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Research Article

Equivalent Property of a Hilbert-Type Integral Inequality Related to the Beta Function in the Whole Plane

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By means of the technique of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane are obtained. The constant factor related to the beta function is proved to be the best possible. As applications, the case of the homogeneous kernel, the operator expressions, and a few corollaries are considered.

1. Introduction

Suppose that $p > 1, 1/p + 1/q = 1, f(x), g(y) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(y)dy < \infty$. We have the following well-known Hardy-Hilbert's integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{x+y} dx\, dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)\, dx\right)^{1/p} \left(\int_0^\infty g^q(y)\, dy\right)^{1/q}, \tag{1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. For $p = q = 2$, (1) reduces to the well-known Hilbert's integral inequality. By using the weight functions, some extensions of (1) were given by [2, 3]. A few Hilbert-type inequalities with the homogenous and nonhomogenous kernels were provided by [4–7]. In 2017, Hong [8] also gave two equivalent statements between Hilbert-type inequalities with the general homogenous kernel and parameters. Some other kinds of Hilbert-type inequalities were obtained by [9–16].

In 2007, Yang [17] gave a Hilbert-type integral inequality in the whole plane as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x) g(y)}{(1+e^{x+y})^\lambda} dx\, dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \cdot \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)\, dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)\, dy\right)^{1/2}, \tag{2}$$

with the best possible constant factor $B(\lambda/2, \lambda/2)$ $(\lambda > 0$, $B(u, v)$ is the beta function) (see [18]). He et al. [19–23] proved a few Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In this paper, by means of the technique of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane similar to (2) are obtained. The constant factor related to the beta function is proved to be the best possible. As applications, the case of the homogeneous kernel, the operator expressions, and a few corollaries are considered.

2. An Example and Two Lemmas

Example 1. For $\mathbf{R} = (-\infty, \infty), \mathbf{R}_+ = (0, \infty)$, we set $h(u) := (\max\{u, 1\})^{\alpha+\beta}/|u-1|^{\lambda+\alpha}(\min\{u, 1\})^\beta$ $(u \in \mathbf{R}_+)$, and then for $a, b \neq 0$,

$$h\left(e^{ax+by}\right) = \frac{\left(\max\left\{e^{ax+by}, 1\right\}\right)^{\alpha+\beta}}{\left|e^{ax+by}-1\right|^{\lambda+\alpha} \left(\min\left\{e^{ax+by}, 1\right\}\right)^\beta} \quad (x, y \in \mathbf{R}). \tag{3}$$

For $\sigma, \mu > \beta, \sigma + \mu = \lambda < 1 - \alpha$ $(\alpha + 2\beta < 1)$, in view of $h(v^{-1})v^{1-\sigma} = h(v)v^{\mu-1}$ $(0 < v < 1)$, we find

$$\begin{aligned}k_\lambda(\sigma) &:= \int_0^\infty h(u)\,u^{\sigma-1}du = \int_0^1 h(u)\left(u^{\sigma-1}+u^{\mu-1}\right)du \\ &= \int_0^1 \frac{(\max\{u,1\})^{\alpha+\beta}}{(1-u)^{\lambda+\alpha}(\min\{u,1\})^{\beta}}\left(u^{\sigma-1}+u^{\mu-1}\right)du \\ &= \int_0^1 \frac{1}{(1-u)^{\lambda+\alpha}}\left(u^{\sigma-\beta-1}+u^{\mu-\beta-1}\right)du \\ &= B(1-\lambda-\alpha,\sigma-\beta)+B(1-\lambda-\alpha,\mu-\beta) \\ &\in \mathbf{R}_+,\end{aligned} \quad (4)$$

where $B(u,v) := \int_0^1 (1-t)^{u-1}t^{v-1}dt$ $(u,v>0)$ is the beta function (cf. [18]).

In particular, (i) for $\alpha=0$, we have $\sigma,\mu>\beta, \sigma+\mu=\lambda<1$ $(\beta<1/2)$, $h_1(u)=(\max\{u,1\})^\beta/|u-1|^\lambda(\min\{u,1\})^\beta$ $(u>0)$, and

$$k_\lambda^{(1)}(\sigma) = B(1-\lambda,\sigma-\beta)+B(1-\lambda,\mu-\beta); \quad (5)$$

(ii) for $\beta=0$, we have $\sigma,\mu>0, \sigma+\mu=\lambda<1-\alpha$ $(\alpha<1)$, $h_2(u)=(\max\{u,1\})^\alpha/|u-1|^{\lambda+\alpha}$ $(u>0)$, and

$$k_\lambda^{(2)}(\sigma) = B(1-\lambda-\alpha,\sigma)+B(1-\lambda-\alpha,\mu); \quad (6)$$

(iii) for $\beta=-\alpha$, we have $\sigma,\mu>-\alpha, \sigma+\mu=\lambda<1-\alpha$ $(\alpha>-1)$, $h_3(u)=(\min\{u,1\})^\alpha/|u-1|^{\lambda+\alpha}$ $(u>0)$, and

$$k_\lambda^{(3)}(\sigma) = B(1-\lambda-\alpha,\sigma+\alpha)+B(1-\lambda-\alpha,\mu+\alpha). \quad (7)$$

In the case of (iii), for $\alpha=0$, we have $\sigma,\mu>0, \sigma+\mu=\lambda<1$, $h_4(u)=1/|u-1|^\lambda$ $(u>0)$, and

$$k_\lambda^{(4)}(\sigma) = B(1-\lambda,\sigma)+B(1-\lambda,\mu). \quad (8)$$

In the following, we assume that $p>1, 1/p+1/q=1, a,b\neq 0, \sigma_1,\sigma\in\mathbf{R}, \sigma,\mu>\beta, \sigma+\mu=\lambda<1-\alpha(\alpha+2\beta<1)$, and

$$\begin{aligned}K_\lambda(\sigma) &:= \frac{1}{|a|^{1/q}|b|^{1/p}}k_\lambda(\sigma) \\ &= \frac{1}{|a|^{1/q}|b|^{1/p}} \\ &\quad\times\left(B(1-\lambda-\alpha,\sigma-\beta)+B(1-\lambda-\alpha,\mu-\beta)\right).\end{aligned} \quad (9)$$

For $n\in\mathbf{N}=\{1,2,\ldots\}$, we define two sets $E_c := \{t\in\mathbf{R}; ct\geq 0\}$, $F_c := \mathbf{R}\setminus E_c = \{t\in\mathbf{R}; ct<0\}$ $(c=a,b)$, and the following two expressions:

$$I_1 := \int_{F_b} e^{(\sigma_1+1/qn)by}\left[\int_{E_a} h\left(e^{ax+by}\right)e^{(\sigma-1/pn)ax}dx\right]dy, \quad (10)$$

$$I_2 := \int_{E_b} e^{(\sigma_1-1/qn)by}\left[\int_{F_a} h\left(e^{ax+by}\right)e^{(\sigma+1/pn)ax}dx\right]dy. \quad (11)$$

Setting $u=e^{ax+by}$ in (10), in view of Fubini theorem (cf. [24]), it follows that

$$\begin{aligned}I_1 &= \frac{1}{|a|}\int_{F_b} e^{(\sigma_1-\sigma+1/n)by}\left(\int_{e^{by}}^\infty (u)\,u^{\sigma-1/pn-1}du\right)dy \\ &= \frac{1}{|ab|}\int_0^1 v^{\sigma_1-\sigma+1/n-1}\left(\int_v^\infty h(u)\,u^{\sigma-1/pn-1}du\right)dv \\ &\qquad\qquad (v=e^{by}).\end{aligned} \quad (12)$$

In the same way, we find that

$$\begin{aligned}I_2 &= \frac{1}{|a|}\int_{E_b} e^{(\sigma_1-\sigma-1/n)by}\left(\int_0^{e^y} h(u)\,u^{\sigma+1/pn-1}dx\right)dy \\ &= \frac{1}{|ab|}\int_1^\infty v^{\sigma_1-\sigma-1/n-1}\left(\int_0^v h(u)\,u^{\sigma+1/pn-1}du\right)dv.\end{aligned} \quad (13)$$

Lemma 2. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned}I &:= \int_{-\infty}^\infty\int_{-\infty}^\infty h\left(e^{ax+by}\right)f(x)\,g(y)\,dx\,dy \\ &\leq M\left[\int_{-\infty}^\infty\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \\ &\quad\cdot\left[\int_{-\infty}^\infty\left(\frac{g(y)}{e^{\sigma_1 by}}\right)^q dy\right]^{1/q}\end{aligned} \quad (14)$$

holds true, then we have $\sigma_1=\sigma$.

Proof. (i) If $\sigma_1<\sigma$, then for $n>1/(\sigma-\sigma_1)$ $(n\in\mathbf{N})$, we set two functions

$$\begin{aligned}f_n(x) &:= \begin{cases} e^{(\sigma-1/pn)ax}, & x\in E_a \\ 0, & x\in F_a,\end{cases} \\ g_n(y) &:= \begin{cases} 0, & y\in E_b \\ e^{(\sigma_1+1/qn)by}, & y\in F_b,\end{cases}\end{aligned} \quad (15)$$

and obtain

$$\begin{aligned}J_2 &:= \left[\int_{-\infty}^\infty e^{-p\sigma ax}f_n^p(x)\,dx\right]^{1/p} \\ &\quad\cdot\left[\int_{-\infty}^\infty e^{-q\sigma_1 by}g_n^q(y)\,dy\right]^{1/q} = \left(\int_{E_a} e^{-ax/n}dx\right)^{1/p} \\ &\quad\cdot\left(\int_{F_b} e^{by/n}dy\right)^{1/q} = \frac{n}{|a|^{1/p}|b|^{1/q}}.\end{aligned} \quad (16)$$

By (12) and (14), we find

$$\begin{aligned}
&\frac{1}{|ab|}\int_0^1 v^{\sigma_1-\sigma+1/n-1}dv\int_1^\infty \frac{u^{\alpha+\beta+\sigma-1/pn-1}}{(u-1)^{\lambda+\alpha}}du \le I_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f_n(x)\, g_n(y)\, dx\, dy \le MJ_2 \\
&= \frac{Mn}{|a|^{1/p}|b|^{1/q}}.
\end{aligned} \tag{17}$$

For any $n > 1/(\sigma - \sigma_1)$ $(n \in \mathbf{N})$, $\sigma_1 - \sigma + 1/n < 0$, it follows that $\int_0^1 v^{\sigma_1-\sigma+1/n-1}dv = \infty$.In view of $\int_1^\infty (u^{\alpha+\beta+\sigma-1/pn-1}/(u-1)^{\lambda+\alpha})du > 0$, by (17), we find that $\infty \le Mn/|a|^{1/p}|b|^{1/q} < \infty$, which is a contradiction.

(ii) If $\sigma_1 > \sigma$, then for $n > (1/(\sigma_1 - \sigma))$ $(n \in \mathbf{N})$, we set functions

$$\begin{aligned}
\widetilde{f}_n(x) &:= \begin{cases} 0, & x \in E_a \\ e^{(\sigma+1/pn)ax}, & x \in F_a, \end{cases} \\
\widetilde{g}_n(y) &:= \begin{cases} e^{(\sigma_1-1/qn)by}, & y \in E_b \\ 0, & y \in F_b, \end{cases}
\end{aligned} \tag{18}$$

and find

$$\begin{aligned}
\widetilde{J}_2 &:= \left[\int_{-\infty}^{\infty} e^{-p\sigma ax}\widetilde{f}_n^p(x)\,dx\right]^{1/p} \\
&\cdot \left[\int_{-\infty}^{\infty} e^{-q\sigma_1 by}\widetilde{g}_n^q(y)\,dy\right]^{1/q} = \left(\int_{F_a} e^{ax/n}dx\right)^{1/p} \\
&\cdot \left(\int_{E_b} e^{-by/n}dy\right)^{1/q} = \frac{n}{|a|^{1/p}|b|^{1/q}}.
\end{aligned} \tag{19}$$

By (13) and (14), we obtain

$$\begin{aligned}
&\frac{1}{|ab|}\int_1^\infty v^{\sigma_1-\sigma-1/n-1}dv\int_0^1 \frac{u^{\sigma-\beta+1/pn-1}}{(1-u)^{\lambda+\alpha}}du \le I_1 \\
&= \int_0^\infty\int_0^\infty h\left(e^{ax+by}\right)\widetilde{f}_n(x)\,\widetilde{g}_n(y)\,dx\,dy \le M\widetilde{J}_2 \\
&= \frac{Mn}{|a|^{1/p}|b|^{1/q}}.
\end{aligned} \tag{20}$$

For $n > 1/(\sigma_1 - \sigma)$ $(n \in \mathbf{N})$, $\sigma_1 - \sigma - 1/n > 0$, it follows that $\int_1^\infty y^{\sigma_1-\sigma-1/n-1}dy = \infty$. By (20), in view of $\int_0^1 (u^{\sigma-\beta+1/pn-1}/(1-u)^{\lambda+\alpha})du > 0$, we have $\infty \le Mn/|a|^{1/p}|b|^{1/q} < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

The lemma is proved. □

For $\sigma_1 = \sigma$, we have the following.

Lemma 3. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\, g(y)\, dx\, dy \\
&\le M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \\
&\cdot \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{1/q}
\end{aligned} \tag{21}$$

holds true, then we have $M \ge K_\lambda(\sigma)(> 0)$.

Proof. By (12), for $\sigma_1 = \sigma$, we obtain

$$\begin{aligned}
I_1 &= \frac{1}{|ab|}\int_0^1 v^{1/n-1}\left(\int_v^1 h(u)\,u^{\sigma-1/pn-1}du\right)dv + \frac{1}{|ab|} \\
&\cdot \int_0^1 v^{1/n-1}\left(\int_1^\infty h(u)\,u^{\sigma-1/pn-1}du\right)dv = \frac{1}{|ab|} \\
&\cdot \int_0^1\left(\int_0^u v^{1/n-1}dv\right)h(u)\,u^{\sigma-1/pn-1}du + \frac{n}{|ab|} \\
&\cdot \int_1^\infty h(u)\,u^{\sigma-1/pn-1}du \\
&= \frac{n}{|ab|}\left(\int_0^1 h(u)\,u^{\sigma+1/qn-1}du\right. \\
&\cdot \left.\int_1^\infty h(u)\,u^{\sigma-1/pn-1}du\right).
\end{aligned} \tag{22}$$

We use inequality $I_1 \le M\widetilde{J}_2$ (for $\sigma_1 = \sigma$) as follows:

$$\begin{aligned}
\frac{|a|^{1/p}|b|^{1/q}}{n}I_1 &= \frac{1}{|a|^{1/q}|b|^{1/p}}\left(\int_0^1 H(u)\,u^{\sigma+1/qn-1}du\right. \\
&+ \left.\int_1^\infty H(u)\,u^{\sigma-1/pn-1}du\right) \le M.
\end{aligned} \tag{23}$$

By Fatou lemma (cf. [24]) and (23), it follows that

$$\begin{aligned}
K_\lambda(\sigma) &= \frac{1}{|a|^{1/q}|b|^{1/p}} \times \left(\int_0^1 \lim_{n\to\infty} h(u)\,u^{\sigma+1/qn-1}du\right. \\
&+ \left.\int_1^\infty \lim_{n\to\infty} h(u)\,u^{\sigma-1/pn-1}du\right) \\
&\le \lim_{n\to\infty}\frac{|a|^{1/p}|b|^{1/q}}{n}I_1 \le M.
\end{aligned} \tag{24}$$

The lemma is proved. □

3. Main Results and Some Corollaries

Theorem 4. *If M is a constant, then the following statements (i), (ii), and (iii) are equivalent:*

(i) For any nonnegative measurable function $f(x)$ in $\mathbf{R}$, we have the following inequality:

$$J := \left[\int_{-\infty}^{\infty} e^{p\sigma_1 by}\left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\,dx\right)^p dy\right]^{1/p} \le M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}. \tag{25}$$

(ii) For any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, we have the following inequality:

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\,g(y)\,dx\,dy \le M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \cdot \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma_1 by}}\right)^q dy\right]^{1/q}. \tag{26}$$

(iii) $\sigma_1 = \sigma$, and $M \ge K_\lambda(\sigma)(> 0)$.

Proof. $(i) => (ii)$. By Hölder's inequality (see [25]), we have

$$I = \int_{-\infty}^{\infty}\left(e^{\sigma_1 by}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\,dx\right) \cdot \left(e^{-\sigma_1 by} g(y)\right) dy \le J\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma_1 by}}\right)^q dy\right]^{1/q}. \tag{27}$$

Then by (25), we have (26).

$(ii) => (iii)$. By Lemma 2, we have $\sigma_1 = \sigma$. Then by Lemma 3, we have $M \ge K_\lambda(\sigma)(> 0)$.

$(iii) => (i)$. Setting $u = e^{ax+by}$, we obtain the following weight functions: for $y, x \in \mathbf{R}$,

$$\omega(\sigma, y) := e^{\sigma by}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) e^{\sigma ax}\,dx = \frac{1}{|a|}\int_0^{\infty} h(u)\,u^{\sigma-1}\,du = \frac{1}{|a|}k_\lambda(\sigma), \tag{28}$$

$$\varpi(\sigma, x) := e^{\sigma ax}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) e^{\sigma by}\,dy = \frac{1}{|b|}k_\lambda(\sigma). \tag{29}$$

By Hölder's inequality with weight and (28), we have

$$\begin{aligned}\left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\,dx\right)^p &= \left[\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) \cdot \left(\frac{e^{\sigma by/p}}{e^{\sigma ax/q}} f(x)\right)\left(\frac{e^{\sigma ax/q}}{e^{\sigma by/p}}\right) dx\right]^p \le \int_{-\infty}^{\infty} h\left(e^{ax+by}\right) \\ &\cdot \frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)\,dx\left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) \cdot \frac{e^{\sigma ax}}{e^{\sigma byq/p}}\,dx\right)^{p/q} = \left[\omega(\sigma, y)\,e^{-q\sigma by}\right]^{p-1} \\ &\cdot \int_{-\infty}^{\infty} h\left(e^{ax+by}\right)\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)\,dx = \left(\frac{1}{|a|}k(\sigma)\right)^{p-1} \\ &\cdot e^{-p\sigma by}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right)\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)\,dx.\end{aligned} \tag{30}$$

For $\sigma_1 = \sigma$, by Fubini theorem (see [24]) and (29), we have

$$\begin{aligned} J &\le \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{1/q} \cdot \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(e^{ax+by}\right)\frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x)\,dx\,dy\right)^{1/p} \\ &= \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{1/q} \cdot \left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right)\frac{e^{\sigma by}}{e^{\sigma axp/q}}\,dy\right) f^p(x)\,dx\right]^{1/p} \\ &= \left(\frac{1}{|a|}k_\lambda(\sigma)\right)^{1/q} \cdot \left(\int_{-\infty}^{\infty}\omega(\sigma, x)\,e^{-p\sigma ax} f^p(x)\,dx\right)^{1/p} = K_\lambda(\sigma) \cdot \left(\int_{-\infty}^{\infty} e^{-p\sigma ax} f^p(x)\,dx\right)^{1/p}.\end{aligned} \tag{31}$$

For $K_\lambda(\sigma) \le M$, we have (25).

Therefore, the statements (i), (ii), and (iii) are equivalent. The theorem is proved. □

Theorem 5. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \ge 0$, satisfying $0 < \int_{-\infty}^{\infty}(f(x)/e^{\sigma ax})^p dx < \infty$, we have the following inequality:

$$J_1 = \left\{\int_{-\infty}^{\infty} e^{p\sigma by}\left[\int_{-\infty}^{\infty}\frac{\left(\max\left\{e^{ax+by}, 1\right\}\right)^{\alpha+\beta} f(x)\,dx}{\left|e^{ax+by} - 1\right|^{\lambda+\alpha}\left(\min\left\{e^{ax+by}, 1\right\}\right)^{\beta}}\right]^p dy\right\}^{1/p} < K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}. \tag{32}$$

(ii) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty}(f(x)/e^{\sigma ax})^p dx < \infty$ and $g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty}(g(y)/e^{\sigma by})^q dy < \infty$, we have the following inequality:

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left(\max\left\{e^{ax+by},1\right\}\right)^{\alpha+\beta} f(x)\, g(y)}{\left|e^{ax+by}-1\right|^{\lambda+\alpha}\left(\min\left\{e^{ax+by},1\right\}\right)^{\beta}}dx\,dy \\ &< K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \\ &\quad \cdot \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{1/q}. \end{aligned} \tag{33}$$

Moreover, the constant factor $K_\lambda(\sigma)$ in (32) and (33) is the best possible.

In particular, for $\alpha = \beta = 0, \sigma, \mu > 0, \sigma + \mu = \lambda < 1$

$$\widetilde{K}_\lambda(\sigma) := \frac{1}{|a|^{1/q}|b|^{1/p}}\left(B(1-\lambda,\sigma) + \zeta(1-\lambda,\mu)\right), \tag{34}$$

we have the following equivalent inequalities with the best possible constant factor $\widetilde{K}_\lambda(\sigma)$:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} e^{p\sigma by}\left(\int_{-\infty}^{\infty}\frac{f(x)}{\left|e^{ax+by}-1\right|^{\lambda}}dx\right)^p dy\right]^{1/p} \\ &\quad < \widetilde{K}_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}, \end{aligned} \tag{35}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)\, g(y)}{\left|e^{ax+by}-1\right|^{\lambda}}dx\,dy < \widetilde{K}_\lambda(\sigma) \\ &\quad \cdot \left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{1/q}. \end{aligned} \tag{36}$$

Proof. We first prove that (32) is valid. If (30) takes the form of equality for a $y \in \mathbf{R}$, then (see [25]), there exist constants A and B, such that they are not all zero, and

$$A\frac{e^{\sigma by}}{e^{\sigma axp/q}}f^p(x) = B\frac{e^{\sigma ax}}{e^{\sigma byq/p}} \quad \text{a.e. in } \mathbf{R}. \tag{37}$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$\left(\frac{f(x)}{e^{\sigma ax}}\right)^p = e^{-q\sigma by}\frac{B}{A} \quad \text{a.e. in } \mathbf{R}, \tag{38}$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty}(f(x)/e^{\sigma ax})^p dx < \infty$. Hence, (30) takes the form of strict inequality. For $\sigma_1 = \sigma$ by the proof of Theorem 4, we obtain (32).

$(i) => (ii)$. By (27) (for $\sigma_1 = \sigma$) and (32), we have (33).

$(ii) => (i)$. We set the following function:

$$g(y) := e^{p\sigma by}\left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x)\, dx\right)^{p-1} \quad (y \in \mathbf{R}). \tag{39}$$

If $J_1 = \infty$, then it is impossible since (32) is valid; if $J_1 = 0$, then (32) is trivially valid. In the following, we suppose that $0 < J_1 < \infty$. By (33), we have

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy = J_1^p = I < K_\lambda(\sigma) \\ &\quad \cdot \left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{1/q} \\ &< \infty, \end{aligned} \tag{40}$$

$$\begin{aligned} J_1 &= \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\sigma by}}\right)^q dy\right]^{1/p} < K_\lambda(\sigma) \\ &\quad \cdot \left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}; \end{aligned}$$

namely, (32) follows, which is equivalent to (33).

Hence, Statements (i) and (ii) are valid and equivalent.

If there exists a constant $M \leq K_\lambda(\sigma)$, such that (33) is valid when replacing $K_\lambda(\sigma)$ by M, then by Lemma 3, we have $K_\lambda(\sigma) \leq M$. Hence, the constant factor $M = K_\lambda(\sigma)$ in (33) is the best possible.

The constant factor $K_\lambda(\sigma)$ in (32) is still the best possible. Otherwise, by (27) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $K_\lambda(\sigma)$ in (33) is not the best possible.

The theorem is proved. □

For $g(y) = e^{-\lambda by}G(y)$, and $\mu_1 = \lambda - \sigma_1$ in Theorems 4 and 5, then replacing b $(G(y))$ by $-b$ $(g(y))$, setting

$$k_\lambda\left(e^{ax}, e^{by}\right) := \frac{\left(\max\left\{e^{ax}, e^{by}\right\}\right)^{\alpha+\beta}}{\left|e^{ax}-e^{by}\right|^{\lambda+\alpha}\left(\min\left\{e^{ax}, e^{by}\right\}\right)^{\beta}} \quad (x, y \in \mathbf{R}), \tag{41}$$

we have the following corollaries.

Corollary 6. *If M is a constant, then the following statements (i), (ii), and (iii) are equivalent:*

(i) For any nonnegative measurable function $f(x)$ in $\mathbf{R}$, we have the following inequality:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} e^{p\mu_1 by}\left(\int_{-\infty}^{\infty} k_\lambda\left(e^{ax}, e^{by}\right) f(x)\, dx\right)^p dy\right]^{1/p} \\ &\quad \leq M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}. \end{aligned} \tag{42}$$

(ii) For any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_\lambda\left(e^{ax},e^{by}\right) f(x)\,g(y)\,dx\,dy \leq M\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \cdot \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu_1 by}}\right)^q dy\right]^{1/q}. \quad (43)$$

(iii) $\mu_1 = \mu$, *and* $M \geq K_\lambda(\sigma)(>0)$.

Corollary 7. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}(f(x)/e^{\sigma ax})^p dx < \infty$, *we have the following inequality:*

$$\left\{\int_{-\infty}^{\infty} e^{pb\mu y}\left[\int_{-\infty}^{\infty}\frac{\left(\max\left\{e^{ax},e^{by}\right\}\right)^{\alpha+\beta} f(x)}{\left|e^{ax}-e^{by}\right|^{\lambda+\alpha}\left(\min\left\{e^{ax},e^{by}\right\}\right)^{\beta}}dx\right]^p dy\right\}^{1/p} < K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}. \quad (44)$$

(ii) For any $f(x) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}(f(x)/e^{\sigma ax})^p dx < \infty$, *and* $g(y) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}(g(y)/e^{\mu by})^q dy < \infty$, *we have the following inequality:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left(\max\left\{e^{ax},e^{by}\right\}\right)^{\alpha+\beta} f(x)\,g(y)}{\left|e^{ax}-e^{by}\right|^{\lambda+\alpha}\left(\min\left\{e^{ax},e^{by}\right\}\right)^{\beta}}dx\,dy < K_\lambda(\sigma)\left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p} \cdot \left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu by}}\right)^q dy\right]^{1/q}. \quad (45)$$

Moreover, the constant factor $K_\lambda(\sigma)$ *in (44) and (45) is the best possible.*

In particular, for $\alpha = \beta = 0, \sigma, \mu > 0, \sigma + \mu = \lambda < 1$, *we have the following equivalent inequalities with the best possible constant factor* $\widetilde{K}_\lambda(\sigma)$:

$$\left[\int_{-\infty}^{\infty} e^{p\mu by}\left(\int_{-\infty}^{\infty}\frac{f(x)}{\left|e^{ax}-e^{by}\right|^{\lambda}}dx\right)^p dy\right]^{1/p} < \widetilde{K}_\lambda(\sigma) \cdot \left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}, \quad (46)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)\,g(y)}{\left|e^{ax}-e^{by}\right|^{\lambda}}dx\,dy < \widetilde{K}_\lambda(\sigma) \cdot \left[\int_{-\infty}^{\infty}\left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx\right]^{1/p}\left[\int_{-\infty}^{\infty}\left(\frac{g(y)}{e^{\mu by}}\right)^q dy\right]^{1/q}. \quad (47)$$

In (35) and (36), setting $F(x) = e^{(\lambda a/2)x}f(x), G(y) = e^{(\lambda b/2)y}g(y)$, then replacing back $F(x)(G(y))$ by $f(x)(g(y))$, and introducing the hyperbolic sine function as $\sinh(s) = (e^s - e^{-s})/2$, we have

Corollary 8. *If* $\sigma, \mu > 0$, $\sigma + \mu = \lambda < 1$, *then the following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}[e^{(\lambda/2-\sigma)ax}f(x)]^p dx < \infty$, *we have the following inequality:*

$$\left[\int_{-\infty}^{\infty} e^{p(\sigma-\lambda/2)by}\left(\int_{-\infty}^{\infty}\frac{f(x)}{\left|\sinh\left((ax+by)/2\right)\right|^{\lambda}}dx\right)^p dy\right]^{1/p} < 2\widetilde{K}_\lambda(\sigma)\left\{\int_{-\infty}^{\infty}\left[e^{(\lambda/2-\sigma)ax}f(x)\right]^p dx\right\}^{1/p}. \quad (48)$$

(ii) For any $f(x) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}[e^{(\lambda/2-\sigma)ax}f(x)]^p dx < \infty$ *and* $g(y) \geq 0$, *satisfying* $0 < \int_{-\infty}^{\infty}[e^{(\lambda/2-\sigma)by}g(y)]^q dy < \infty$, *we have the following inequality:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)\,g(y)}{\left|\sinh\left((ax+by)/2\right)\right|^{\lambda}}dx\,dy < 2\widetilde{K}_\lambda(\sigma) \cdot \left\{\int_{-\infty}^{\infty}\left[e^{(\lambda/2-\sigma)ax}f(x)\right]^p dx\right\}^{1/p} \cdot \left\{\int_{-\infty}^{\infty}\left[e^{(\lambda/2-\sigma)by}g(y)\right]^q dy\right\}^{1/q}. \quad (49)$$

Moreover, the constant factor $2\widetilde{K}_\lambda(\sigma)$ *in (48) and (49) is the best possible.*

4. Operator Expressions

We set the following functions: $\varphi(x) := e^{-p\sigma ax}, \psi(y) := e^{-q\sigma by}, \phi(y) := e^{-q\mu by}$, wherefrom, $\psi^{1-p}(y) = e^{p\sigma by}, \phi^{1-p}(y) = e^{p\mu by}$ $(x, y \in \mathbf{R})$, and define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{f : \|f\|_{p,\varphi} := \left(\int_{-\infty}^{\infty}\varphi(x)\left|f(x)\right|^p dx\right)^{1/p} < \infty\right\}, \quad (50)$$

wherefrom,

$$L_{q,\psi}(\mathbf{R}) = \left\{ g : \|g\|_{q,\psi} := \left(\int_{-\infty}^{\infty} \psi(y) |g(y)|^q \, dy \right)^{1/q} < \infty \right\},$$

$$L_{q,\phi}(\mathbf{R}) = \left\{ g : \|g\|_{q,\phi} := \left(\int_{-\infty}^{\infty} \phi(y) |g(y)|^q \, dy \right)^{1/q} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}) = \left\{ h : \|h\|_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) |h(y)|^p \, dy \right)^{1/p} < \infty \right\},$$

$$L_{q,\phi^{1-p}}(\mathbf{R}) = \left\{ h : \|h\|_{p,\phi^{1-p}} = \left(\int_{-\infty}^{\infty} \phi^{1-p}(y) |h(y)|^p \, dy \right)^{1/p} < \infty \right\}. \tag{51}$$

(a) In view of Theorem 5, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x) \, dx \quad (y \in \mathbf{R}), \tag{52}$$

by (34), we have

$$\|h_1\|_{p,\psi^{1-p}} = \left[\int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) \, dy \right]^{1/p} \leq K_\lambda(\sigma) \|f\|_{p,\varphi} < \infty. \tag{53}$$

Definition 9. Define a Hilbert-type integral operator with the nonhomogeneous kernel $T^{(1)} : L_{p,\varphi}(\mathbf{R}) \longrightarrow L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: for any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(1)} f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}, T^{(1)} f(y) = h_1(y)$.

In view of (53), it follows that

$$\left\| T^{(1)} f \right\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq K_\lambda(\sigma) \|f\|_{p,\varphi}, \tag{54}$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\left\| T^{(1)} \right\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R})} \frac{\left\| T^{(1)} f \right\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq K_\lambda(\sigma). \tag{55}$$

If we define the formal inner product of $T^{(1)} f$ and g as follows:

$$\left(T^{(1)} f, g \right) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h\left(e^{ax+by}\right) f(x) \, dx \right) g(y) \, dy, \tag{56}$$

then we can rewrite Theorem 5 as follows.

Theorem 10. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, we have the following inequality:

$$\left\| T^{(1)} f \right\|_{p,\psi^{1-p}} < K_\lambda(\sigma) \|f\|_{p,\varphi}. \tag{57}$$

(ii) For any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, and $\|g\|_{q,\psi} > 0$, we have the following inequality:

$$\left(T^{(1)} f, g \right) < K_\lambda(\sigma) \|f\|_{p,\varphi} \|g\|_{q,\psi}. \tag{58}$$

Moreover, the constant factor $K_\lambda(\sigma)$ in (57) and (58) is the best possible, namely,

$$\left\| T^{(1)} \right\| = K_\lambda(\sigma). \tag{59}$$

(b) In view of Corollary 7, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_2(y) := \int_{-\infty}^{\infty} k_\lambda\left(e^{ax}, e^{by}\right) f(x) \, dx \quad (y \in \mathbf{R}), \tag{60}$$

by (44), we have

$$\|h_2\|_{p,\phi^{1-p}} = \left[\int_{-\infty}^{\infty} \phi^{1-p}(y) h_2^p(y) \, dy \right]^{1/p} \leq K_\lambda(\sigma) \|f\|_{p,\varphi} < \infty. \tag{61}$$

Definition 11. Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbf{R}) \longrightarrow L_{p,\phi^{1-p}}(\mathbf{R})$ as follows: for any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(2)} f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}, T^{(2)} f(y) = h_2(y)$.

In view of (61), it follows that

$$\left\| T^{(2)} f \right\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq K_\lambda(\sigma) \|f\|_{p,\varphi}, \tag{62}$$

and then the operator $T^{(2)}$ is bounded satisfying

$$\left\| T^{(2)} \right\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R})} \frac{\left\| T^{(2)} f \right\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq K_\lambda(\sigma). \tag{63}$$

If we define the formal inner product of $T^{(2)} f$ and g as follows:

$$\left(T^{(2)} f, g \right) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} k_\lambda\left(e^{ax}, e^{by}\right) f(x) \, dx \right) g(y) \, dy, \tag{64}$$

then we can rewrite Corollary 7 as follows.

Corollary 12. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, we have the following inequality:

$$\left\| T^{(2)} f \right\|_{p,\phi^{1-p}} < K_\lambda(\sigma) \|f\|_{p,\varphi}. \tag{65}$$

(ii) For any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R})$, $g \in L_{q,\phi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, and $\|g\|_{q,\phi} > 0$, we have the following inequality:

$$\left(T^{(2)} f, g\right) < K_{\lambda}(\sigma) \|f\|_{p,\varphi} \|g\|_{q,\phi}. \tag{66}$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (65) and (66) is the best possible, namely,

$$\left\|T^{(2)}\right\| = K_{\lambda}(\sigma). \tag{67}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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