Research Article

The Eigenvalue Problem for Caputo Type Fractional Differential Equation with Riemann-Stieltjes Integral Boundary Conditions

Wenjie Ma¹ and Yujun Cui²

¹Department of Applied Mathematics, Shandong University of Science and Technology, Qingdao 266590, China

²State Key Laboratory of Mining Disaster Prevention and Control Co-Founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Yujun Cui; cuj720201@163.com

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In this paper, we investigate the eigenvalue problem for Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions ${}^c D_{0+}^{\theta} p(y) + \mu f(t, p(y)) = 0$, $y \in [0, 1]$, $p(0) = p''(0) = 0$, $p(1) = \int_0^1 p(y) dA(y)$, where ${}^c D_{0+}^{\theta}$ is Caputo fractional derivative, $\theta \in (2, 3]$, and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. By using the Guo-Krasnoselskii's fixed point theorem on cone and the properties of the Green's function, some new results on the existence and nonexistence of positive solutions for the fractional differential equation are obtained.

1. Introduction

The experience of the last few years has fully borne out the fact that the integer order calculus is not as widely used as fractional order calculus in some fields such as chemistry, control theory, and signal processing. On the remarkable survey of Agarwal, Benchohra, and Hamani [1] it is pointed out that fractional differential equations constitute a fundamental tool in the modeling of some phenomena (see also [2–4]). The use of fractional order is more accurate for the description of phenomena, so the study of fractional differential equations becomes the mainstream with the help of techniques of nonlinear analysis. We refer the reader to [5–36] for recent results. For example, in [9], the author studied the following fractional differential equation:

$${}^c D^{\alpha} x(y) + f(y, x(y)) = 0, \quad 0 < y < 1, \quad (1)$$

with boundary conditions

$$\begin{aligned} x'(0) &= x''(0) = 0, \\ x(1) &= \mu \int_0^1 x(s) ds, \end{aligned} \quad (2)$$

where $\alpha \in (2, 3]$, $\mu \in [0, 1]$, and ${}^c D^{\alpha}$ is the Caputo derivative. They solved the above problem by means of classical fixed point theorems.

In [5], the boundary value problem for the following nonlinear fractional differential equation was discussed:

$$D_{0+}^{\alpha} x(y) + a(y) f(y, x(y), x''(y)) = 0, \quad (3)$$

$$0 < y < 1,$$

with boundary conditions

$$x(0) = x'(0) = x''(0) = x''(1) = 0, \quad (4)$$

where D_{0+}^{α} is the Riemann-Liouville differentiation, $\alpha \in (3, 4]$. By using a fixed point theorem, a new result of the existence of three positive solutions is obtained.

In [15], the authors investigated the following class of BVP:

$${}^c D_{0+}^q x(y) = f(y, x(y), {}^c D_{0+}^{\sigma} x(y)), \quad 0 < y < 1, \quad (5)$$

with boundary conditions

$$\begin{aligned} x(0) &= x''(0) = 0, \\ x'(1) &= \alpha x''(1), \end{aligned} \quad (6)$$

where $q \in (2, 3)$, $\sigma \in (1, 2)$, $f : [0, 1] \times R \times R \rightarrow R$ is a given function, and ${}^c D_{0+}^q$ denotes the Caputo differentiation. The author investigated this problem by using Banach contraction principle, Leray-Schauder nonlinear alternative, properties of the Green's function, and Guo-Krasnoselskii fixed point theorem on cone. Similar problems can be referred to in [25].

In this paper, we investigate the eigenvalue problem for Caputo fractional boundary value problem with Riemann-Stieltjes integral boundary conditions

$$\begin{aligned} {}^c D_{0+}^\theta p(y) + \mu f(y, p(y)) &= 0, \quad y \in [0, 1], \\ p(0) &= p''(0) = 0, \\ p(1) &= \int_0^1 p(y) dA(y), \end{aligned} \quad (7)$$

where $\theta \in (2, 3)$, $f : [0, 1] \times [0, \infty) \rightarrow (0, \infty)$ is continuous, $\mu > 0$ is a parameter, ${}^c D_{0+}^\theta$ is the Caputo fractional derivative, and A is a bounded variation function with positive measures with

$$B = \int_0^1 y dA(y) < 1. \quad (8)$$

Our proof is based on the properties of the Green's function and the Guo-Krasnoselskii fixed point theorem on cone.

2. Preliminaries

In order to solve problem (7), we provide the properties related to problem (7).

Definition 1 (see [3]). The Caputo's fractional derivative of order $\theta > 0$ for a function $x \in C^n[0, +\infty)$ is defined as

$${}^c D_{0+}^\theta x(y) = \frac{1}{\Gamma(n-\theta)} \int_0^y (y-s)^{n-\theta-1} x^{(n)}(s) ds, \quad (9)$$

$n-1 < \theta < n,$

where n is the smallest integer greater than or equal to θ .

Lemma 2 (see [3]). Let $\theta > 0$. If we assume $x \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation

$${}^c D_{0+}^\theta x(y) = 0 \quad (10)$$

has the general solution $x(y) = C_0 + C_1 y + \dots + C_{n-1} y^{n-1}$, $C_k \in R$, $k = 0, 1, \dots, n-1$, where n is the smallest integer greater than or equal to θ .

Lemma 3 (see [3]). Given that $x \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order θ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^\theta {}^c D_{0+}^\theta x(y) = x(y) + C_0 + C_1 y + \dots + C_{n-1} y^{n-1}, \quad (11)$$

for $C_k \in R$, $k = 0, 1, \dots, n-1$,

where n is the smallest integer greater than or equal to θ .

Firstly, we consider the following linear Caputo fractional differential equation:

$$\begin{aligned} {}^c D_{0+}^\theta p(y) + \sigma(y) &= 0, \quad y \in [0, 1], \\ p(0) &= p''(0) = 0, \end{aligned} \quad (12)$$

$$p(1) = \int_0^1 p(y) dA(y).$$

Lemma 4. Let $\theta \in (2, 3)$ and assume that $\sigma \in C[0, 1]$. Then p is the solution of the above boundary value problem (12), if and only if p satisfies the following integral equation:

$$p(y) = \int_0^1 G(y, s) \sigma(s) ds, \quad (13)$$

where

$$G(y, s) = \frac{1}{\Gamma(\theta)} \begin{cases} \frac{y}{1-B} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right] - (y-s)^{\theta-1}, & 0 \leq s \leq y \leq 1, \\ \frac{y}{1-B} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right], & 0 \leq y \leq s \leq 1, \end{cases} \quad (14)$$

and

$$B = \int_0^1 y dA(y) < 1. \quad (15)$$

Proof. Applying the fractional integral of order θ to both sides of (12) for $y \in [0, 1]$, we get the following formula:

$$I_{0+}^\theta {}^c D_{0+}^\theta p(y) + I_{0+}^\theta \sigma(y) = 0. \quad (16)$$

According to $p(0) = p''(0) = 0$ and Lemma 3, we obtain

$$p(y) = cy - \frac{1}{\Gamma(\theta)} \int_0^y (y-s)^{\theta-1} \sigma(s) ds, \quad (17)$$

where $c \in R$. Since $p(1) = \int_0^1 p(y) dA(y)$, we deduce that

$$\begin{aligned} c - \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} \sigma(s) ds \\ = \int_0^1 \left(cy - \frac{1}{\Gamma(\theta)} \int_0^y (y-s)^{\theta-1} \sigma(s) ds \right) dA(y). \end{aligned} \quad (18)$$

Therefore,

$$c = \frac{1}{\Gamma(\theta)(1-B)} \left[\int_0^1 (1-s)^{\theta-1} \sigma(s) ds - \int_0^1 \int_0^y (y-s)^{\theta-1} \sigma(s) ds dA(y) \right]. \quad (19)$$

Substituting the above equality into (17), one has

$$p(y) = \frac{y}{\Gamma(\theta)(1-B)} \left[\int_0^1 (1-s)^{\theta-1} \sigma(s) ds - \int_0^1 \int_0^y (y-s)^{\theta-1} \sigma(s) ds dA(y) \right] - \frac{1}{\Gamma(\theta)}$$

and

$$\begin{aligned} & \cdot \int_0^y (y-s)^{\theta-1} \sigma(s) ds \\ &= \frac{y}{\Gamma(\theta)(1-B)} \left[\int_0^1 (1-s)^{\theta-1} \sigma(s) ds - \int_0^1 \int_s^1 (y-s)^{\theta-1} dA(y) \sigma(s) ds \right] - \frac{1}{\Gamma(\theta)} \\ & \cdot \int_0^y (y-s)^{\theta-1} \sigma(s) ds \end{aligned} \quad (20)$$

$$G(y, s) = \frac{1}{\Gamma(\theta)} \begin{cases} \frac{y}{1-B} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right] - (y-s)^{\theta-1}, & 0 \leq s \leq y \leq 1, \\ \frac{y}{1-B} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right], & 0 \leq y \leq s \leq 1. \end{cases} \quad (21)$$

The proof is completed. $\square$

For $s \leq y$, we have

Lemma 5. The Green's function $G(y, s)$ has the following properties:

- (i) $\Gamma(\theta)G(y, s) \leq (1/(1-B))(1-s)^{\theta-1}$, for $y, s \in [0, 1]$;
- (ii) $\Gamma(\theta)G(y, s) \geq N(1-s)^{\theta-1}$, for $y \in [1/4, 3/4]$ and $s \in [0, 1]$, where

$$N = \min \left\{ \frac{1 - \int_0^1 y^{\theta-1} dA(y)}{4(1-B)}, \min_{y \in [1/4, 3/4]} y(1-y)^{\theta-2} \right\}. \quad (22)$$

Proof. (i) Obviously, the inequality $\Gamma(\theta)G(y, s) \leq (1/(1-B))(1-s)^{\theta-1}$ holds from the representation of $G(y, s)$.

(ii) In view of $B = \int_0^1 y dA(y) < 1$ and $\theta \in (2, 3)$, we have $1 - \int_s^1 y^{\theta-1} dA(y) > 0$.

For $y \leq s$, we have

$$\begin{aligned} & \frac{y}{(1-B)} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right] \\ & \geq \frac{y}{(1-B)} \left[(1-s)^{\theta-1} - \int_s^1 (y-ys)^{\theta-1} dA(y) \right] \\ & \geq \frac{y}{(1-B)} (1-s)^{\theta-1} \left[1 - \int_s^1 y^{\theta-1} dA(y) \right] \\ & \geq \frac{y \left(1 - \int_0^1 y^{\theta-1} dA(y) \right)}{(1-B)} (1-s)^{\theta-1}. \end{aligned} \quad (23)$$

$$\begin{aligned} & \frac{y}{(1-B)} \left[(1-s)^{\theta-1} - \int_s^1 (y-s)^{\theta-1} dA(y) \right] - (y-s)^{\theta-1} \\ & \geq \frac{y}{(1-B)} \left[(1-s)^{\theta-1} - \int_s^1 (y-ys)^{\theta-1} dA(y) \right] - (y-ys)^{\theta-1} \\ & = \frac{y(1-B+B)}{(1-B)} (1-s)^{\theta-1} - \frac{y}{(1-B)} \\ & \cdot \int_s^1 (y-ys)^{\theta-1} dA(y) - y^{\theta-1} (1-s)^{\theta-1} \\ & = y(1-s)^{\theta-1} + \frac{y \int_0^1 y dA(y)}{(1-B)} (1-s)^{\theta-1} \\ & - \frac{y}{(1-B)} \int_s^1 (y-ys)^{\theta-1} dA(y) - y^{\theta-1} (1-s)^{\theta-1} \\ & = (1-s)^{\theta-1} (y - y^{\theta-1}) \\ & + \frac{y}{(1-B)} \left((1-s)^{\theta-1} \int_0^1 y dA(y) - \int_s^1 (y-ys)^{\theta-1} dA(y) \right) = (1-s)^{\theta-1} (y - y^{\theta-1}) \\ & + \frac{y}{(1-B)} (1-s)^{\theta-1} \left(\int_0^1 y dA(y) \right) \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 y^{\theta-1} dA(y) \Big) = (1-s)^{\theta-1} (y - y^{\theta-1}) \\
& + \frac{y}{(1-B)} (1-s)^{\theta-1} \left(\int_0^1 (y - y^{\theta-1}) dA(y) \right) \geq (1 \\
& - s)^{\theta-1} (y - y^{\theta-1}) = (1-s)^{\theta-1} y (1 - y^{\theta-2}). \quad (24)
\end{aligned}$$

Thus, the above two inequalities yield the inequality in (ii). The proof is completed. $\square$

Let $X = C[0, 1]$, $\|p\| = \max_{y \in [0, 1]} |p(y)|$; then $(X, \|\cdot\|)$ is a Banach space. We define the cone $P \subset X$ by

$$P = \left\{ p \in X : p(y) \geq N(1-B)\|p\|, \ y \in \left[\frac{1}{4}, \frac{3}{4} \right] \right\}. \quad (25)$$

Let $A_\mu : X \rightarrow X$ be the operator defined as

$$(A_\mu p)(y) = \mu \int_0^1 G(y, s) f(s, p(s)) ds. \quad (26)$$

Thus, the fixed point of the above integral equation is equivalent to the solution of the BVP (7).

Lemma 6. $A_\mu(P) \subset P$ and $A_\mu : P \rightarrow P$ is a completely continuous operator, where A_μ is defined in (26).

Proof. By Lemma 5, for $\forall p \in P$, we have

$$\begin{aligned}
(A_\mu p)(y) &= \mu \int_0^1 G(y, s) f(s, p(s)) ds \\
&\geq \frac{\mu N}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} f(s, p(s)) ds \\
&\geq \mu N(1-B) \max_{t \in [0, 1]} \int_0^1 G(t, s) f(s, p(s)) ds \\
&= N(1-B) \|A_\mu p\|, \quad y \in \left[\frac{1}{4}, \frac{3}{4} \right]. \quad (27)
\end{aligned}$$

Hence we have $A_\mu(P) \subset P$. Let $\Omega \subset P$ be bounded. Then there exists a constant $M > 0$ such that $\|p\| \leq M$ for $\forall p \in \Omega$. Let $L_1 = \max_{y \in [0, 1], p \in [0, M]} (\mu f(y, p) + 1)$. Then

$$\begin{aligned}
(A_\mu p)(y) &\leq \frac{L_1}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} ds, \\
&\quad y \in [0, 1]. \quad (28)
\end{aligned}$$

Thus, $A_\mu(\Omega)$ is bounded. Put $p \in \Omega$ and $y_1, y_2 \in [0, 1]$. We deduce that

$$\begin{aligned}
& |(A_\mu p)(y_1) - (A_\mu p)(y_2)| = \mu \left| \int_0^1 G(y_1, s) \right. \\
& \cdot f(s, p(s)) ds - \int_0^1 G(y_2, s) f(s, p(s)) ds \Big|
\end{aligned}$$

$$\begin{aligned}
&= \mu \left| \int_0^{y_1} \left[\frac{y_1}{1-B} \left((1-s)^{\theta-1} - \int_s^1 (y_1-s)^{\theta-1} dA(y) \right) \right. \right. \\
& - (y_1-s)^{\theta-1} \Big] f(s, p(s)) ds + \int_{y_1}^{y_2} \frac{y_1}{1-B} \left[(1-s)^{\theta-1} \right. \\
& - \int_s^1 (y_1-s)^{\theta-1} dA(y) \Big] f(s, p(s)) ds \\
& + \int_{y_2}^1 \frac{y_1}{1-B} \left[(1-s)^{\theta-1} - \int_s^1 (y_1-s)^{\theta-1} dA(y) \right] \\
& \cdot f(s, p(s)) ds \\
& - \int_0^{y_1} \left[\frac{y_2}{1-B} \left((1-s)^{\theta-1} - \int_s^1 (y_2-s)^{\theta-1} dA(y) \right) \right. \\
& - (y_2-s)^{\theta-1} \Big] f(s, p(s)) ds \\
& - \int_{y_1}^{y_2} \left[\frac{y_2}{1-B} \left((1-s)^{\theta-1} - \int_s^1 (y_2-s)^{\theta-1} dA(y) \right) \right. \\
& - (y_2-s)^{\theta-1} \Big] f(s, p(s)) ds - \int_{y_2}^1 \frac{y_2}{1-B} \left[(1-s)^{\theta-1} \right. \\
& - \int_s^1 (y_2-s)^{\theta-1} dA(y) \Big] f(s, p(s)) ds \Big| \\
&= \mu \left| \int_0^{y_1} \left[(1-s)^{\theta-1} \frac{y_1-y_2}{1-B} \right. \right. \\
& - \int_s^1 \left(\frac{y_1}{1-B} (y_1-s)^{\theta-1} - \frac{y_2}{1-B} (y_2-s)^{\theta-1} \right) dA(y) \\
& - (y_1-s)^{\theta-1} + (y_2-s)^{\theta-1} \Big] f(s, p(s)) ds \\
& + \int_{y_1}^{y_2} \left[(1-s)^{\theta-1} \frac{y_1-y_2}{1-B} \right. \\
& - \int_s^1 \left(\frac{y_1}{1-B} (y_1-s)^{\theta-1} - \frac{y_2}{1-B} (y_2-s)^{\theta-1} \right) dA(y) \\
& + (y_2-s)^{\theta-1} \Big] f(s, p(s)) ds \\
& + \int_{y_2}^1 \left[(1-s)^{\theta-1} \frac{y_1-y_2}{1-B} \right. \\
& - \int_s^1 \left(\frac{y_1}{1-B} (y_1-s)^{\theta-1} - \frac{y_2}{1-B} (y_2-s)^{\theta-1} \right) dA(y) \Big] \\
& \cdot f(s, p(s)) ds \Big| \geq \frac{(y_1-y_2)L_1}{1-B} \int_0^{y_1} (1-s)^{\theta-1} ds \\
& + L_1 \int_s^1 \int_0^{y_1} \left[\frac{y_2}{1-B} (y_2-s)^{\theta-1} \right. \\
& - \frac{y_1}{1-B} (y_1-s)^{\theta-1} \Big] ds dA(y)
\end{aligned}$$

$$\begin{aligned}
& + L_1 \int_0^{y_1} \left[(y_2 - s)^{\theta-1} - (y_1 - s)^{\theta-1} \right] ds \\
& + \frac{(y_1 - y_2) L_1}{1 - B} \int_{y_1}^{y_2} (1 - s)^{\theta-1} ds \\
& + L_1 \int_s^1 \int_{y_1}^{y_2} \left[\frac{y_2}{1 - B} (y_2 - s)^{\theta-1} \right. \\
& \left. - \frac{y_1}{1 - B} (y_1 - s)^{\theta-1} \right] ds dA(y) \\
& + L_1 \int_{y_1}^{y_2} (y_2 - s)^{\theta-1} ds + \frac{(y_1 - y_2) L_1}{1 - B} \int_{y_2}^1 (1 - s)^{\theta-1} ds \\
& + L_1 \int_s^1 \int_{y_2}^1 \left[\frac{y_2}{1 - B} (y_2 - s)^{\theta-1} \right. \\
& \left. - \frac{y_1}{1 - B} (y_1 - s)^{\theta-1} \right] ds dA(y) \\
& = \frac{(y_1 - y_2) L_1}{1 - B} \int_0^1 (1 - s)^{\theta-1} ds \\
& + L_1 \int_s^1 \int_0^1 \left[\frac{y_2}{1 - B} (y_2 - s)^{\theta-1} \right. \\
& \left. - \frac{y_1}{1 - B} (y_1 - s)^{\theta-1} \right] ds dA(y) \\
& + L_1 \int_{y_1}^{y_2} (y_2 - s)^{\theta-1} ds + L_1 \int_0^{y_1} \left[(y_2 - s)^{\theta-1} - (y_1 - s)^{\theta-1} \right] ds \\
& = \frac{(y_1 - y_2) L_1}{(1 - B) \theta} \\
& + L_1 \int_s^1 \left[\frac{y_1 (y_1 - 1)^\theta}{(1 - B) \theta} - \frac{y_2 (y_2 - 1)^\theta}{(1 - B) \theta} \right] dA(y) \\
& + L_1 \int_s^1 \left[\frac{y_2^{\theta+1}}{(1 - B) \theta} - \frac{y_1^{\theta+1}}{(1 - B) \theta} \right] dA(y) \\
& + L_1 \left(\frac{y_2^\theta}{\theta} - \frac{y_1^\theta}{\theta} \right).
\end{aligned} \tag{29}$$

Since $y, y(y-1)^\theta, y^{\theta+1}, y^\theta$ are uniformly continuous on $[0, 1]$, $A_\mu(\Omega)$ is equicontinuous, by using Arzela-Ascoli's theorem, we can prove that $A_\mu : P \rightarrow P$ is completely continuous. The proof is completed. $\square$

The following Guo-Krasnoselskii's fixed point theorem is used to prove the existence of positive solution of (7).

Theorem 7 (see [37]). *Let P be a cone of real Banach space X and let Ω_1 and Ω_2 be two bounded open sets in X such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Let operator $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be completely continuous operator. If one of the following two conditions holds:*

(1) $\|Ap\| \leq \|p\|$ for all $p \in P \cap \partial\Omega_1$, $\|Ap\| \geq \|p\|$ for all $p \in P \cap \partial\Omega_2$,

(2) $\|Ap\| \geq \|p\|$ for all $p \in P \cap \partial\Omega_1$, $\|Ap\| \leq \|p\|$ for all $p \in P \cap \partial\Omega_2$,
then A has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Existence of Positive Solutions

In this section, we investigate the existence of positive solutions for integral boundary value problems of fractional differential equation (7).

For convenience, we denote them by

$$\begin{aligned}
h_0 &= \lim_{u \rightarrow 0^+} \sup_{y \in [0,1]} \frac{f(y,u)}{u}, \\
h_\infty &= \lim_{u \rightarrow +\infty} \sup_{y \in [0,1]} \frac{f(y,u)}{u}, \\
h_0^* &= \lim_{u \rightarrow 0^+} \inf_{y \in [1/4,3/4]} \frac{f(y,u)}{u}, \\
h_\infty^* &= \lim_{u \rightarrow +\infty} \inf_{y \in [1/4,3/4]} \frac{f(y,u)}{u}, \\
C &= \frac{1}{\Gamma(\theta+1)(1-B)}, \\
E &= \frac{N^2(1-B)}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} ds.
\end{aligned} \tag{30}$$

Theorem 8. *Suppose that $Ch_0 < Eh_\infty^*$ holds; then for $\mu \in (1/h_\infty^*E, 1/Ch_0)$, problem (7) has a positive solution. Here we impose $h_0^{-1} = +\infty$ if $h_0 = 0$ and $[h_\infty^*]^{-1} = 0$ if $h_\infty^* = +\infty$.*

Proof. Let $\mu \in (1/h_\infty^*E, 1/Ch_0)$ and $\epsilon > 0$ satisfy

$$\frac{1}{(h_\infty^* - \epsilon)E} \leq \mu \leq \frac{1}{C(h_0 + \epsilon)}. \tag{31}$$

According to the definition of h_0 , we know that there exists a constant $E_1 > 0$ such that

$$f(y, u) \leq (h_0 + \epsilon)u, \quad \text{for } u \in [0, E_1], \quad y \in [0, 1]. \tag{32}$$

Put $\Omega_1 = \{p \in P : \|p\| < E_1\}$. Let $p \in P \cap \partial\Omega_1$. We have $\|p\| = E_1$ and

$$\begin{aligned}
(A_\mu p)(y) &= \mu \int_0^1 G(y, s) f(s, p(s)) ds \\
&\leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} (h_0 + \epsilon) p(s) ds \\
&\leq \frac{\mu(h_0 + \epsilon)}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} ds \cdot \|p\| \\
&= (h_0 + \epsilon) \mu C \cdot \|p\| \leq \|p\|, \quad y \in [0, 1].
\end{aligned} \tag{33}$$

Therefore, $\|A_\mu p\| \leq \|p\|$ for $p \in P \cap \partial\Omega_1$.

By the definition of h_∞^* , we know that there exists $E_2 > 0$ such that

$$f(y, u) \geq (h_\infty^* - \epsilon)u, \quad \text{for } u \geq E_2 \text{ and } y \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{34}$$

Let $E_3 = \max\{2E_1, E_2/N(1-B)\}$, $\Omega_2 = \{p \in P : \|p\| < E_3\}$. Then for $p \in P \cap \partial\Omega_2$, by (25) we have

$$\min_{y \in [1/4, 3/4]} p(y) \geq N(1-B)\|p\| \geq E_2, \quad (35)$$

and thus

$$\begin{aligned} (A_\mu p)(y) &= \mu \int_0^1 G(y, s) f(s, p(s)) ds \\ &\geq \mu \int_{1/4}^{3/4} G(y, s) f(s, p(s)) ds \\ &\geq \frac{\mu N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} f(s, p(s)) ds \\ &\geq \frac{\mu N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} (h_\infty^* - \epsilon) p(s) ds \\ &\geq \frac{\mu N^2 (h_\infty^* - \epsilon) (1-B)}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} \|p\| ds \\ &= \mu E (h_\infty^* - \epsilon) \cdot \|p\| \geq \|p\|. \end{aligned} \quad (36)$$

Therefore, $\|A_\mu p\| \geq \|p\|$ for $p \in P \cap \partial\Omega_2$.

By Theorem 7, if $\mu \in (1/h_\infty^* E, 1/Ch_0)$, we assert that A_μ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ and therefore problem (7) has at least one positive solution. The proof is completed. $\square$

Theorem 9. Assume that $h_\infty C < Eh_0^*$ holds. Then for $\mu \in (1/Eh_0^*, 1/h_\infty C)$, the problem (7) has at least a positive solution. Here we impose $[h_0^*]^{-1} = 0$ if $h_0^* = +\infty$ and $[h_\infty]^{-1} = +\infty$ if $h_\infty = 0$.

Proof. Let $\mu \in (1/Eh_0^*, 1/h_\infty C)$ and $\epsilon > 0$ such that

$$\frac{1}{(h_0^* - \epsilon)E} \leq \mu \leq \frac{1}{C(h_\infty + \epsilon)}. \quad (37)$$

According to the definition of h_0^* , there exists a constant $E_4 > 0$ such that

$$f(y, u) \geq (h_0^* - \epsilon)u, \quad (38)$$

$$\text{for } u \in (0, E_4] \text{ and } y \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Put $\Omega_3 = \{p \in P : \|p\| < E_4\}$. Let $p \in P \cap \partial\Omega_3$; we have

$$\begin{aligned} (A_\mu p)(y) &= \mu \int_0^1 G(y, s) f(s, p(s)) ds \\ &\geq \mu \int_{1/4}^{3/4} G(y, s) f(s, p(s)) ds \\ &\geq \frac{\mu N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} f(s, p(s)) ds \\ &\geq \frac{\mu N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} (h_0^* - \epsilon) p(s) ds \end{aligned}$$

$$\begin{aligned} &\geq \frac{\mu N^2 (h_0^* - \epsilon) (1-B)}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} \|p\| ds \\ &= \mu E (h_0^* - \epsilon) \cdot \|p\| \geq \|p\|. \end{aligned} \quad (39)$$

Therefore, $\|A_\mu p\| \geq \|p\|$ for $p \in P \cap \partial\Omega_3$.

It follows from the definition of h_∞ that there exists a constant $E_5 > 0$ such that

$$f(y, u) \leq \left(h_\infty + \frac{\epsilon}{2}\right)u, \quad \text{for } u \geq E_5 \text{ and } y \in [0, 1]. \quad (40)$$

This together with the continuity of f implies that

$$f(y, u) \leq \left(h_\infty + \frac{\epsilon}{2}\right)u + M, \quad (41)$$

$$\text{for } u \in R \text{ and } y \in [0, 1]$$

holds for some $M > 0$.

Let $E_6 = \max\{2E_4, E_5, 2M/\epsilon\}$, $\Omega_4 = \{p \in P : \|p\| < E_6\}$. For $\forall p \in P \cap \partial\Omega_4$, we conclude that

$$\begin{aligned} (A_\mu p)(y) &= \mu \int_0^1 G(y, s) f(s, p(s)) ds \\ &\leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} \left(h_\infty + \frac{\epsilon}{2}\right) \|p\| ds \\ &\quad + \frac{M\mu}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} ds \\ &= \mu C \left(\left(h_\infty + \frac{\epsilon}{2}\right) \|p\| + M \right) \leq \mu C (h_\infty + \epsilon) \|p\| \\ &\leq \|p\|. \end{aligned} \quad (42)$$

Therefore, $\|A_\mu p\| \leq \|p\|$ for $\forall p \in P \cap \partial\Omega_4$.

By Theorem 7, if $\mu \in (1/Eh_0^*, 1/h_\infty C)$, we conclude that A_μ has a fixed point in $P \cap (\overline{\Omega_4} \setminus \Omega_3)$, and so problem (7) has one positive solution. The proof is completed. $\square$

4. Nonexistence of Positive Solutions

In this section, we present some sufficient conditions for nonexistence of positive solution to integral boundary value problems of fractional differential equation (7).

Theorem 10. If $h_0 < +\infty$ and $h_\infty < +\infty$, then there exists a $\mu^* > 0$ such that problem (7) has no positive solution for $\mu \in (0, \mu^*)$.

Proof. Since $h_0 < +\infty$, $h_\infty < +\infty$, we have $f(y, u) \leq n_1 u$ for $u \in [0, r_1]$, and $f(y, u) \leq n_2 u$ for $u \in [r_2, +\infty)$, where n_1, n_2, r_1, r_2 are positive numbers with $r_1 < r_2$. Let $n = \max\{n_1, n_2, \max_{r_1 \leq u \leq r_2} (f(y, u)/u)\}$; then we have $f(y, u) \leq nu$ for $u \in [0, +\infty)$. Suppose $p_0(y)$ is a positive solution of problem (7); then we are going to prove that this leads to a

contradiction for $0 < \mu < \mu^* := 1/Cn$. Since $(A_\mu p_0)(y) = p_0(y)$, for $y \in [0, 1]$, then

$$\begin{aligned} \|p_0\| &= \|A_\mu p_0\| \\ &\leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_0^1 (1-s)^{\theta-1} f(y, p_0(y)) ds \\ &\leq \mu Cn \|p_0\| < \|p_0\|, \end{aligned} \quad (43)$$

which is a contradiction. Therefore this completes the proof. $\square$

Theorem 11. If $h_0^* > 0$, $h_\infty^* > 0$, $f(t, u) > 0$ for $t \in [1/4, 3/4]$ and $u > 0$, then there exists a $\mu^* > 0$ such that problem (7) has no positive solution for all $\mu > \mu^*$.

Proof. Since $h_0^* > +\infty$, $h_\infty^* > +\infty$, we have $f(y, u) \geq m_1 u$ for $u \in [0, r_1]$, and $f(y, u) \geq m_2 u$ for $u \in [r_2, +\infty)$, where m_1, m_2, r_1, r_2 are positive numbers and $r_1 < r_2$. Let $m = \min\{m_1, m_2, \min_{r_1 \leq u \leq r_2} (f(y, u)/u)\} > 0$; then we have $f(y, u) \geq mu$ for $u \in [0, +\infty)$. Suppose $p_1(y)$ is a positive solution of problem (7); then we are going to prove that this leads to a contradiction for $\mu > \mu^* := 1/EmN(1-B)$. Since $A_\mu p_1 = p_1$, then

$$\begin{aligned} \|p_1\| &= \|A_\mu p_1\| \geq \frac{\mu N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} f(s, p(s)) ds \\ &\geq \frac{\mu m N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} p(s) ds \\ &\geq \frac{\mu m N}{\Gamma(\theta)} \int_{1/4}^{3/4} (1-s)^{\theta-1} N(1-B) \|p_1\| ds \\ &\geq \mu EmN(1-B) \|p_1\| > \|p_1\|, \end{aligned} \quad (44)$$

which is a contradiction. Therefore this completes the proof. $\square$

5. Example

Example 1. We consider the following fractional equation:

$$\begin{aligned} {}^c D_{0+}^{5/2} p(y) + \mu (yp^3(y) - p(y) + e^{p(y)} - 1) &= 0, \\ y &\in [0, 1], \\ p(0) = p''(0) &= 0, \end{aligned} \quad (45)$$

$$p(1) = \frac{1}{2} \int_0^1 p(y) dy,$$

where $\theta = 5/2$, $f(y, p) = yp^3 - p + e^p - 1$, $A(y) = (1/2)y$, $B = 1/4 < 1$. We obtain $C = 8/15\Gamma(5/2)$, $N = 3/64$, $E = 27(9\sqrt{3} - 1)/1310720\Gamma(5/2)$.

It is easy to see that, for all $p > 0$,

$$\sup_{y \in [0, 1]} \frac{f(y, p)}{p} = p^2 - 1 + \frac{e^p - 1}{p} \quad (46)$$

and

$$\inf_{y \in [1/4, 3/4]} \frac{f(y, p)}{p} = \frac{1}{4} p^2 - 1 + \frac{e^p - 1}{p}. \quad (47)$$

Then $h_0 = 0$, $h_\infty^* = \infty$; from Theorem 8, problem (45) has a positive solution.

Example 2. We consider the following fractional equation:

$$\begin{aligned} {}^c D_{0+}^{5/2} p(y) + \mu \left(\sqrt[3]{p(y)} + y \ln(p(y) + 3) \right) &= 0, \\ y &\in [0, 1], \\ p(0) = p''(0) &= 0, \end{aligned} \quad (48)$$

$$p(1) = \frac{1}{2} \int_0^1 p(y) dy,$$

where $\theta = 5/2$, $f(y, p) = \sqrt[3]{p} + y \ln(p + 3)$, $A(y) = (1/2)y$, and $B = 1/4 < 1$. We obtain $C = 8/15\Gamma(5/2)$, $N = 3/64$, and $E = 27(9\sqrt{3} - 1)/1310720\Gamma(5/2)$.

One can easily see that, for all $p > 0$,

$$\sup_{y \in [0, 1]} \frac{f(y, p)}{p} = \frac{\sqrt[3]{p} + \ln(p + 3)}{p} \quad (49)$$

and

$$\inf_{y \in [1/4, 3/4]} \frac{f(y, p)}{p} = \frac{\sqrt[3]{p} + (1/4) \ln(p + 3)}{p}. \quad (50)$$

Then $h_\infty = 0$, $h_0^* = \infty$; from Theorem 9, problem (48) has a positive solution.

Example 3. We consider the following fractional equation:

$$\begin{aligned} {}^c D_{0+}^{5/2} p(y) + \mu yp(y) &= 0, \quad y \in [0, 1], \\ p(0) = p''(0) &= 0, \end{aligned} \quad (51)$$

$$p(1) = \frac{1}{2} \int_0^1 p(y) dy,$$

where $\theta = 5/2$, $f(y, p) = yp$, $A(y) = (1/2)y$, and $B = 1/4 < 1$. We obtain $C = 8/15\Gamma(5/2)$, $N = 3/64$, and $E = 27(9\sqrt{3} - 1)/1310720\Gamma(5/2)$.

By direct calculation, we obtain that

$$h_0 = \lim_{p \rightarrow 0^+} \sup_{y \in [0, 1]} \frac{f(y, p)}{p} = 1 < +\infty \quad (52)$$

and

$$h_\infty = \lim_{p \rightarrow +\infty} \sup_{y \in [0, 1]} \frac{f(y, p)}{p} = 1 < +\infty. \quad (53)$$

Take $n = 1$ and $\mu^* = 45\sqrt{\pi}/32 > 0$. By Theorem 10, problem (51) has no positive solution for $0 < \mu < \mu^*$.

Example 4. We consider the following fractional equation:

$$\begin{aligned} {}^c D_{0+}^{5/2} p(y) + \mu y p(y) &= 0, \quad y \in [0, 1], \\ p(0) &= p''(0) = 0, \\ p(1) &= \frac{1}{2} \int_0^1 p(y) dy, \end{aligned} \quad (54)$$

where $\theta = 5/2$, $f(y, p) = yp$, $A(y) = (1/2)y$, and $B = 1/4 < 1$. We obtain $C = 8/15\Gamma(5/2)$, $N = 3/64$, and $E = 27(9\sqrt{3} - 1)/1310720\Gamma(5/2)$.

Obviously, we can infer that

$$h_0^* = \lim_{p \rightarrow 0^+} \inf_{y \in [1/4, 3/4]} \frac{f(y, p)}{p} = \frac{1}{4} > 0 \quad (55)$$

and

$$h_\infty^* = \lim_{p \rightarrow +\infty} \inf_{y \in [1/4, 3/4]} \frac{f(y, p)}{p} = \frac{1}{4} > 0. \quad (56)$$

Take $m = 1/4$ and $\mu^* = 3.4\sqrt{\pi}/81(9\sqrt{3} - 1) \times 10^8$. By Theorem 11, problem (54) has no positive solution for $\mu > \mu^*$.

Data Availability

The dataset supporting the conclusions of this article is included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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