004.

[3] J. B. Perot and C. J. Zusi, "Differential forms for scientists and engineers," *Journal of Computational Physics*, vol. 257, no. part B, pp. 1373–1393, 2014.

[4] C. Scott, "L^p-theory of differential forms on manifolds theory of differential forms on manifolds," *Transactions of the American Mathematical Society*, vol. 347, no. 6, pp. 2075–2096, 1995.

[5] S. Ding, "Two-weight Caccioppoli inequalities for solutions of nonhomogeneous A-harmonic equations on Riemannian manifolds," *Proceedings of the American Mathematical Society*, vol. 132, no. 8, pp. 2367–2375, 2004.

[6] S. Ding and B. Liu, "A singular integral of the composite operator," *Applied Mathematics Letters*, vol. 22, no. 8, pp. 1271–1275, 2009.

[7] S. Ding and B. Liu, "Global estimates for singular integrals of the composite operator," *Illinois Journal of Mathematics*, vol. 53, no. 4, pp. 1173–1185, 2009.

[8] S. Ding and B. Liu, "Global integrability of the Jacobian of a composite mapping," *Journal of Inequalities and Applications*, Art. ID 89134, 9 pages, 2006.

[9] H. Bi and S. Ding, "Some strong (p,q)-type inequalities for the homotopy operator," *Computers and Mathematics with Applications. An International Journal*, vol. 62, no. 4, pp. 1780–1789, 2011.

[10] H. Bi, "Weighted inequalities for potential operators on differential forms," *Journal of Inequalities and Applications*, Art. ID 713625, 13 pages, 2010.

[11] Y. Ling and B. Gejun, "Some local Poincaré inequalities for the composition of the sharp maximal operator and the Green's operator," *Computers and Mathematics with Applications*, vol. 63, no. 3, pp. 720–727, 2012.

[12] S. Ding, "Integral estimates for the Laplace-Beltrami and Green's operators applied to differential forms on manifolds," *Journal for Analysis and its Applications*, vol. 22, no. 4, pp. 939–957, 2003.

[13] G. Lu and R. L. Wheeden, "High order representation formulas and embedding theorems on stratified groups and generalizations," *Studia Mathematica*, vol. 142, no. 2, pp. 101–133, 2000.

[14] G. Lu, "Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-STEin

spaces on stratified groups," *Acta Mathematica Sinica*, vol. 16, no. 3, pp. 405–444, 2000.

[15] E. W. Stredulinsky, "Higher integrability from reverse Hölder inequalities," *Indiana University Mathematics Journal*, vol. 29, no. 3, pp. 407–413, 1980.

[16] F. W. Gehring, "The L^p-integrability of the partial derivatives of a quasiconformal mapping," *Acta Mathematica*, vol. 130, pp. 265–277, 1973.

[17] W. S. Cohn, G. Lu, and S. Lu, "Higher order Poincaré inequalities associated with linear operators on stratified groups and applications," *Mathematische Zeitschrift*, vol. 244, no. 2, pp. 309–335, 2003.

[18] S. Ding and B. Liu, "Dirac-harmonic equations for differential forms," *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal*, vol. 122, pp. 43–57, 2015.

[19] C. A. Nolder, "Hardy-Littlewood theorems for A-harmonic tensors," *Illinois Journal of Mathematics*, vol. 43, no. 4, pp. 613–632, 1999.

[20] A. Carbonaro, A. McIntosh, and A. J. Morris, "Local Hardy spaces of differential forms on Riemannian manifolds," *The Journal of Geometric Analysis*, vol. 23, no. 1, pp. 106–169, 2013.

[21] K. Kodaira, "Harmonic fields in Riemannian manifolds (generalized potential theory)," *Annals of Mathematics: Second Series*, vol. 50, pp. 587–665, 1949.

[22] R. R. Coifman, R. Rochberg, and G. Weiss, "Factorization theorems for Hardy spaces in several variables," *Annals of Mathematics*, vol. 103, no. 3, pp. 611–635, 1976.

[23] L. Chaffee and D. Cruz-Uribe, "Necessary conditions for the boundedness of linear and bilinear commutators on Banach function spaces," *Mathematical Inequalities and Applications*, vol. 21, no. 1, pp. 1–16, 2018.

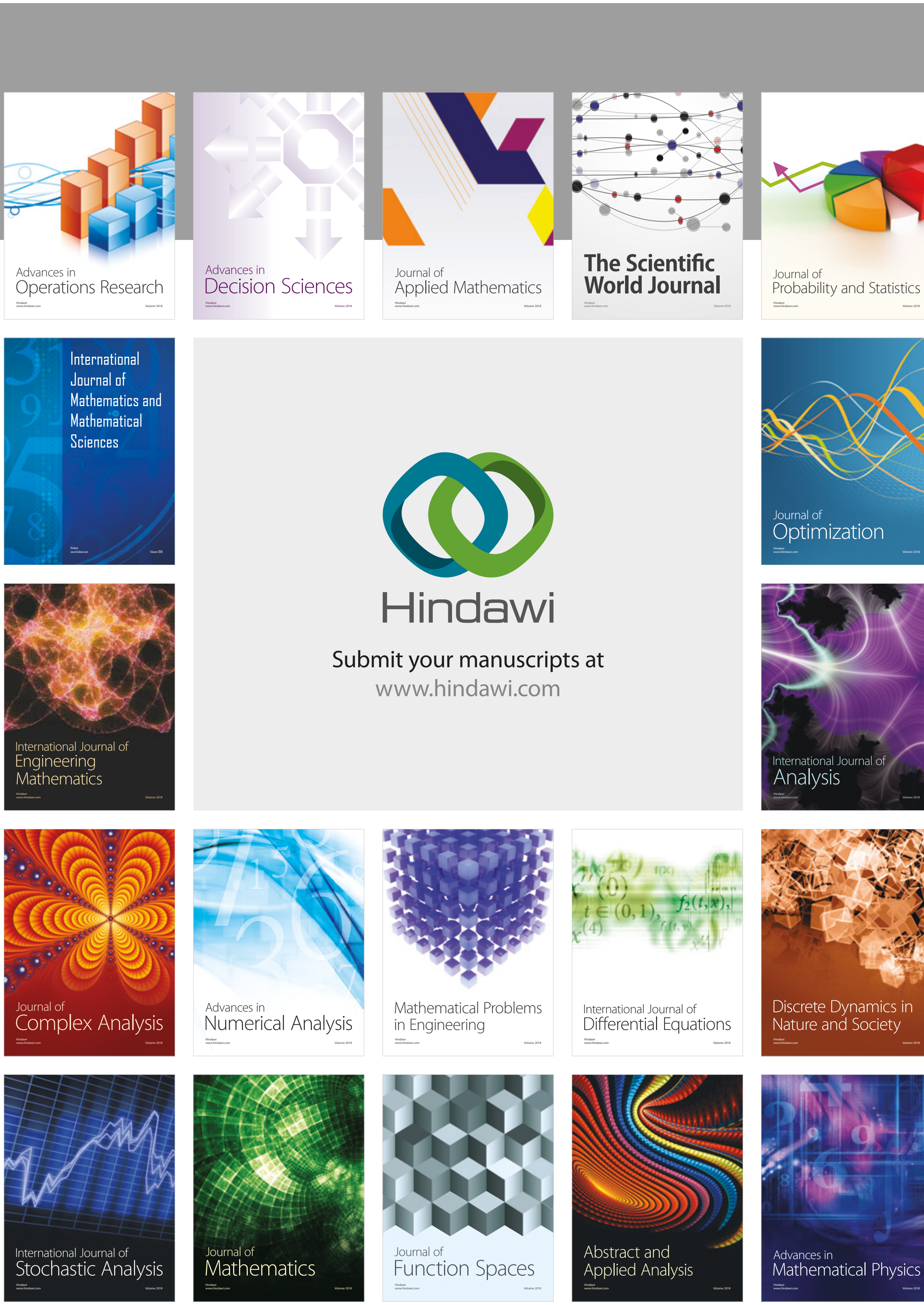
[24] C. Segovia and J. L. Torrea, "Weighted inequalities for commutators of fractional and singular integrals," *Publicacions Matemàtiques*, vol. 35, no. 1, pp. 209–235, 1991.

[25] C. Pérez, "Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function," *Journal of Fourier Analysis and Applications*, vol. 3, no. 6, pp. 743–756, 1997.

[26] T. Iwaniec and A. Lutoborski, "Integral estimates for null Lagrangians," *Archive for Rational Mechanics and Analysis*, vol. 125, no. 1, pp. 25–79, 1993.

[27] S. Ding, "$L^{\varphi}(\mu)$-averaging domains and the quasi-hyperbolic metric," *Computers and Mathematics with Applications. An International Journal*, vol. 47, no. 10-11, pp. 1611–1618, 2004.

[28] A. P. Calderón, "Commutators of singular integral operators," *Proceedings of the National Acadamy of Sciences of the United States of America*, vol. 53, pp. 1092–1099, 1965.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
The Scientific
World Journal
Journal of
Probability and Statistics
International
Journal of
Mathematics and
Mathematical
Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of
Optimization
International Journal of
Engineering
Mathematics
International Journal of
Analysis
Journal of
Complex Analysis
Advances in
Numerical Analysis
Mathematical Problems
in Engineering
International Journal of
Differential Equations
Discrete Dynamics in
Nature and Society
International Journal of
Stochastic Analysis
Journal of
Mathematics
Journal of
Function Spaces
Abstract and
Applied Analysis
Advances in
Mathematical Physics