Research Article

Ground State Solutions for Schrödinger Problems with Magnetic Fields and Hardy-Sobolev Critical Exponents

Min Liu¹ and Xiaorui Yue²

¹School of Sciences, Liaoning Shihua University, Fushun 113001, China

²College of Information Science & Technology, Hainan University, Haikou 570228, China

Correspondence should be addressed to Xiaorui Yue; yxr@hainu.edu.cn

Received 25 June 2018; Accepted 24 July 2018; Published 1 August 2018

Academic Editor: Mitsuru Sugimoto

Copyright © 2018 Min Liu and Xiaorui Yue. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A Schrödinger equation and system with magnetic fields and Hardy-Sobolev critical exponents are investigated in this paper, and, under proper conditions, the existence of ground state solutions to these two problems is given.

1. Introduction

In the present paper, we consider the following semilinear stationary Schrödinger equation and system with magnetic fields and Hardy-Sobolev critical exponents:

$$(-i\nabla + A)^2 u = \lambda \frac{|u|^{p-2} u}{|x|^s} + \mu \frac{|u|^{2^*(s_1)-2} u}{|x|^{s_1}}, \quad (1)$$

$$u \in H_A^1(\Omega),$$

$$u = 0, \quad \text{on } \partial\Omega,$$

$$(-i\nabla + A)^2 u = \lambda_1 \frac{|u|^{p-2} u}{|x|^s} + \mu_1 \frac{|u|^{2^*(s_1)-2} u}{|x|^{s_1}} + \frac{\alpha\gamma}{2^*(s_2)} \frac{|u|^{\alpha-2} u |v|^\beta}{|x|^{s_2}},$$

$$(-i\nabla + B)^2 v = \lambda_2 \frac{|v|^{p-2} v}{|x|^s} + \mu_2 \frac{|v|^{2^*(s_1)-2} v}{|x|^{s_1}} + \frac{\beta\gamma}{2^*(s_2)} \frac{|u|^\alpha |v|^{\beta-2} v}{|x|^{s_2}}, \quad (2)$$

$$u \in H_A^1(\Omega), \quad v \in H_B^1(\Omega), \quad u = v = 0, \quad \text{on } \partial\Omega,$$

where $u, v : \mathbb{R}^N \rightarrow \mathbb{C}, N \geq 4, A = (A_1, \dots, A_N), B = (B_1, \dots, B_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are magnetic potentials, $0 \leq s < 2, 0 < s_1, s_2 < 2, \lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2, \gamma > 0, \alpha, \beta > 1$ with $\alpha + \beta = 2^*(s_2) := 2(N - s_2)/(N - 2)$, and Ω is a smooth bounded domain containing the origin as a interior point. Set $-\Delta_A := (-i\nabla + A)^2, \nabla_A := \nabla + iA$, and

$$H_A^1(\Omega) := \{u \in L^2(\Omega) : |\nabla_A u| \in L^2(\Omega)\}. \quad (3)$$

Then $-\Delta_A u = -\Delta u - iu \operatorname{div} A - 2iA \cdot \nabla u + |A|^2 u$; $H_A^1(\Omega)$ is a Hilbert space obtained by the closure of $C_c^\infty(\Omega, \mathbb{C})$ with respect to scalar product:

$$\operatorname{Re} \left(\int_{\Omega} \nabla_A u \cdot \overline{\nabla_A v} \right), \quad (4)$$

where the bar denotes complex conjugation. Here and in the following, $\int \cdot$ means $\int \cdot dx$. We regard the range of function as $\mathbb{C}$, except for emphasizing, that is, $\mathbb{R}$. $L^p(\Omega, dx/|x|^s)$ denotes the space of L^p -integrable functions with respect to the measure $dx/|x|^s$, endowed with norm

$$|u|_{p,s} := \left(\int_{\Omega} \frac{|u|^p}{|x|^s} \right)^{1/p}. \quad (5)$$

For $\Omega = \mathbb{R}^N$, denote the $L^p(\Omega, dx/|x|^s)$ norm by

$$|u|_{p,s,\mathbb{R}^N} := \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^s} \right)^{1/p}. \quad (6)$$

Write $|u|_p := |u|_{p,0}$ and $|u|_{p,\mathbb{R}^N} := |u|_{p,0,\mathbb{R}^N}$ for simplicity.

The first existence results for this kind of problems with a nonsingular magnetic potential were established in [1]. Motivated by the seminal work [1], some papers dealt with the Schrödinger equations with nonsingular magnetic field, for example, [2–11] and references therein. For the singular magnetic potential, we refer to [12, 13]. There are a few works about the nonsingular magnetic problems with critical exponents, such as [9, 14, 15] for Sobolev critical exponent and [16] for Hardy critical exponent.

As far as we have known, there is no result about the problems of this type with Hardy-Sobolev critical exponents, especially for the system case. The current paper is mainly motivated by [15] and applies the least energy solution obtained in [17, 18] to show the existence of the ground state solution to the critical system (2) under proper conditions. Precisely, we get the following results.

Theorem 1. Assume that $A \in L_{loc}^N(\mathbb{R}^N, \mathbb{R}^N)$. If $\text{curl} A \equiv 0$ or A is continuous at 0, where the $\text{curl} A$ is the $N \times N$ skew-symmetric matrix with entries $a_{jk} = \partial_j A_k - \partial_k A_j$, then (1) has a nontrivial ground state solution.

Theorem 2. Assume that $A, B \in L_{loc}^N(\mathbb{R}^N, \mathbb{R}^N)$ and $\alpha, \beta < 2$. If $\text{curl} A \equiv 0 \equiv \text{curl} B$ or A and B are both continuous at 0, then (2) has a nontrivial ground state solution.

The corresponding energy functional $E : H_{A,B} \rightarrow \mathbb{R}$ of (2) is

$$\begin{aligned} E(u, v) = & \frac{1}{2} \|(u, v)\|_{H_{A,B}}^2 - \frac{1}{p} (\lambda_1 |u|_{p,s}^p + \lambda_2 |v|_{p,s}^p) \\ & - \frac{1}{2^*(s_1)} (\mu_1 |u|_{2^*(s_1),s_1}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1),s_1}^{2^*(s_1)}) \\ & - \frac{\gamma}{2^*(s_2)} \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}}, \end{aligned} \quad (7)$$

where $H_{A,B} := H_A^1(\Omega) \times H_B^1(\Omega)$, endowed with norm

$$\|(u, v)\|_{H_{A,B}}^2 := |\nabla_A u|_2^2 + |\nabla_B v|_2^2. \quad (8)$$

Define

$$\begin{aligned} \mathcal{M} := & \left\{ (u, v) \in H_{A,B} \setminus \{(0, 0)\} : \|(u, v)\|_{H_{A,B}}^2 = \lambda_1 |u|_{p,s}^p \right. \\ & + \lambda_2 |v|_{p,s}^p + \mu_1 |u|_{2^*(s_1),s_1}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1),s_1}^{2^*(s_1)} \\ & \left. + \gamma \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}} \right\} \end{aligned} \quad (9)$$

and $M := \inf_{(u,v) \in \mathcal{M}} E(u, v)$. By nontrivial solutions $(u, v) \in H_{A,B}$ of (2), we mean $u \neq 0, v \neq 0$. A solution of (2) is called a ground state solution if $(u, v) \in \mathcal{M}$ and $E(u, v) = M$. A ground state solution is semitrivial if it is of type $(u, 0)$ or $(0, v)$. Similar definitions are verified for single equation (1).

Remark 3. Although it has been studied that the existence of ground state solutions is heavily dependent on the type of equations without magnetic fields, the symmetric and decaying information about ground state solutions of equations with magnetic fields is not known. In this paper, we find that the latter equation and system also indeed have the ground state solutions provided suitable conditions.

2. Preliminaries

Consider

$$\begin{aligned} -\Delta u = & \mu_1 \frac{|u|^{2^*(s_1)-2} u}{|x|^{s_1}} + \frac{\alpha \gamma}{2^*(s_2)} \frac{|u|^{\alpha-2} u |v|^\beta}{|x|^{s_2}}, \\ -\Delta v = & \mu_2 \frac{|v|^{2^*(s_1)-2} v}{|x|^{s_1}} + \frac{\beta \gamma}{2^*(s_2)} \frac{|u|^\alpha |v|^{\beta-2} v}{|x|^{s_2}}, \end{aligned} \quad (10)$$

$$u, v \in D^{1,2}(\mathbb{R}^N),$$

and the corresponding energy functional $I : D \rightarrow \mathbb{R}$

$$\begin{aligned} I(u, v) = & \frac{1}{2} \|(u, v)\|_D^2 \\ & - \frac{1}{2^*(s_1)} (\mu_1 |u|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)}) \\ & - \frac{\gamma}{2^*(s_2)} \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}}, \end{aligned} \quad (11)$$

where $D := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, endowed with the norm

$$\|(u, v)\|_D^2 := |\nabla u|_{2,\mathbb{R}^N}^2 + |\nabla v|_{2,\mathbb{R}^N}^2. \quad (12)$$

Define

$$\begin{aligned} \mathcal{N} := & \left\{ (u, v) \in D \setminus \{(0, 0)\} : \|(u, v)\|_D^2 \right. \\ & = \mu_1 |u|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} \\ & \left. + \gamma \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}} \right\} \end{aligned} \quad (13)$$

and $M_0 := \inf_{(u,v) \in \mathcal{N}} I(u, v)$. Then, by [18], we see that, under the condition $\alpha, \beta < 2$, (10) has a positive ground state solution (U, V) , where U and V are radial symmetric decreasing with the following decay information:

$$\begin{aligned} U(x) + V(x) & \leq C(1 + |x|^{2-N}), \\ |\nabla U(x)| + |\nabla V(x)| & \leq C(1 + |x|^{1-N}). \end{aligned} \quad (14)$$

That is, M_0 is attained by (U, V) . Define

$$\begin{aligned} (U_\varepsilon(x), V_\varepsilon(x)) = & \left(\varepsilon^{-(N-2)/2} U\left(\frac{x}{\varepsilon}\right), \varepsilon^{-(N-2)/2} V\left(\frac{x}{\varepsilon}\right) \right). \end{aligned} \quad (15)$$

As proved in [1, 19], for any $u \in H_A^1(\Omega)$, the following *diamagnetic inequality* holds pointwise for almost every $x \in \Omega$:

$$|\nabla |u|| = \left| \operatorname{Re} \left(\nabla u \frac{\bar{u}}{|u|} \right) \right| = \left| \operatorname{Re} \left((\nabla u + iAu) \frac{\bar{u}}{|u|} \right) \right| \leq |\nabla_A u|. \quad (16)$$

Then, for $u \in H_A^1(\Omega)$, we see that $|u|$ belongs to the usual Sobolev space $H_0^1(\Omega)$. Moreover, we have the following.

Lemma 4 (see [20]). *The embedding $H_A^1(\Omega) \hookrightarrow L^p(\Omega, dx/|x|^s)$ is continuous for $1 \leq p \leq 2^*(s)$, and it is compact for $1 \leq p < 2^*(s)$, where $0 \leq s < 2$.*

Lemma 5. *If $u_n \rightharpoonup 0$ weakly in $H_A^1(\Omega)$, where $A \in L_{loc}^N(\mathbb{R}^N, \mathbb{R}^N)$, then $\int_{\Omega} |\nabla_A u_n|^2 = \int_{\Omega} |\nabla u_n|^2 + o(1)$.*

Proof. Obviously, $|u_n| \in H_0^1(\Omega)$. Lemma 4 ensures that $u_n \rightharpoonup 0$ weakly in $L^{2^*}(\Omega)$ and $u_n \rightarrow 0$ a.e. in Ω . Then, $u_n^2 \rightharpoonup 0$ weakly in $L^{2^*/2}(\Omega)$. Since $|A|^2 \in L_{loc}^{N/2}(\mathbb{R}^N) = (L_{loc}^{2^*/2}(\mathbb{R}^N))'$, the dual space of $L_{loc}^{2^*/2}(\mathbb{R}^N)$, we see that

$$\int_{\Omega} |A|^2 |u_n|^2 = \langle |A|^2, u_n^2 \rangle \rightarrow 0, \quad (17)$$

where the duality product is taken with respect to $L^{N/2}(\Omega)$ and $L^{2^*/2}(\Omega)$. It is easy to see that

$$\begin{aligned} \int_{\Omega} |\nabla_A u_n|^2 &= \int_{\Omega} |\nabla u_n|^2 + \int_{\Omega} |A|^2 |u_n|^2 \\ &\quad + 2\operatorname{Re} \int_{\Omega} iAu_n \overline{\nabla u_n}. \end{aligned} \quad (18)$$

By Hölder inequality and (17), we have

$$\left| \operatorname{Re} \int_{\Omega} iAu_n \overline{\nabla u_n} \right| \leq \left(\int_{\Omega} |\nabla u_n|^2 \right)^{1/2} \left(\int_{\Omega} |A|^2 |u_n|^2 \right)^{1/2} \rightarrow 0. \quad (19)$$

Thus, the proof is completed by (17), (18), and (19). $\square$

Remark 6. It is easy to see that $\mathcal{M}$ is closed and bounded away from 0, for any $(u, v) \in H_{A,B} \setminus \{(0, 0)\}$, there exists unique $t_{u,v} > 0$ such that $(t_{u,v}u, t_{u,v}v) \in \mathcal{M}$. $M > 0$ follows from Lemma 4. It is standard to see that any (PS) sequence of E with constraint $\mathcal{M}$ is a (PS) sequence of E without constraints and every (PS) sequence of E is bounded in $H_{A,B}$. Define

$$\begin{aligned} \mathbb{M} &= \inf_{(u,v) \in H_{A,B} \setminus \{(0,0)\}} \sup_{t>0} E(tu, tv), \\ M &= \inf_{h \in \Gamma} \max_{t \in [0,1]} E(h(t)), \end{aligned} \quad (20)$$

where $\Gamma := \{h \in C([0, 1], H_{A,B}) : h(0) = (0, 0), I(h(1)) < 0\}$. Then it can be proved that $M = \mathbb{M} = M_0$.

3. Ground States

In this section, we firstly suppose that Theorem 1 holds, and then we show Theorem 2 below. In fact, the proof of Theorem 1 is similar to and easier than that of Theorem 2 by the application of the ground state solutions to $-\Delta u = |u|^{2^*(s_1)-2}u/|x|^{s_1}$ (cf. [17]), so we shall show Theorem 1 briefly as a remark in the end of this section.

By Theorem 1, we see that

$$-\Delta_A u = \lambda_1 \frac{|u|^{p-2}u}{|x|^s} + \mu_1 \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}}, \quad u \in H_0^1(\Omega) \quad (21)$$

and

$$-\Delta_B u = \lambda_2 \frac{|u|^{p-2}u}{|x|^s} + \mu_2 \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}}, \quad u \in H_0^1(\Omega) \quad (22)$$

have ground state solutions u_{μ_1} and u_{μ_2} with energies

$$\begin{aligned} M_{\mu_1} &:= \frac{1}{2} \int_{\Omega} |\nabla_A u_{\mu_1}|^2 - \frac{\lambda_1}{p} \int_{\Omega} \frac{|u_{\mu_1}|^p}{|x|^s} \\ &\quad - \frac{\mu_1}{2^*(s_1)} \int_{\Omega} \frac{|u_{\mu_1}|^{2^*(s_1)}}{|x|^{s_1}} \end{aligned} \quad (23)$$

and

$$\begin{aligned} M_{\mu_2} &:= \frac{1}{2} \int_{\Omega} |\nabla_B u_{\mu_2}|^2 - \frac{\lambda_2}{p} \int_{\Omega} \frac{|u_{\mu_2}|^p}{|x|^s} \\ &\quad - \frac{\mu_2}{2^*(s_1)} \int_{\Omega} \frac{|u_{\mu_2}|^{2^*(s_1)}}{|x|^{s_1}} \end{aligned} \quad (24)$$

Then, we have the following.

Lemma 7. *Assume that $A, B \in L_{loc}^N(\mathbb{R}^N, \mathbb{R}^N)$ and $\alpha, \beta < 2$. If $\operatorname{curl} A \equiv 0 \equiv \operatorname{curl} B$ or A and B are continuous at 0, then $M < \min\{M_{\mu_1}, M_{\mu_2}, M_0\}$.*

Proof. Firstly, we prove that $M < M_0$. There exists $\rho > 0$ satisfying $B_{2\rho} := \{x : |x| \leq 2\rho\} \subset \Omega$.

(i) If $\operatorname{curl} A \equiv 0 \equiv \operatorname{curl} B$ holds, then by [21], there exist $\varphi, \psi \in W_{loc}^{1,N}(\mathbb{R}^N, \mathbb{R})$ such that $\nabla \varphi = A, \nabla \psi = B$. Let

$$(u_{\varepsilon}(x), v_{\varepsilon}(x)) = (\phi U_{\varepsilon}(x) e^{-i\varphi(x)}, \phi V_{\varepsilon}(x) e^{-i\psi(x)}), \quad (25)$$

where $(U_\varepsilon, V_\varepsilon)$ is defined by (15) and $\phi \in C_0^1(B_{2\rho})$ is a cut-off function with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ for $|x| \leq \rho$. By direct computation, we have

$$\begin{aligned} \int_{\Omega} |\nabla_A u_\varepsilon|^2 &= \int_{\Omega} |\nabla (\phi U_\varepsilon)|^2 \\ &\leq \int_{\mathbb{R}^N} |\nabla U|^2 + O(\varepsilon^{N-2}), \end{aligned} \quad (26)$$

$$\int_{\Omega} \frac{|u_\varepsilon|^{2^*(s_1)}}{|x|^{s_1}} \geq \int_{\mathbb{R}^N} \frac{U^{2^*(s_1)}}{|x|^{s_1}} + O(\varepsilon^{N-s_1}), \quad (27)$$

$$\int_{\Omega} \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^\beta}{|x|^{s_2}} \geq \int_{\mathbb{R}^N} \frac{U^\alpha V^\beta}{|x|^{s_2}} + O(\varepsilon^{N-s_2}), \quad (28)$$

$$\int_{\Omega} \frac{|u_\varepsilon|^p}{|x|^s} \geq C\varepsilon^{N-(N-2)p/2-s} + O(\varepsilon^{(N-2)p/2}), \quad (29)$$

$$\begin{aligned} \int_{\Omega} |\nabla_B v_\varepsilon|^2 &= \int_{\Omega} |\nabla (\phi V_\varepsilon)|^2 \\ &\leq \int_{\mathbb{R}^N} |\nabla V|^2 + O(\varepsilon^{N-2}), \end{aligned} \quad (30)$$

$$\int_{\Omega} \frac{|v_\varepsilon|^{2^*(s_1)}}{|x|^{s_1}} \geq \int_{\mathbb{R}^N} \frac{V^{2^*(s_1)}}{|x|^{s_1}} + O(\varepsilon^{N-s_1}), \quad (31)$$

$$\int_{\Omega} \frac{|v_\varepsilon|^p}{|x|^s} \geq C\varepsilon^{N-(N-2)p/2-s} + O(\varepsilon^{(N-2)p/2}). \quad (32)$$

By (26)–(32), for any $t > 0$, we deduce that

$$\begin{aligned} E(tu_\varepsilon, tv_\varepsilon) &= \frac{t^2}{2} \|(u_\varepsilon, v_\varepsilon)\|_{H_{A,B}}^2 - \frac{t^p}{p} (\lambda_1 |u_\varepsilon|_{p,s}^p + \lambda_2 |v_\varepsilon|_{p,s}^p) \\ &\quad - \frac{t^{2^*(s_1)}}{2^*(s_1)} (\mu_1 |u_\varepsilon|_{2^*(s_1),s_1}^{2^*(s_1)} + \mu_2 |v_\varepsilon|_{2^*(s_1),s_1}^{2^*(s_1)}) \\ &\quad - \frac{t^{2^*(s_2)} \gamma}{2^*(s_2)} \int_{\Omega} \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^\beta}{|x|^{s_2}} \\ &\leq \frac{t^2}{2} A_\varepsilon - \frac{t^p}{p} B_\varepsilon - \frac{t^{2^*(s_1)}}{2^*(s_1)} C_\varepsilon - \frac{t^{2^*(s_2)}}{2^*(s_2)} D_\varepsilon \\ &\leq \frac{t(\varepsilon)^2}{2} A_\varepsilon - \frac{t(\varepsilon)^p}{p} B_\varepsilon - \frac{t(\varepsilon)^{2^*(s_1)}}{2^*(s_1)} C_\varepsilon \\ &\quad - \frac{t(\varepsilon)^{2^*(s_2)}}{2^*(s_2)} D_\varepsilon, \end{aligned} \quad (33)$$

where

$$\begin{aligned} A_\varepsilon &= \|(U, V)\|_D^2 + O(\varepsilon^{N-2}), \\ B_\varepsilon &= d\varepsilon^{N-(N-2)p/2-s} + O(\varepsilon^{(N-2)p/2}), \\ C_\varepsilon &= \mu_1 |U|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} + \mu_2 |V|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} + O(\varepsilon^{N-s_1}), \\ D_\varepsilon &= \gamma \int_{\mathbb{R}^N} \frac{U^\alpha V^\beta}{|x|^{s_2}} + O(\varepsilon^{N-s_2}), \end{aligned} \quad (34)$$

d is a positive constant, and $t(\varepsilon)$ solves

$$A_\varepsilon - t(\varepsilon)^{p-2} B_\varepsilon - t(\varepsilon)^{2^*(s_1)-2} C_\varepsilon - t(\varepsilon)^{2^*(s_2)-2} D_\varepsilon = 0. \quad (35)$$

Then, (35) yields that $t'(\varepsilon) = P(\varepsilon)/Q(\varepsilon)$, where

$$\begin{aligned} P(\varepsilon) &= (O(\varepsilon^{N-3})) \\ &\quad - t^{p-2} \left(d \left(N - \frac{(N-2)p}{2} - s \right) \varepsilon^{N-(N-2)p/2-s-1} \right. \\ &\quad \left. + O(\varepsilon^{(N-2)p/2-1}) \right) - t^{2^*(s_1)-2} O(\varepsilon^{N-s_1-1}) \\ &\quad - t^{2^*(s_2)-2} O(\varepsilon^{N-s_2-1}), \end{aligned} \quad (36)$$

$$\begin{aligned} Q(\varepsilon) &= (p-2)t^{p-3}B_\varepsilon + (2^*(s_1)-2)t^{2^*(s_1)-3}C_\varepsilon \\ &\quad + (2^*(s_2)-2)t^{2^*(s_2)-3}D_\varepsilon. \end{aligned}$$

Since $t(0) = 1$, we see that as $\varepsilon \rightarrow 0$,

$$\begin{aligned} P(\varepsilon) &= -d \left(N - \frac{(N-2)p}{2} - s \right) \varepsilon^{N-(N-2)p/2-s-1} (1 \\ &\quad + o(1)), \\ Q(\varepsilon) &= \left((2^*(s_1)-2) \right. \\ &\quad \cdot (\mu_1 |U|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)} + \mu_2 |V|_{2^*(s_1),s_1,\mathbb{R}^N}^{2^*(s_1)}) \\ &\quad \left. + (2^*(s_2)-2) \gamma \int_{\mathbb{R}^N} \frac{U^\alpha V^\beta}{|x|^{s_2}} \right) (1 + o(1)). \end{aligned} \quad (37)$$

Setting $\kappa = N - (N - 2)p/2 - s$ and

$$\Lambda = \frac{d}{(2^*(s_1) - 2) \left(\mu_1 |U|_{2^*(s_1), s_1, \mathbb{R}^N}^{2^*(s_1)} + \mu_2 |V|_{2^*(s_1), s_1, \mathbb{R}^N}^{2^*(s_1)} \right) + (2^*(s_2) - 2) \gamma \int_{\mathbb{R}^N} (U^\alpha V^\beta / |x|^{s_2})}, \quad (38)$$

we have

$$\begin{aligned} t'(\varepsilon) &= -\Lambda \kappa \varepsilon^{\kappa-1} (1 + o(1)), \\ t(\varepsilon) &= 1 - \Lambda \varepsilon^\kappa (1 + o(1)), \\ t(\varepsilon)^2 &= 1 - 2\Lambda \varepsilon^\kappa (1 + o(1)), \\ t(\varepsilon)^p &= 1 - p\Lambda \varepsilon^\kappa (1 + o(1)), \\ t(\varepsilon)^{2^*(s_1)} &= 1 - 2^*(s_1) \Lambda \varepsilon^\kappa (1 + o(1)), \\ t(\varepsilon)^{2^*(s_2)} &= 1 - 2^*(s_2) \Lambda \varepsilon^\kappa (1 + o(1)). \end{aligned} \quad (39)$$

Then, by (33), we deduce that

$$\begin{aligned} E(tu_\varepsilon, tv_\varepsilon) &\leq \frac{1 - 2\Lambda \varepsilon^\kappa (1 + o(1))}{2} A_\varepsilon \\ &\quad - \frac{1 - p\Lambda \varepsilon^\kappa (1 + o(1))}{p} B_\varepsilon \\ &\quad - \frac{1 - 2^*(s_1) \Lambda \varepsilon^\kappa (1 + o(1))}{2^*(s_1)} C_\varepsilon \\ &\quad - \frac{1 - 2^*(s_2) \Lambda \varepsilon^\kappa (1 + o(1))}{2^*(s_2)} D_\varepsilon \\ &= \frac{1}{2} \|(U, V)\|_D^2 \\ &\quad - \frac{1}{2^*(s_1)} \left(\mu_1 |U|_{2^*(s_1), s_1, \mathbb{R}^N}^{2^*(s_1)} + \mu_2 |V|_{2^*(s_1), s_1, \mathbb{R}^N}^{2^*(s_1)} \right) \\ &\quad - \frac{\gamma}{2^*(s_2)} \int_{\mathbb{R}^N} \frac{U^\alpha V^\beta}{|x|^{s_2}} - \frac{d\varepsilon^\kappa}{p} (1 + o(1)) < M_0, \end{aligned} \quad (40)$$

which implies that

$$M \leq \sup_{t>0} I(tu_\varepsilon, tv_\varepsilon) < M_0. \quad (41)$$

(ii) If A and B are continuous at 0, then setting

$$\begin{aligned} \theta(x) &:= -\sum_{j=1}^N A_j(0) x_j, \\ \vartheta(x) &:= -\sum_{j=1}^N B_j(0) x_j, \end{aligned} \quad (42)$$

we have

$$\begin{aligned} \nabla \theta(x) &= (-A_1(0), \dots, -A_N(0)) = -A(0), \\ \nabla \vartheta(x) &= (-B_1(0), \dots, -B_N(0)) = -B(0), \end{aligned} \quad (43)$$

which imply that $(\nabla \theta + A)(0) = 0$ and $(\nabla \vartheta + B)(0) = 0$. Then, there exists $\delta > 0$ such that

$$\begin{aligned} |(\nabla \theta + A)(x)|^2 &\leq m, \\ |(\nabla \vartheta + B)(x)|^2 &\leq m, \\ \forall |x| &< \delta, \end{aligned} \quad (44)$$

where m is a positive number. Let $2r := \min\{\delta, 2\rho\}$ and

$$\begin{aligned} (u_\varepsilon(x), v_\varepsilon(x)) &= (\phi(x) U_\varepsilon(x) e^{i\theta(x)}, \phi(x) V_\varepsilon(x) e^{i\vartheta(x)}), \end{aligned} \quad (45)$$

where $\phi \in C_0^1(B_{2r})$ is a cut-off function such that $\phi(x) = 1$ in B_r and $(U_\varepsilon, V_\varepsilon)$ is defined by (15). By (44), we deduce that

$$\begin{aligned} \int_{\Omega} (|\nabla_A u_\varepsilon|^2 + |\nabla_B v_\varepsilon|^2) &= \int_{\Omega} (|\nabla(\phi U_\varepsilon)|^2 \\ &\quad + \phi^2 U_\varepsilon^2 |\nabla \theta + A|^2 + |\nabla(\phi V_\varepsilon)|^2 + \phi^2 V_\varepsilon^2 |\nabla \vartheta + B|^2) \\ &\leq \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2) + O(\varepsilon^{N-2}) \\ &\quad + m \int_{B_{2r}} (\phi^2 (U_\varepsilon^2 + V_\varepsilon^2)) \leq \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2) \\ &\quad + O(\varepsilon^2). \end{aligned} \quad (46)$$

Hence, replacing A_ε with $\|(U, V)\|_D^2 + O(\varepsilon^2)$ and following the previous proof for case $\text{curl} A \equiv 0 \equiv \text{curl} B$, we may prove that $M < M_0$ with minor modifications.

Next to prove $M < M_{\mu_1}$, recalling (9), choose $t(\varepsilon) := t_{u_{\mu_1}, \varepsilon u_{\mu_1}}$ satisfying

$$\begin{aligned} |\nabla_A u_{\mu_1}|^2 + \varepsilon^2 |\nabla_B u_{\mu_1}|^2 &= t(\varepsilon)^{p-2} (\lambda_1 + \lambda_2 |\varepsilon|^p) |u_{\mu_1}|_{p,s}^p \\ &\quad + t(\varepsilon)^{2^*(s_1)-2} \left(\mu_1 + \mu_2 |\varepsilon|^{2^*(s_1)} \right) |u_{\mu_1}|_{2^*(s_1), s_1}^{2^*(s_1)} \\ &\quad + t(\varepsilon)^{2^*(s_2)-2} \gamma |u_{\mu_1}|_{2^*(s_2), s_2}^{2^*(s_2)} |\varepsilon|^\beta. \end{aligned} \quad (47)$$

Then, $(t(\epsilon)u_{\mu_1}, t(\epsilon)u_{\mu_1}) \in \mathcal{M}$. Noting that $|\nabla_A u_{\mu_1}|^2 = \lambda_1 |u_{\mu_1}|_{p,s}^p + \mu_1 |u_{\mu_1}|_{2^*(s_1),s}^{2^*(s_1)}$ and $t(0) = 1$, we deduce that

$$\begin{aligned} t'(\epsilon) &= -\Lambda_1 \beta |\epsilon|^{\beta-2} \epsilon (1 + o(1)), \\ t(\epsilon) &= 1 - \Lambda_1 |\epsilon|^\beta (1 + o(1)), \\ t(\epsilon)^p &= 1 - p\Lambda_1 |\epsilon|^\beta (1 + o(1)), \\ t(\epsilon)^{2^*(s_1)} &= 1 - 2^*(s_1) \Lambda_1 |\epsilon|^\beta (1 + o(1)), \\ t(\epsilon)^{2^*(s_2)} &= 1 - 2^*(s_2) \Lambda_1 |\epsilon|^\beta (1 + o(1)), \end{aligned} \quad (48)$$

where

$$\begin{aligned} \Lambda_1 &= \frac{\gamma |u_{\mu_1}|_{2^*(s_2),s_2}^{2^*(s_2)}}{(p-2)\lambda_1 |u_{\mu_1}|_{p,s}^p + (2^*(s_1)-2)\mu_1 |u_{\mu_1}|_{2^*(s_1),s_1}^{2^*(s_1)}}. \end{aligned} \quad (49)$$

Then,

$$\begin{aligned} M &\leq I(t(\epsilon)u_{\mu_1}, t(\epsilon)u_{\mu_1}) = \left(\frac{1}{2} - \frac{1}{p}\right)(\lambda_1 + \lambda_2 |\epsilon|^p) \\ &\quad \cdot |u_{\mu_1}|_{p,s}^p t(\epsilon)^p + \left(\frac{1}{2} - \frac{1}{2^*(s_1)}\right)(\mu_1 + \mu_2 |\epsilon|^{2^*(s_1)}) \\ &\quad \cdot |u_{\mu_1}|_{2^*(s_1),s_1}^{2^*(s_1)} t(\epsilon)^{2^*(s_1)} + \left(\frac{1}{2} - \frac{1}{2^*(s_2)}\right)\gamma |\epsilon|^\beta \\ &\quad \cdot |u_{\mu_1}|_{2^*(s_2),s_2}^{2^*(s_2)} t(\epsilon)^{2^*(s_2)} = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1 |u_{\mu_1}|_{p,s}^p \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^*(s_1)}\right)\mu_1 |u_{\mu_1}|_{2^*(s_1),s_1}^{2^*(s_1)} \\ &\quad - \left[\left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1 |u_{\mu_1}|_{p,s}^p p\Lambda_1 \right. \\ &\quad \left. + \left(\frac{1}{2} - \frac{1}{2^*(s_2)}\right)\mu_1 |u_{\mu_1}|_{2^*(s_1),s_1}^{2^*(s_1)} 2^*(s_1) \Lambda_1 \right. \\ &\quad \left. - \left(\frac{1}{2} - \frac{1}{2^*(s_2)}\right)\gamma |u_{\mu_1}|_{2^*(s_2),s_2}^{2^*(s_2)}\right] |\epsilon|^\beta + o(|\epsilon|^\beta) \\ &< M_{\mu_1} - \left(\frac{1}{2} + \frac{1}{2^*(s_2)}\right)\gamma |u_{\mu_1}|_{2^*(s_2),s_2}^{2^*(s_2)} |\epsilon|^\beta \\ &\quad + o(|\epsilon|^\beta) < M_{\mu_1} \quad \text{for } |\epsilon| \text{ small enough.} \end{aligned} \quad (50)$$

Similarly, $M < M_{\mu_2}$. $\square$

Proof of Theorem 2. Note that the functional E has a mountain pass structure. It follows from the mountain pass theorem (see [22]) that there exists $\{(u_n, v_n)\} \subset H_{A,B}$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} E(u_n, v_n) &= M, \\ \lim_{n \rightarrow \infty} E'(u_n, v_n) &= 0. \end{aligned} \quad (51)$$

By Remark 6, we see that $\{(u_n, v_n)\}$ is bounded in $H_{A,B}$. By Lemma 4, we may assume that

$$u_n \rightharpoonup u \quad \text{weakly in } H_A^1(\Omega),$$

$$v_n \rightharpoonup v \quad \text{weakly in } H_B^1(\Omega),$$

$$u_n \rightarrow u,$$

$$v_n \rightarrow v$$

strongly in $L^2(\Omega)$,

$$u_n \rightharpoonup u,$$

$$v_n \rightharpoonup v$$

$$\text{weakly in } L^{2^*(s_1)}\left(\Omega, \frac{dx}{|x|^{s_1}}\right), \quad (52)$$

$$|u_n|^{q-1} u_n \rightharpoonup |u|^{q-1} u,$$

$$|v_n|^{q-1} v_n \rightharpoonup |v|^{q-1} v$$

$$\text{weakly in } L^{2^*(s_1)/q}\left(\Omega, \frac{dx}{|x|^{s_1}}\right),$$

$$|u_n|^{\alpha-2} u_n |v_n|^\beta \rightharpoonup |u|^{\alpha-2} u |v|^\beta,$$

$$|u_n|^\alpha |v_n|^{\beta-2} v_n \rightharpoonup |u|^\alpha |v|^{\beta-2} v$$

$$\text{weakly in } L^{2^*(s_2)/(2^*(s_2)-1)}\left(\Omega, \frac{dx}{|x|^{s_2}}\right),$$

where $q \in (1, 2^*(s_1))$. From

$$\begin{aligned} &\int_{\Omega} (\nabla_A u_n \overline{\nabla_A \varphi} + \nabla_B v_n \overline{\nabla_B \psi}) - \lambda_1 \int_{\Omega} \frac{|u_n|^{p-2} u_n \overline{\varphi}}{|x|^s} \\ &\quad - \lambda_2 \int_{\Omega} \frac{|v_n|^{p-2} v_n \overline{\psi}}{|x|^s} - \mu_1 \int_{\Omega} \frac{|u_n|^{2^*(s_1)-2} u_n \overline{\varphi}}{|x|^{s_1}} \\ &\quad - \mu_2 \int_{\Omega} \frac{|v_n|^{2^*(s_1)-2} v_n \overline{\psi}}{|x|^{s_1}} \\ &\quad - \frac{\alpha\gamma}{2^*(s_2)} \int_{\Omega} \frac{|u_n|^{\alpha-2} u_n \overline{\varphi} |v_n|^\beta}{|x|^{s_2}} \\ &\quad - \frac{\beta\gamma}{2^*(s_2)} \int_{\Omega} \frac{|u_n|^\alpha |v_n|^{\beta-2} v_n \overline{\psi}}{|x|^{s_2}} = o(1), \end{aligned} \quad (53)$$

$$\forall (\varphi, \psi) \in H,$$

we get that

$$\begin{aligned}
& \int_{\Omega} (\nabla_A u \overline{\nabla_A \varphi} + \nabla_B v \overline{\nabla_B \psi}) - \lambda_1 \int_{\Omega} \frac{|u|^{p-2} u \overline{\varphi}}{|x|^s} \\
& - \lambda_2 \int_{\Omega} \frac{|v|^{p-2} v \overline{\psi}}{|x|^s} - \mu_1 \int_{\Omega} \frac{|u|^{2^*(s_1)-2} u \overline{\varphi}}{|x|^{s_1}} \\
& - \mu_2 \int_{\Omega} \frac{|v|^{2^*(s_1)-2} v \overline{\psi}}{|x|^{s_1}} \\
& - \frac{\alpha \gamma}{2^*(s_2)} \int_{\Omega} \frac{|u|^{\alpha-2} u \overline{\varphi} |v|^{\beta}}{|x|^{s_2}} \\
& - \frac{\beta \gamma}{2^*(s_2)} \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta-2} v \overline{\psi}}{|x|^{s_2}} = 0,
\end{aligned} \tag{54}$$

i.e., $E'(u, v) = 0$ in $H'_{A,B}$, the dual space of $H_{A,B}$. Setting $\omega_n = u_n - u$ and $\sigma_n = v_n - v$, then $\omega_n \rightharpoonup 0$ in $H_B^1(\Omega)$ and $\sigma_n \rightharpoonup 0$ in $H_B^1(\Omega)$. Thus, $\omega_n \rightarrow 0, \sigma_n \rightarrow 0$ strongly in $L^2(\Omega)$, $\omega_n \rightarrow 0, \sigma_n \rightarrow 0$ a.e. in Ω , up to a subsequence. Brezis-Lieb Lemma guarantees that

$$\begin{aligned}
|u_n|_{2^*(s_1), s_1}^{2^*(s_1)} &= |u|_{2^*(s_1), s_1}^{2^*(s_1)} + |\omega_n|_{2^*(s_1), s_1}^{2^*(s_1)} + o(1), \\
|v_n|_{2^*(s_1), s_1}^{2^*(s_1)} &= |v|_{2^*(s_1), s_1}^{2^*(s_1)} + |\sigma_n|_{2^*(s_1), s_1}^{2^*(s_1)} + o(1), \\
\int_{\Omega} \frac{|u_n|^{\alpha} |v_n|^{\beta}}{|x|^{s_2}} &= \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta}}{|x|^{s_2}} + \int_{\Omega} \frac{|\omega_n|^{\alpha} |\sigma_n|^{\beta}}{|x|^{s_2}} + o(1).
\end{aligned} \tag{55}$$

Noting that $E'(u_n, v_n)(u_n, 0) = o(1)$, $E'(u_n, v_n)(0, v_n) = o(1)$, and $E'(u, v)(u, v) = 0$, by (55) and Lemma 5, we deduce that

$$\begin{aligned}
& \int_{\Omega} |\nabla \omega_n|^2 - \int_{\Omega} \left(\mu_1 \frac{|\omega_n|^{2^*(s_1)}}{|x|^{s_1}} + \frac{\alpha \gamma}{2^*(s_2)} \frac{|\omega_n|^{\alpha} |\sigma_n|^{\beta}}{|x|^{s_2}} \right) \\
& = o(1), \\
& \int_{\Omega} |\nabla \sigma_n|^2 - \int_{\Omega} \left(\mu_2 \frac{|\sigma_n|^{2^*(s_1)}}{|x|^{s_1}} + \frac{\beta \gamma}{2^*(s_2)} \frac{|\omega_n|^{\alpha} |\sigma_n|^{\beta}}{|x|^{s_2}} \right) \\
& = o(1),
\end{aligned} \tag{56}$$

and

$$I(\omega_n, \sigma_n) = E(u_n, v_n) - E(u, v) + o(1). \tag{57}$$

Assume that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{\Omega} \mu_1 \frac{|\omega_n|^{2^*(s_1)}}{|x|^{s_1}} &= a_1, \\
\lim_{n \rightarrow \infty} \int_{\Omega} \mu_2 \frac{|\sigma_n|^{2^*(s_1)}}{|x|^{s_1}} &= a_2, \\
\lim_{n \rightarrow \infty} \int_{\Omega} \gamma \frac{|\omega_n|^{\alpha} |\sigma_n|^{\beta}}{|x|^{s_2}} &= b,
\end{aligned} \tag{58}$$

up to a subsequence. By (56), we see that

$$\begin{aligned}
I(\omega_n, \sigma_n) &= \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) (a_1 + a_2) \\
&+ \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) b + o(1).
\end{aligned} \tag{59}$$

Letting $n \rightarrow \infty$, we have

$$\begin{aligned}
M &= \lim_{n \rightarrow \infty} E(u_n, v_n) \\
&= \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) (a_1 + a_2) + \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) b \\
&+ E(u, v) \geq E(u, v) \geq 0.
\end{aligned} \tag{60}$$

Case 1 ($u \equiv 0, v \equiv 0$). By $M > 0$ and (60), we see that $a_1 + a_2 + b > 0$. Then, we may assume that $(\omega_n, \sigma_n) \neq (0, 0)$ for n large enough. Recalling (56), (57), and $\mathcal{N}$ in (13), there exists $t_n > 0$ with $t_n \rightarrow 1$ as $n \rightarrow \infty$ such that $(t_n \omega_n, t_n \sigma_n) \in \mathcal{M}$. Then, by (60), we obtain that

$$\begin{aligned}
M &= \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) (a_1 + a_2) + \left(\frac{1}{2} - \frac{1}{2^*(s_1)} \right) b \\
&= \lim_{n \rightarrow \infty} I(\omega_n, \sigma_n) = \lim_{n \rightarrow \infty} I(t_n \omega_n, t_n \sigma_n) \geq M_0,
\end{aligned} \tag{61}$$

which contradicts Lemma 7. Hence, $u \equiv 0, v \equiv 0$ is impossible.

Case 2 ($u \neq 0, v \equiv 0$ or $u \equiv 0, v \neq 0$). Without loss of generality, we may assume that $u \neq 0, v \equiv 0$. Thus, u is a nontrivial solution of $\Delta_A u + \lambda_1(|u|^{p-2}u/|x|^s) + \mu_1(|u|^{2^*(s_1)-2}u/|x|^{s_1}) = 0$. Then, by (60), we deduce that $M \geq E(u, 0) \geq M_{\mu_1}$, a contradiction to Lemma 7. Therefore, Case 2 is impossible.

Since Cases 1 and 2 are both impossible, we have $u \neq 0, v \neq 0$. Remark 6 yields that $E(u, v) = M = M$. It follows from (9) and (20) that there exists a Lagrange multiplier $L \in \mathbb{R}$ such that

$$E'(u, v) + LF'(u, v) = 0, \tag{62}$$

where

$$\begin{aligned}
F(u, v) &= \|u, v\|_{H_{A,B}}^2 - (\lambda_1 |u|_{p,s}^p + \lambda_2 |v|_{p,s}^p) \\
&- \left(\mu_1 |u|_{2^*(s_1), s_1}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1), s_1}^{2^*(s_1)} \right) \\
&- \gamma \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta}}{|x|^{s_2}},
\end{aligned} \tag{63}$$

Since $E'(u, v)(u, v) = F(u, v) = 0$ and

$$\begin{aligned} F'(u, v)(u, v) &= -(p-2) \left(\lambda_1 |u|_{p,s}^p + \lambda_2 |v|_{p,s}^p \right) \\ &\quad - (2^*(s_1) - 2) \left(\mu_1 |u|_{2^*(s_1), s_1}^{2^*(s_1)} + \mu_2 |v|_{2^*(s_1), s_1}^{2^*(s_1)} \right) \\ &\quad - (2^*(s_2) - 2) \gamma \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^{s_2}} < 0, \end{aligned} \quad (64)$$

we obtain that $L = 0$ and, hence, $E'(u, v) = 0$ in $H'_{A,B}$. That is, (u, v) is a ground state solution of (2). $\square$

Remark 8. It follows from Remark 6 that the ground state solution of (2) is also a mountain pass solution.

Remark 9. Note that we only use the result in Theorem 1 to construct the values of M_{μ_1} and M_{μ_2} which are used in Lemma 7 and then in Case 2 at last of the proof of Theorem 2 to exclude the semitrivial solutions as $(u, 0)$ and $(0, v)$ to system (2), that is to say that the part of proof of Theorem 2 before Case 2 is independent of Theorem 1. So we can repeat the proof above with a little modification to prove Theorem 1. Indeed, the existence of a ground state solution to $-\Delta u = |u|^{2^*(s_1)-2}u/|x|^{s_1}$ is ensured in [17], and we only need to exclude the trivial solution of $u = 0$ to (1) which is similar to Case 1 in the proof for the system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This paper is supported by NSFC (11601109), Hainan Association of Science and Technology Plans for Youth Innovation (201503), and Natural Science Fund of Hainan Province (20161001).

References

- [1] M. J. Esteban and P.-L. Lions, "Stationary solutions of nonlinear Schrödinger equations with an external magnetic field," in *Partial differential equations and the calculus of variations*, vol. 1 of *Progr. Nonlinear Differential Equations Appl.*, pp. 401–449, Birkhäuser Boston, Boston, MA, 1989.
- [2] C. O. Alves and G. M. Figueiredo, "Multiplicity of solutions for a NLS equations with magnetic fields in $\mathbb{R}^N$," *Monatshefte für Mathematik*, vol. 175, no. 1, pp. 1–23, 2014.
- [3] C. O. Alves, G. M. Figueiredo, and M. F. Furtado, "Multiple solutions for a nonlinear Schrödinger equation with magnetic fields," *Communications in Partial Differential Equations*, vol. 36, no. 9, pp. 1565–1586, 2011.
- [4] C. O. Alves, G. M. Figueiredo, and M. F. Furtado, "On the number of solutions of NLS equations with magnetic fields in expanding domains," *Journal of Differential Equations*, vol. 251, no. 9, pp. 2534–2548, 2011.
- [5] D. Cao and Z. Tang, "Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields," *Journal of Differential Equations*, vol. 222, no. 2, pp. 381–424, 2006.
- [6] S. Cingolani, "Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field," *Journal of Differential Equations*, vol. 188, no. 1, pp. 52–79, 2003.
- [7] S. Cingolani, M. Clapp, and S. Secchi, "Multiple solutions to a magnetic nonlinear Choquard equation," *Journal of Applied Mathematics and Physics*, vol. 63, no. 2, pp. 233–248, 2012.
- [8] M. Clapp, R. Iturriaga, and A. Szulkin, "Periodic and Bloch solutions to a magnetic nonlinear Schrödinger equation," *Advanced Nonlinear Studies*, vol. 9, no. 4, pp. 639–655, 2009.
- [9] M. Clapp and A. Szulkin, "Multiple solutions to a nonlinear Schrödinger equation with Aharonov-Bohm magnetic potential," *Nonlinear Differential Equations and Applications NoDEA*, vol. 17, no. 2, pp. 229–248, 2010.
- [10] K. Kurata, "Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 41, no. (5-6, Ser. A: Theory Methods), pp. 763–778, 2000.
- [11] Z. Tang, "Multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields and critical frequency," *Journal of Differential Equations*, vol. 245, no. 10, pp. 2723–2748, 2008.
- [12] L. Abatangelo and S. Terracini, "Solutions to nonlinear Schrödinger equations with singular electromagnetic potential and critical exponent," *Journal of Fixed Point Theory and Applications*, vol. 10, no. 1, pp. 147–180, 2011.
- [13] V. Felli, A. Ferrero, and S. Terracini, "Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential," *Journal of the European Mathematical Society*, vol. 13, no. 1, pp. 119–174, 2011.
- [14] C. O. Alves and G. M. Figueiredo, "Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field," *Milan Journal of Mathematics*, vol. 82, no. 2, pp. 389–405, 2014.
- [15] G. Arioli and A. Szulkin, "A semilinear Schrödinger equation in the presence of a magnetic field," *Archive for Rational Mechanics and Analysis*, vol. 170, no. 4, pp. 277–295, 2003.
- [16] M. Clapp and A. Szulkin, "Multiple solutions to nonlinear Schrödinger equations with singular electromagnetic potential," *Journal of Fixed Point Theory and Applications*, vol. 13, no. 1, pp. 85–102, 2013.
- [17] N. Ghoussoub and C. Yuan, "Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents," *Transactions of the American Mathematical Society*, vol. 352, no. 12, pp. 5703–5743, 2000.
- [18] X. Zhong and W. Zou, "On Elliptic Systems involving critical Hardy-Sobolev exponents. Preprint, 2015, <http://arxiv.org/>.
- [19] E.-H. Lieb and M. Loss, *Analysis*, vol. 14 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2nd edition, 2001.
- [20] Z. Guo, M. Melgaard, and W. Zou, "Schrödinger equations with magnetic fields and Hardy-Sobolev critical exponents," *Electronic Journal of Differential Equations*, vol. 199, pp. 1–18, 2017.

- [21] H. Leinfelder, "Gauge invariance of Schrödinger operators and related spectral properties," *The Journal of Operator Theory*, vol. 9, no. 1, pp. 163–179, 1983.
- [22] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.

Submit your manuscripts at
www.hindawi.com