Hindawi
Journal of Function Spaces
Volume 2018, Article ID 2793673, 11 pages
https://doi.org/10.1155/2018/2793673

Research Article

Synchronization of Different Uncertain Fractional-Order Chaotic Systems with External Disturbances via T-S Fuzzy Model

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Received 22 May 2018; Accepted 18 July 2018; Published 12 August 2018

Academic Editor: Liguang Wang

This paper presents an adaptive fuzzy synchronization control strategy for a class of different uncertain fractional-order chaotic/hyperchaotic systems with unknown external disturbances via T-S fuzzy systems, where the parallel distributed compensation technology is provided to design adaptive controller with fractional adaptation laws. T-S fuzzy models are employed to approximate the unknown nonlinear systems and tracking error signals are used to update the parametric estimates. The asymptotic stability of the closed-loop system and the boundedness of the states and parameters are guaranteed by fractional Lyapunov theory. This approach is also valid for synchronization of fractional-order chaotic systems with the same system structure. One constructive example is given to verify the feasibility and superiority of the proposed method.

1. Introduction

Fractional calculus is a mathematical topic being more than 300 years old, which can be traced back to the birth of integer-order calculus. The fundamentals results of fractional calculus were concluded in [1]. At present, researchers found that fractional differential equations not only improve the veracity in modeling physical systems but also generate a lot of applications in physics, electrical engineering, robotics, control systems, and chemical mixing [2–11]. In addition, the chaotic behavior has been discovered in many fractional-order systems, for instance, the fractional-order Chen's system, the fractional-order Chua's system, and the fractional-order Liu system. In view of chaotic potential value in control systems and secure communication [12], chaos synchronization was studied by more and more researches [13, 14].

The conventional nonlinear systems control approaches suffer from discontented performance resulting from structure and parametric uncertainties, external disturbances. Usually, it is very hard to provide accurate mathematical models [15–25]. To control these uncertain systems, adaptive fuzzy/neural-network control was proposed [26, 27]. This method is effective and superior for handling parametric and structure uncertainties, external disturbances in integer-order nonlinear systems [28, 29], where tracking error is developed to update adjusted parameters and fuzzy logic systems or neural networks are introduced to model unknown physical systems as well as to approximate unknown nonlinear functions. There are two types of fuzzy logic systems: Mamdani type and T-S type. T-S fuzzy logic system is first proposed by Takagi and Sugeno [30]. Subsequently, many works found that T-S fuzzy systems can uniformly approximate any continuous functions on a compact set with random accuracy based on the Weierstrass approximation theorem [31]. Moreover, it was also shown that the approximation ability of T-S fuzzy systems was better than the Mamdani fuzzy systems [32]. Therefore, many studies focused on the chaos synchronization of fractional-order chaotic systems via T-S fuzzy models. For example, synchronization of fractional-order modified chaotic system via new linear control, backstepping control, and T-S fuzzy approaches was investigated in [33]. Impulsive control for fractional-order chaotic system was presented in [34]. Other results about the synchronization of a fractional-order chaotic system via T-S fuzzy approaches can be found in [35, 36]. However, only chaos synchronization of fractional-order nonlinear systems with same structure based on T-S fuzzy systems is considered in above previous works.

This work investigates the chaos synchronization of fractional-order chaotic systems with different structures based on T-S fuzzy systems, where external disturbances in slaves system are considered. T-S fuzzy systems with random rule consequents are introduced to model controlled systems, whereas T-S fuzzy systems that have the same rule consequents with Mamdani fuzzy systems are used to approximate unknown nonlinear functions. The asymptotic stability of closed-loop system is proofed based on fractional Lyapunov stability theory. Compared to previous literature, the main contributions of this paper are as follows:

(1) This paper first considers the chaos synchronization of the master system and slave system with different structure based on T-S fuzzy systems, and the external disturbances are assumed to be unknown. The required knowledge of the disturbances is weaker than above previous works, for example, in [34–36]. In these works, the external disturbances are assumed to be bounded with known upper bounds. However, in our control method, we do not need to know the exact value of the upper bounds of external disturbances.

(2) T-S fuzzy logic systems are used to model the controlled system and the final outputs of system can be obtained. By combining the adaptive fuzzy control method and parallel distributed compensation technique, an adaptive controller with fractional-order laws is designed. The proposed method is superior to some works based on linear matrix inequality (LMI) and modified LMI [37].

2. Fundamentals of Fractional Calculus and Fuzzy Logic Systems

2.1. Fractional Calculus. There are two frequently used definitions for fractional integration and differentiation: Riemann-Liouville (denote R-L) and Caputo definitions. In this paper, we will consider Caputo's definition, whose initial conditions are as the same form of the integer-order one [38–40]. The fractional integral is designed as [1]

$$ {}_0^C D_t^{-\mu} u(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{u(\xi)}{(t-\xi)^{1-\mu}} d\xi, \tag{1}$$

where $\mu > 0$, $n-1 \le \mu < n$, and $\Gamma(\cdot)$ is Euler's Gamma function, which is defined as $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$. The fractional derivative operator is given as

$$ {}_0^C D_t^{\mu} u(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{u^{(n)}(\xi)}{(t-\xi)^{\mu+1-n}} d\xi. \tag{2}$$

Some useful properties of fractional calculus that will be used in the controller design are listed as follows.

Property 1 (see [1, 41–44]). Caputo's fractional derivative and integral are linear operations with $\lambda_1, \lambda_2, \mu \in R$

$$\begin{aligned} & {}_0^C D_t^{\mu} (\lambda_1 u_1(t) + \lambda_2 u_2(t)) \\ & \quad = \lambda_1 {}_0^C D_t^{\mu} u_1(t) + \lambda_2 {}_0^C D_t^{\mu} u_2(t). \end{aligned} \tag{3}$$

Property 2. Let $x(t) \in C^n[0,T]$. Then we have

$$ {}_0^C D_t^{-\mu} {}_0^C D_t^{\mu} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k. \tag{4}$$

Property 3 (see [1, 45–47]). The Laplace transform of (2) is

$$ \mathscr{L}\left[{}_0^C D_t^{\mu} f(t) \right] = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0), \tag{5}$$

with $F(s) = \mathscr{L}[f(t)]$.

Definition 4. The two-parameter Mittag-Leffler function was defined by [1]

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{6}$$

with $\alpha, \beta > 0$ and $z \in C$. The Laplace transform of the Mittag-Leffler function is given as

$$ \mathscr{L}\left[t^{\beta-1} E_{\alpha,\beta}(-at^{\alpha}) \right] = \frac{s^{\alpha-\beta}}{s^{\alpha}+a}. \tag{7}$$

In the subsequent paper, we only consider the case that $\mu \in (0,1)$.

2.2. Takagi-Sugeno Fuzzy Logic Systems. Unlike the Mamdani fuzzy logic systems, the ith rule of a Multi-Input and Multioutput general fractional-order Takagi-Sugeno (T-S) fuzzy systems can be expressed as follows ($i = 1, 2, \cdots, N$):

R^i: If $x_1(t)$ is F_1^i and $\cdots$ and $x_n(t)$ is F_n^i, then ${}_0^C D_t^{\mu} \mathbf{x}(t) = f_i(t, \mathbf{x}(t))$,

with $F_j^i \in R$, $j = 1, 2, \cdots, n$ are fuzzy sets, $\mathbf{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$ is a state vector, and $f_i(t, \mathbf{x}(t))$ is a random function. In this paper, singleton fuzzification, center average defuzzification, and product inference are adopted and a general fractional-order T-S fuzzy system can be rewritten in the form

$$ {}_0^C D_t^{\mu} \mathbf{x}(t) = \sum_{i=1}^{N} \mu_i(\mathbf{x}(t)) f_i(t, \mathbf{x}(t)), \tag{8}$$

where $\mu_i(\mathbf{x}(t)) = \prod_{j=1}^n F_j^i(x_j(t)) / \sum_{i=1}^N (\prod_{j=1}^n F_j^i(x_j(t)))$ satisfying $\sum_{i=1}^N \mu_i(\mathbf{x}(t)) = 1$ and $\mu_i(\mathbf{x}(t)) \ge 0$.

Depending on the above statements, a main difference of Mamdani fuzzy logic systems and T-S fuzzy systems is that the rule consequents are functions for T-S fuzzy system whereas the rule consequents are fuzzy sets for Mamdani fuzzy logic systems. Moreover, the T-S fuzzy logic systems are also universal approximators [31].

3. Adaptive Fuzzy Synchronization Control

3.1. Problem Statement. Consider the following fractional-order chaotic system as the master system via T-S type fuzzy systems. The ith rule can be expressed as ($i = 1, 2, \cdots, N$)

R^i: If $x_1(t)$ is F_1^i and $\cdots$ and $x_n(t)$ is F_n^i, then ${}_0^CD_t^\mu \mathbf{x}(t) = A_i\mathbf{x}(t) + \mathbf{b}_1$,

where A_i is a constant matrix, $\mathbf{x}(t) \in D_1 \subseteq R^n$ is the state vector (D_1 is a compact set), $\mathbf{b}_1$ is a constant vector, and F_j^i, $j = 1, 2, \cdots, n$, are fuzzy sets. Hence, the final output of master system can be rewritten as

$$ {}_0^CD_t^\mu \mathbf{x}(t) = \sum_{i=1}^{N} \mu_i(\mathbf{x}(t))\left[A_i\mathbf{x}(t) + \mathbf{b}_1\right], \quad (9)$$

with $\mu_i(\mathbf{x}(t)) = \prod_{j=1}^n F_j^i(x_j(t)) / \sum_{i=1}^N [\prod_{j=1}^n F_j^i(x_j(t))]$ satisfying $\mu_i(\mathbf{x}(t)) \geq 0$ and $\sum_{i=1}^N \mu_i(\mathbf{x}(t)) = 1$.

Consider the following fractional-order chaotic system with external disturbances in the equation as the slave system based on T-S fuzzy models. The ith rule can be written in the following form ($i = 1, \cdots, N$):

R^i: If $y_1(t)$ is $\widehat{F}_1^i$ and $y_2(t)$ is $\widehat{F}_2^i$ and $\cdots$ and $y_n(t)$ is $\widehat{F}_n^i$, then ${}_0^CD_t^\mu \mathbf{y}(t) = B_i\mathbf{y}(t) + \mathbf{b}_2 + \mathbf{u}(t) + \mathbf{d}_i(t,\mathbf{y})$,

where B_i is a constant matrix, $\mathbf{y}(t) \in D_2 \subseteq R^n$ is the state vector (D_2 is a compact set), $\mathbf{b}_2$ is a constant vector, $\widehat{F}_j^i$, $j = 1, 2, \cdots, n$, are fuzzy sets, $u(t) \in R$ is control input, and $\mathbf{d}_i(t,\mathbf{y}) \in R^n$ are unknown external disturbances. Hence, the final output of slave system can be obtained as

$$ \begin{aligned} &{}_0^CD_t^\mu \mathbf{y}(t) \\ &= \sum_{i=1}^{N} \mu_i(\mathbf{y}(t))\left[B_i\mathbf{y}(t) + \mathbf{b}_2 + \mathbf{u}(t) + \mathbf{d}_i(t,\mathbf{y})\right], \end{aligned} \quad (10)$$

with $\mu_i(\mathbf{y}(t)) = \prod_{j=1}^n \widehat{F}_j^i(y_j(t)) / \sum_{i=1}^N [\prod_{j=1}^n \widehat{F}_j^i(y_j(t))]$ satisfying $\mu_i(\mathbf{y}(t)) \geq 0$ and $\sum_{i=1}^N \mu_i(\mathbf{y}(t)) = 1$.

The control objective of this work is to design a proper adaptive controller $u(t)$ to synchronize the above chaotic systems (9) and (10) with the tracking error signal

$$ \mathbf{e}(t) = \mathbf{y}(t) - \mathbf{x}(t) \quad (11)$$

asymptotically converging to zero with random accuracy, that is, $\lim_{t\longrightarrow+\infty}\|\mathbf{e}(t)\| = 0$. The norm adopts Euclid norm in this paper. In addition, all states and parameters in the closed-loop system are bounded. The following assumptions are necessary.

Assumption 5. The structure of master system (9) and slave system (10) is different. The parameters and the structure of the master system are complete unknown or partial unknown, but the parameters and structure of the slave system are known.

Assumption 6. The unknown disturbances $\mathbf{d}_i(t,\mathbf{y}) = (d_i^1(t,\mathbf{y}), \cdots, d_i^n(t,\mathbf{y}))^T$, $(i = 1, 2, \cdots, N)$ satisfying $|d_i^j(t,\mathbf{y})| \leq \rho_i^j(\mathbf{y})$ with $\rho_i^j(\mathbf{y})$ being a continuous function, where $\rho_i^j(\mathbf{y})$ is the estimated value of the observed value for $|d_i^j(t,\mathbf{y})|$, for all $\mathbf{y} \in D_2$.

Remark 7. It is worth pointing out that Assumptions 5 and 6 are rational. Due to the boundedness of chaos systems, we assume that D_1 and D_2 are compact sets. Since $\mathbf{d}_i(t,\mathbf{y})$ are unknown external disturbances and may be not continuous, they are assumed to be unknown measurable nonlinear functions. The slave systems and the controller lie on the receiving terminal; hence, the parameters and the structure of the master system may be complete unknown or partial unknown, but the parameters and structure of the slave system are known.

3.2. Control Design. The synchronization error dynamic equation can be obtained from (11) as

$$ \begin{aligned} {}_0^CD_t^\mu \mathbf{e}(t) &= {}_0^CD_t^\mu \mathbf{y}(t) - {}_0^CD_t^\mu \mathbf{x}(t) \\ &= \sum_{i=1}^{N} \mu_i(\mathbf{y}(t)) B_i\mathbf{y}(t) - \sum_{i=1}^{N} \mu_i(\mathbf{x}(t)) A_i\mathbf{x}(t) \\ &\quad + \mathbf{m} + \mathbf{u}(t) + \sum_{i=1}^{N} \mu_i(\mathbf{y}(t))\mathbf{d}_i(t,\mathbf{y}), \end{aligned} \quad (12)$$

with $\mathbf{m} = \mathbf{b}_2 - \mathbf{b}_1$ being a constant vector.

Based on T-S fuzzy logic system universal approximation theorem, T-S fuzzy systems $\widehat{\rho}_i^j(\mathbf{y}, \theta_i^j(t))$ that have the same rule consequents with the Mamdani type fuzzy logic systems are used to approximate to $\rho_i^j(\mathbf{y})$ in the Assumption 6, $j \in 1, 2, \cdots, n$ and $i \in 1, 2, \cdots, N$, where $\theta_i^j(t)$ are adjusted parameters in fuzzy systems. Denote $\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) = (\widehat{\rho}_i^1(\mathbf{y}, \theta_i^1(t)). \cdots, \widehat{\rho}_i^n(\mathbf{y}, \theta_i^n(t)))^T$. Using [48–50], we obtain the ideal parameter as

$$ \theta_i^{*j} = \arg\min_{\theta_i^j(t)\in R}\sup_{t\geq 0}\left|\widehat{\rho}_i^j\left(\mathbf{y}, \theta_i^j(t)\right) - \rho_i^j(\mathbf{y})\right|. \quad (13)$$

Then we obtain the optimal parameter vector as $\boldsymbol{\theta}_i^* = (\theta_i^{*1}, \cdots, \theta_i^{*n})^T$. Hence, $\widehat{\rho}_i^j(\mathbf{y}, \theta_i^{*j})$ is the ideal approximator of $\rho_i^j(\mathbf{y})$; that is, $\widehat{\boldsymbol{\rho}}_i(y, \boldsymbol{\theta}_i^*)$ is the ideal approximator of $\boldsymbol{\rho}_i(\mathbf{y})$. The minimum approximation errors and the ideal parameter errors of the fuzzy systems are defined as ($j = 1, 2, \cdots, n$)

$$ \varepsilon_i^j(\mathbf{y}) = \rho_i^j(\mathbf{y}) - \widehat{\rho}_i^j\left(\mathbf{y}, \theta_i^{*j}\right), \quad (14)$$

$$ \widehat{\theta}_i^j(t) = \theta_i^j(t) - \theta_i^{*j}. \quad (15)$$

According to [29, 51, 52], the approximation errors $\varepsilon_i^j(\mathbf{y})$, $j = 1, 2, \cdots, n$) are assumed to be bounded, that is, $|\varepsilon_i^j(\mathbf{y})| \leq \varepsilon_i^{*j}$ with the ε_i^{*j} being constants and $\widehat{\varepsilon}_i^{*j}(\mathbf{y})$ being the estimate value of ε_i^{*j}. Thus, from the above analysis, we can obtain the equations $\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) = \boldsymbol{\theta}_i^T(t)\varphi_i(\mathbf{y})$ and $\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i^*) = \boldsymbol{\theta}_i^{*T}\varphi_i(\mathbf{y})$, where $\varphi_i(\mathbf{y})$ is fuzzy base functions. Denoting $\widetilde{\boldsymbol{\theta}}_i(t) = \boldsymbol{\theta}_i(t) - \boldsymbol{\theta}_i^*$ and $\boldsymbol{\varepsilon}_i(\mathbf{y}) = (\varepsilon_i^1(\mathbf{y}), \cdots, \varepsilon_i^n(\mathbf{y}))^T$, one has

$$\begin{aligned}\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) - \boldsymbol{\rho}_i(\mathbf{y}) &= \widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) - \widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i^*) \\ &\quad + \widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i^*) - \boldsymbol{\rho}_i(\mathbf{y}) \\ &= \widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) - \widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i^*) \\ &\quad - \boldsymbol{\varepsilon}_i(\mathbf{y}) \\ &= \widetilde{\boldsymbol{\theta}}_i^T(t)\varphi_i(\mathbf{y}) - \boldsymbol{\varepsilon}_i(\mathbf{y}).\end{aligned} \tag{16}$$

Remark 8. As shown in [53], if the rule consequences of T-S fuzzy systems have the same form with the rule consequences of Mamdani type logic systems, then T-S type is equivalent to Mamdani type fuzzy system.

Based on above discussion, the controller is designed with the fuzzy system $\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t))$ as well as the estimate value $\widehat{\varepsilon}_i^{*j}(\mathbf{y})$ as

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_d(t) + \widetilde{\mathbf{u}}(t) + \mathbf{u}_1(t) = -\sum_{i=1}^N \mu_i(\mathbf{y}(t))\,[K\mathbf{e}(t) \\ &\quad + B_i\mathbf{y}(t) + H_i\operatorname{sign}(\mathbf{e}(t)) + T\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) + \mathbf{m}] \\ &\quad + \sum_{i=1}^N \mu_i(\mathbf{x}(t))\,A_i\mathbf{x}(t),\end{aligned} \tag{17}$$

where the ith rule of $\mathbf{u}_d(t)$ and $\widetilde{\mathbf{u}}(t)$ can be written as follows, respectively, $(i = 1, 2, \cdots, N)$:

> R^i: If $y_1(t)$ is $\widehat{F}_1^i$ and $y_2(t)$ is $\widehat{F}_2^i$ and $\cdots$ and $y_n(t)$ is $\widehat{F}_n^i$, then $\mathbf{u}_d(t) = -K\mathbf{e}(t) - B_i\mathbf{y}(t) - \mathbf{m}$, with K being an adjusted control gain matrix.
>
> R^i: If $x_1(t)$ is F_1^i and $\cdots$ and $x_n(t)$ is F_n^i, then $\widetilde{\mathbf{u}}(t) = A_i\mathbf{x}(t)$.

Let us denote $\mathbf{u}_1(t) = -\sum_{i=1}^N \mu_i(\mathbf{y}(t))[T\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) + H_i\operatorname{sign}(\mathbf{e}(t))]$, where $T = \operatorname{diag}[\operatorname{sign}(\mathbf{e}_1(t)), \cdots, \operatorname{sign}(\mathbf{e}_n(t))]$, $H_i = \operatorname{diag}[\widehat{\varepsilon}_i^{*1}(\mathbf{y}), \cdots, \widehat{\varepsilon}_i^{*n}(\mathbf{y})]$.

In order to update parametric estimates, the fractional adaptation laws are designed as

$${}_0^C D_t^\mu \widetilde{\theta}_i^j(t) = \alpha_i \mu_i(\mathbf{y}(t))\left|e_j(t)\right|\varphi_i^j(\mathbf{y}), \tag{18}$$

$${}_0^C D_t^\mu \widehat{\varepsilon}_i^{*j}(\mathbf{y}) = \beta_i \mu_i(\mathbf{y}(t))\left|e_j(t)\right|, \tag{19}$$

with $\alpha_i, \beta_i > 0$ being adaptation rates which are constant parameters. Taking the control law (17) into (12) and letting $K = \operatorname{diag}[k_1, \cdots, k_n]$ $(k_j > 0)$, we have

$$\begin{aligned}{}_0^C D_t^\mu \mathbf{e}(t) &= -K\mathbf{e}(t) + \sum_{i=1}^N \mu_i(\mathbf{y}(t)) \\ &\quad \cdot [\mathbf{d}_i(t, \mathbf{y}) - T\widehat{\boldsymbol{\rho}}_i(\mathbf{y}, \boldsymbol{\theta}_i(t)) - H_i\operatorname{sign}(\mathbf{e}(t))].\end{aligned} \tag{20}$$

Multiplying both sides of (20) by $\mathbf{e}^T(t)$ and letting $\widetilde{\varepsilon}_i^{*j}(\mathbf{y}) = \widehat{\varepsilon}_i^{*j}(\mathbf{y}) - \varepsilon_i^{*j}$, one gets

$$\begin{aligned}\mathbf{e}^T(t)\,{}_0^C D_t^\mu \mathbf{e}(t) &\le -\mathbf{e}^T(t)K\mathbf{e}(t) - \sum_{i=1}^N \mu_i(\mathbf{y}(t)) \\ &\quad \cdot \sum_{j=1}^n \left|e_j(t)\right|\left[\widehat{\varepsilon}_i^{*j}(\mathbf{y}) + \widehat{\rho}_i^j(\mathbf{y}, \theta_i^j(t)) - \rho_i^j(\mathbf{y})\right] \\ &\le -\mathbf{e}^T(t)K\mathbf{e}(t) + \sum_{i=1}^N \mu_i(\mathbf{y}(t)) \\ &\quad \cdot \sum_{j=1}^n \left|e_j(t)\right|\left[\varepsilon_i^{*j} - \widehat{\varepsilon}_i^{*j}(\mathbf{y}) - \widetilde{\theta}_i^j(t)\varphi_i^j(\mathbf{y})\right] = -\mathbf{e}^T(t) \\ &\quad \cdot K\mathbf{e}(t) - \sum_{i=1}^N \mu_i(\mathbf{y}(t)) \\ &\quad \cdot \sum_{j=1}^n \left|e_j(t)\right|\left[\widetilde{\varepsilon}_i^{*j}(\mathbf{y}) + \widetilde{\theta}_i^j(t)\varphi_i^j(\mathbf{y})\right].\end{aligned} \tag{21}$$

3.3. Stability Analysis

3.3. Stability Analysis. Here, fractional Lyapunov's theory is used to analyze the stability in closed-loop system. The following Lemmas are proposed to simplify the stability analysis.

Lemma 9 (see [54]). *If $\mu \in (0, 1)$ and $\mathbf{x}(t) \in C^1$, then ${}_0^C D_t^\mu \mathbf{x}(t)^T\mathbf{x}(t) \le 2\mathbf{x}(t)^T\,{}_0^C D_t^\mu \mathbf{x}(t)$.*

Lemma 10. *If ${}_0^C D_t^\mu x(t) \le 0$, then one gets $\mathbf{x}(t) \le x(0)$; if ${}_0^C D_t^\mu x(t) \ge 0$, then one gets $x(t) \ge x(0)$, with $\mu \in (0, 1)$ and $t \in [0, +\infty)$.*

Proof. We only consider the front part. If ${}_0^C D_t^\mu x(t) \le 0$, let

$$h(t) = -{}_0^C D_t^\mu x(t). \tag{22}$$

Both sides of (22) take Laplace transform and one obtains

$$\mathscr{L}\left[{}_0^C D_t^\mu x(t)\right] + H(s) = 0. \tag{23}$$

Using Property 3, one obtains the following:

$$s^\mu X(s) - s^{\mu-1}x(0) + H(s) = 0. \tag{24}$$

Further, one gets

$$X(s) = \frac{x(0)}{s} - \frac{H(s)}{s^\mu}, \tag{25}$$

with $X(s) = \mathscr{L}[x(t)], H(s) = \mathscr{L}[h(t)]$. Both sides of (25) make Laplace inverse transform and using the fractional integral definition, one obtains

$$\begin{aligned}x(t) &= x(0) - {}_0^C D_t^{-\mu}h(t) \\ &= x(0) - \frac{1}{\Gamma(\mu)}\int_0^t (t-\xi)^{\mu-1}h(\xi)\,d\xi.\end{aligned} \tag{26}$$

From the above equation, we get $x(t) \le x(0), t \in [0, +\infty)$. □

Lemma 11. *Let* $V(t) = (1/2)\mathbf{x}(t)^T\mathbf{x}(t) + (1/2)\mathbf{y}(t)^T\mathbf{y}(t)$ *with* $\mathbf{x}(t), \mathbf{y}(t) \in R^n$ *be continuous and derivable functions. If there exists a constant* $h > 0$ *such that*

$$ {}_0^C D_t^{\mu} V(t) \leq -h\mathbf{x}(t)^T \mathbf{x}(t), \tag{27} $$

then $\|\mathbf{x}(t)\|$ *and* $\|\mathbf{y}(t)\|$ *are bounded and* $\lim_{t\longrightarrow+\infty}\|\mathbf{x}(t)\| = 0$.

Proof. According to Lemma 10 and ${}_0^C D_t^{\mu} V(t) \leq -h\mathbf{x}(t)^T\mathbf{x}(t) \leq 0$, we obtain $V(t) \leq V(0)$. Further, we get the following:

$$ \begin{aligned} \|\mathbf{x}(t)\| &\leq \sqrt{2V(0)}, \\ \|\mathbf{y}(t)\| &\leq \sqrt{2V(0)}. \end{aligned} \tag{28} $$

This means that $\|\mathbf{x}(t)\|$ and $\|\mathbf{y}(t)\|$ are bounded.

We will proof that $\mathbf{x}(t)$ tends to $\mathbf{0}$ asymptotically below. Both sides of (27) commute with μ-order integral; based on Property 2, one gets

$$ V(t) - V(0) \leq -h\, {}_0^C D_t^{-\mu} \mathbf{x}(t)^T \mathbf{x}(t). \tag{29} $$

Further, one obtains

$$ \mathbf{x}(t)^T \mathbf{x}(t) \leq 2V(0) - 2h\, {}_0^C D_t^{-\mu} \mathbf{x}(t)^T \mathbf{x}(t). \tag{30} $$

Hence, the following can be obtained from (30) with a nonnegative function $z(t)$ as

$$ \mathbf{x}(t)^T \mathbf{x}(t) + z(t) = 2V(0) - 2h\, {}_0^C D_t^{-\mu} \mathbf{x}(t)^T \mathbf{x}(t). \tag{31} $$

Applying the Laplace transform to formula (31) and according to the Definition 4, we have

$$ \begin{aligned} \mathscr{L}\left[\mathbf{x}(t)^T \mathbf{x}(t)\right] &= 2V(0)\frac{s^{\mu-1}}{s^{\mu}+2h} - Z(s)\frac{s^{\mu}}{s^{\mu}+2h} \\ &= \mathscr{L}\left[2V(0) E_{\mu,1}\left(-2ht^{\mu}\right)\right] \\ &\quad - \mathscr{L}[z(t)]\, \mathscr{L}\left[t^{-1} E_{\mu,0}\left(-2ht^{\mu}\right)\right]. \end{aligned} \tag{32} $$

Hence, using the Laplace inverse transform to (32), we have

$$ \begin{aligned} \mathbf{x}(t)^T \mathbf{x}(t) &= 2V(0) E_{\mu,1}\left(-2ht^{\mu}\right) - z(t) \\ &\quad * \left[t^{-1}\right] E_{\mu,0}\left(-2ht^{\mu}\right), \end{aligned} \tag{33} $$

with $*$ being the convolution operator. Since t^{-1} and $E_{\mu,0}(-2ht^{\mu})$ are nonnegative functions, then $\mathbf{x}(t)^T\mathbf{x}(t) \leq 2V(0)E_{\mu,1}(-2ht^{\mu})$. According to the results in [55], one obtains that $\mathbf{x}(t)$ is M-L stability and $\mathbf{x}(t)$ tends to 0 asymptotically (namely, $\lim_{t\longrightarrow\infty}\|\mathbf{x}(t)\| = 0$). □

From above discussion, the boundedness of all signals in closed-loop system and the convergence of tracking error based on adaptive fuzzy control scheme via T-S fuzzy logic systems is presented in the following theorem.

Theorem 12. *For the master system (9) and slave system (10) under the known initial conditions, if Assumptions 5 and 6 are satisfied and the adaptive controller is given as (17) with the fractional adaptation laws (18) and (19), then all signals in the closed-loop system are bounded and the tracking error signal tends to zero asymptotically.*

Proof. Define the following Lyapunov function:

$$ \begin{aligned} V(t) &= \frac{1}{2}\mathbf{e}^T(t)\mathbf{e}(t) + \sum_{i=1}^{N}\frac{1}{2\alpha_i}\widetilde{\boldsymbol{\theta}}_i^T(t)\widetilde{\boldsymbol{\theta}}_i(t) \\ &\quad + \sum_{i=1}^{N}\frac{1}{2\beta_i}\widetilde{\boldsymbol{\varepsilon}}_i^{*T}(\mathbf{y})\widetilde{\boldsymbol{\varepsilon}}_i^{*}(\mathbf{y}), \end{aligned} \tag{34} $$

with $\widetilde{\boldsymbol{\theta}}_i(t) = \boldsymbol{\theta}_i(t) - \boldsymbol{\theta}_i^*$ and $\widetilde{\boldsymbol{\varepsilon}}_i^*(\mathbf{y}) = \widehat{\boldsymbol{\varepsilon}}_i^*(\mathbf{y}) - \boldsymbol{\varepsilon}_i^*$. Hence, using the Lemma 9, the μ-order derivative of $V(t)$ with respect to time t is obtained as

$$ \begin{aligned} {}_0^C D_t^{\mu} V(t) &\leq \mathbf{e}^T(t)\, {}_0^C D_t^{\mu}\mathbf{e}(t) + \sum_{i=1}^{N}\frac{1}{\alpha_i}\widetilde{\boldsymbol{\theta}}_i^T(t)\, {}_0^C D_t^{\mu}\widetilde{\boldsymbol{\theta}}_i(t) \\ &\quad + \sum_{i=1}^{N}\frac{1}{\beta_i}\widetilde{\boldsymbol{\varepsilon}}_i^{*T}(\mathbf{y})\, {}_0^C D_t^{\mu}\widetilde{\boldsymbol{\varepsilon}}_i^{*}(\mathbf{y}) \end{aligned} \tag{35} $$

Substituting (21) into (35), one gets

$$ \begin{aligned} {}_0^C D_t^{\mu} V(t) &\leq -\mathbf{e}^T(t) K\mathbf{e}(t) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{n}\left[\frac{1}{\alpha_i}\widetilde{\theta}_i^j(t)\, {}_0^C D_t^{\mu}\widetilde{\theta}_i^j(t)\right. \\ &\quad - \mu_i(\mathbf{y}(t))\left|e_j(t)\right|\widetilde{\theta}_i^j(t)\varphi_i^j(\mathbf{y}) \\ &\quad + \frac{1}{\beta_i}\widetilde{\varepsilon}_i^{*j}(\mathbf{y})\, {}_0^C D_t^{\mu}\widetilde{\varepsilon}_i^{*j}(\mathbf{y}) \\ &\quad \left. - \mu_i(\mathbf{y}(t))\left|e_j(t)\right|\widetilde{\varepsilon}_i^{*j}(\mathbf{y})\right], \end{aligned} \tag{36} $$

Taking (18) and (19) into (36), one gets the following inequality:

$$ {}_0^C D_t^{\mu} V \leq -\mathbf{e}^T(t) K\mathbf{e}(t) \leq -\lambda_{min}\mathbf{e}^T(t)\mathbf{e}(t), \tag{37} $$

where λ_{min} is the least eigenvalue of the positive definite matrix K. According to Lemma 11 and above discussion, we know that the tracking error signal $\mathbf{e}(t)$ tends to 0 asymptotically (that is, $\lim_{t\longrightarrow\infty}\|\mathbf{e}(t)\| = 0$) and $\widetilde{\boldsymbol{\theta}}_i(t)$ and $\widetilde{\boldsymbol{\varepsilon}}_i^*(\mathbf{y})$ are bounded. Further, it means that $\boldsymbol{\theta}_i(t)$ and $\widehat{\boldsymbol{\varepsilon}}_i^*(\mathbf{y})$ are bounded. Because of the boundedness of $\mathbf{e}(t)$ and $\mathbf{x}(t)$, we know that $\mathbf{y}(t)$ is bounded. Based on the control design, $\mathbf{u}(t)$ is bounded. Therefore, we know that all signals in the closed-loop system are bounded. □

4. Simulation Example

In this section, in order to further illustrate the effectiveness of the proposed control method designed in previous sections, one example about the synchronization for two different uncertain fractional-order chaotic system is given. The master system of a fractional-order chaotic system via T-S fuzzy model is given as

$$ {}_0^C D_t^{\mu}\mathbf{x}(t) = \sum_{i=1}^{N}\mu_i(\mathbf{x}(t))[A_i\mathbf{x}(t)+\mathbf{b}_1], \quad (38)$$

the ith rule of master system is given by

R^1: If x_1 is F_1^1 and x_2 is F_2^1 and x_3 is F_3^1, then ${}_0^C D_t^{0.8}\mathbf{x}(t) = A_1\mathbf{x}(t)+\mathbf{b}_1$,

R^2: If x_1 is F_1^2 and x_2 is F_2^2 and x_3 is F_3^2, then ${}_0^C D_t^{0.8}\mathbf{x}(t) = A_2\mathbf{x}(t)+\mathbf{b}_1$.

The upper system is formulated to the alike form in (9) with

$$\begin{gathered} A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & -62.6 \\ 25.04 & 0 & -8 \end{bmatrix}, \\ A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & 60.7 \\ -24.28 & 0 & -8 \end{bmatrix}, \\ \mathbf{b}_1(t) = (0,0,0)^T. \end{gathered} \quad (39)$$

Figure 1 depicts the simulation results of the master system with the parameters $N = 2, \mu = 0.8$ with time step $h = 0.005$. Figure 1 shows $x_1(t)$, $x_2(t)$ and $x_3(t) \in [-6.07, 6.26]$, $[-10, 10.6]$, and $[0.2, 9.05]$, respectively, that is, $\mathbf{x}(t) \in D_1 = [-6.07, 6.26]\times[-10, 10.6]\times[0.2, 9.05]$. Obviously, Chaos was found in system (38) with $\mu = 0.8$.

Two fuzzy sets are defined for the state x_1 over the interval $[-6.07, 6.26]$ with the membership functions as

$$\begin{gathered} F_1^1(x_1(t)) = \frac{1}{2}\left(1-\frac{0.095-x_1(t)}{6.165}\right), \\ F_1^2(x_1(t)) = \frac{1}{2}\left(1+\frac{0.095-x_1(t)}{6.165}\right). \end{gathered} \quad (40)$$

Two fuzzy sets are defined for the state x_2 over the interval $[-10, 10.6]$ with the membership functions as

$$\begin{gathered} F_2^1(x_2(t)) = \frac{1}{2}\left(1-\frac{0.3-x_2(t)}{10.3}\right), \\ F_2^2(x_2(t)) = \frac{1}{2}\left(1-\frac{0.3-x_2(t)}{10.3}\right). \end{gathered} \quad (41)$$

Two fuzzy sets are defined for the state x_3 over the interval $[0.2, 9.05]$ with the membership functions as

$$\begin{gathered} F_3^1(x_3(t)) = \frac{1}{2}\left(1-\frac{4.625-x_3(t)}{4.425}\right), \\ F_3^2(x_3(t)) = \frac{1}{2}\left(1-\frac{4.625-x_3(t)}{4.425}\right). \end{gathered} \quad (42)$$

The slave system of a fractional-order chaotic system with unknown disturbances via T-S fuzzy model is given as

$$\begin{aligned} &{}_0^C D_t^{0.8}\mathbf{y}(t) \\ &\quad = \sum_{i=1}^{2}\mu_i(\mathbf{y}(t))[B_i\mathbf{y}(t)+\mathbf{b}_2+\mathbf{u}(t)+\mathbf{d}_i(t,\mathbf{y})]. \end{aligned} \quad (43)$$

The ith rule of slave system is given by

R^1: If y_1 is $\widehat{F}_1^1$ and y_2 is $\widehat{F}_2^1$ and y_3 is $\widehat{F}_3^1$, then ${}_0^C D_t^{0.8}\mathbf{y}(t) = B_1\mathbf{y}(t)+\mathbf{b}_2+\mathbf{u}(t)+\mathbf{d}_1(t,\mathbf{y})$,

R^2: If y_1 is $\widehat{F}_1^2$ and y_2 is $\widehat{F}_2^2$ and y_3 is $\widehat{F}_3^2$, then ${}_0^C D_t^{0.8}\mathbf{y}(t) = B_2\mathbf{y}(t)+\mathbf{b}_2+\mathbf{u}(t)+\mathbf{d}_2(t,\mathbf{y})$.

The upper system is formulated to the alike form in (10) with

$$\begin{gathered} B_1 = \begin{bmatrix} -30 & 30 & 0 \\ 0 & 22.2 & -21.52 \\ 0 & 21.52 & -2.94 \end{bmatrix}, \\ B_2 = \begin{bmatrix} -30 & 30 & 0 \\ 0 & 22.2 & 29.21 \\ 0 & -29.21 & -2.94 \end{bmatrix}, \\ \mathbf{b}_2(t) = (0,0,0)^T. \end{gathered} \quad (44)$$

Figure 2 with $\mathbf{u}(t) = \mathbf{0}$ and without the external disturbance is depicted the simulation results of the slave system with the parameters below: $N = 2, \mu = 0.8$, for time step $h = 0.005$. Moreover, Chaos was found in system (43) with $\mu = 0.8$. Figure 2 shows $y_1(t), y_2(t)$ and $y_3(t) \in [-29.21, 21.52], [-35.6, 26.5]$ and $[0, 53.6]$, respectively, that is, $\mathbf{y}(t) \in D_2 = [-29.21, 21.52]\times[-35.6, 26.5]\times[0, 53.6]$.

Two fuzzy sets are defined for the state y_1 over the interval $[-29.21, 21.52]$ with the membership functions as follows:

$$\begin{gathered} \widehat{F}_1^1(y_1(t)) = \frac{1}{2}\left(1-\frac{-3.845-y_1(t)}{25.365}\right), \\ \widehat{F}_1^2(y_1(t)) = \frac{1}{2}\left(1+\frac{-3.845-y_1(t)}{25.365}\right). \end{gathered} \quad (45)$$

Two fuzzy sets are defined for the state y_2 over the interval $[-35.6, 26.5]$ with the membership functions as follows:

$$\begin{gathered} \widehat{F}_2^1(y_2(t)) = \frac{1}{2}\left(1-\frac{-4.55-y_2(t)}{31.05}\right), \\ \widehat{F}_2^2(y_2(t)) = \frac{1}{2}\left(1-\frac{-4.55-y_2(t)}{31.05}\right). \end{gathered} \quad (46)$$

Two fuzzy sets are defined for the state y_3 over the interval $[0, 53.6]$ with the membership functions as follows:

$$\begin{gathered} \widehat{F}_3^1(y_3(t)) = \frac{1}{2}\left(1-\frac{26.8-y_3(t)}{26.8}\right), \\ \widehat{F}_3^2(y_3(t)) = \frac{1}{2}\left(1-\frac{26.8-y_3(t)}{26.8}\right). \end{gathered} \quad (47)$$

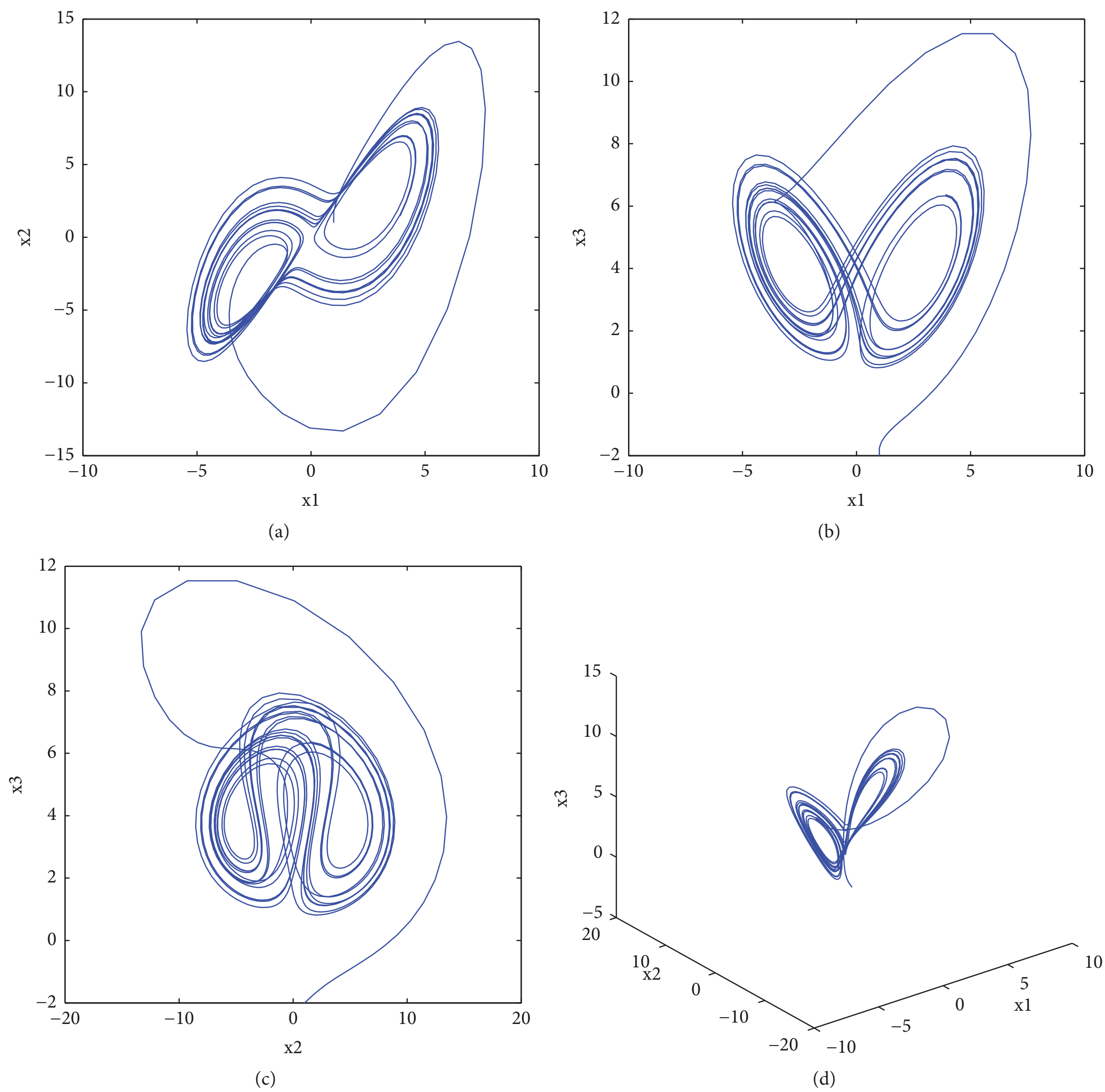


FIGURE 1: Master system.

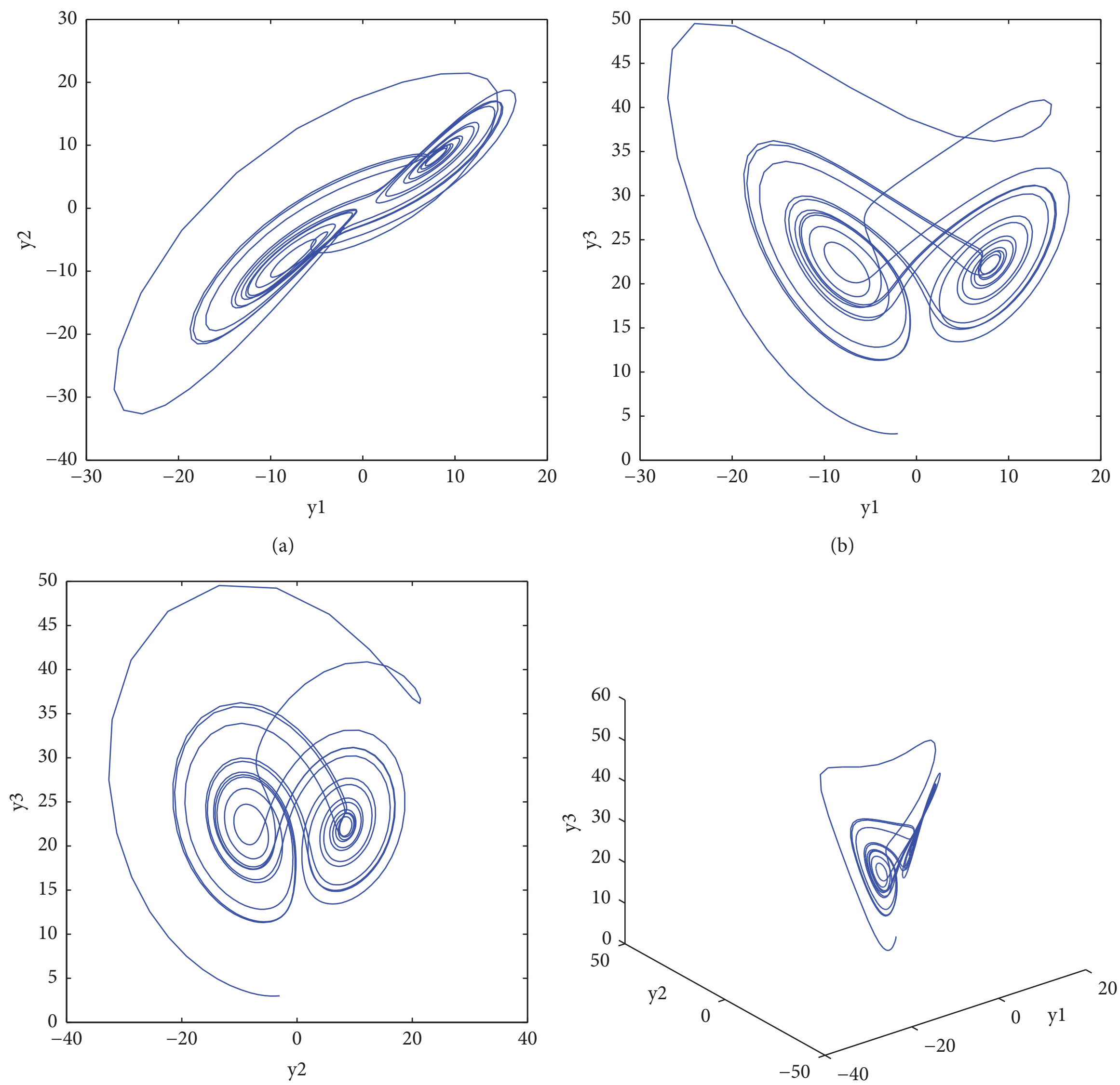
In the simulation, the initial conditions of master system and slave system are selected as $\mathbf{x}(0) = (1, 1, -2)^T$ and $\mathbf{y}(0) = (-2, -3, 3)^T$. The parameters relating the synchronization problem are set to $K = E$ and $\boldsymbol{\rho}_1(y) = \boldsymbol{\rho}_2(y) = (1.5y_1, 1.5y_2, 1.5y_3)^T$. Let $\mathbf{e}(t) = (e_1(t), e_2(t), e_3(t))^T$.

The controller is designed as

$$\begin{aligned}\mathbf{u}(t) &= (u_1(t), u_2(t), u_3(t))^T \\ &= -\widehat{F}_1^1(y_1(t)) B_1\mathbf{y}(t) - T\widehat{\boldsymbol{\rho}}(\mathbf{y}, \boldsymbol{\theta}(t)) - \widehat{F}_1^2 B_2\mathbf{y}(t) \\ &\quad - H\text{sign}(\mathbf{e}(t)) - \mathbf{e}(t) + F_1^1 A_1\mathbf{x}(t) \\ &\quad + F_1^2 A_2\mathbf{x}(t).\end{aligned} \tag{48}$$

Let $\widehat{\boldsymbol{\rho}}(\mathbf{y}, \boldsymbol{\theta}(t)) = (\widehat{\rho}^1, \widehat{\rho}^2, \widehat{\rho}^3)^T$, $H = (\widehat{\varepsilon}^{*1}(\mathbf{y}), \widehat{\varepsilon}^{*2}(\mathbf{y})$, and $\widehat{\varepsilon}^{*3}(\mathbf{y}))^T$; then

$$\begin{aligned}u_1(t) &= -e_1(t) - 10x_1 + 10x_2 + 30y_1 - 30y_2 - T\widehat{\rho}^1 \\ &\quad - \widehat{\varepsilon}^{*1}(\mathbf{y}), \\ u_2(t) &= -e_2(t) + 40x_1 - 10x_1x_3 - 22.2y_2 + y_1y_3 \\ &\quad - T\widehat{\rho}^2 - \widehat{\varepsilon}^{*2}(\mathbf{y}), \\ u_3(t) &= -e_3(t) - 8x_3 + 4x_1^2 - y_1y_2 + 2.94y_3 - T\widehat{\rho}^3 \\ &\quad - \widehat{\varepsilon}^{*3}(\mathbf{y}).\end{aligned} \tag{49}$$



(a) (b) (c) (d)

Figure 2: Slave system.

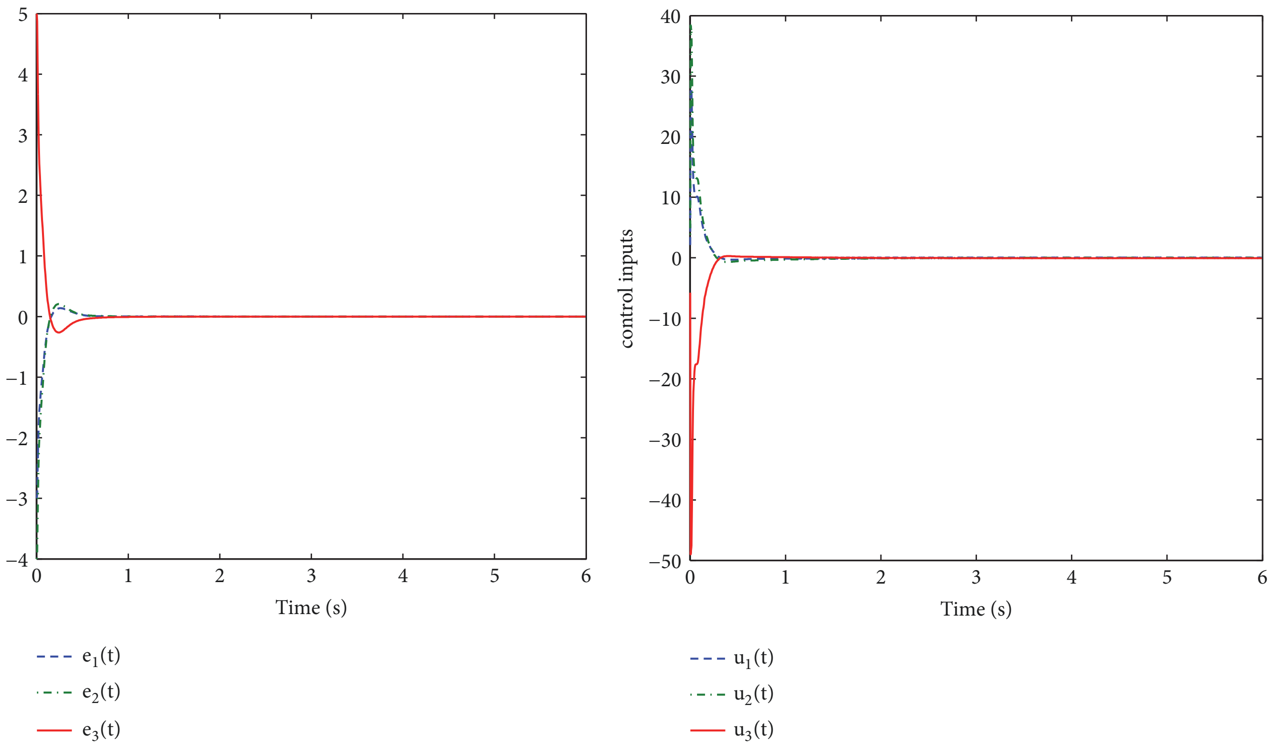
The fractional adaptation laws of $\widetilde{\theta}^j(\mathbf{y})$ and $\widetilde{\varepsilon}^{*j}(\mathbf{y}), (j = 1, 2, 3)$, with $\alpha = 700, \beta = 0.05$ are designed to be

$$\begin{aligned}
{}_0^C D_t^{\mu} \widetilde{\theta}^1(t) &= \alpha \left| e_1(t) \right| \varphi^1(\mathbf{y}), \\
{}_0^C D_t^{\mu} \widetilde{\varepsilon}^{*1}(\mathbf{y}) &= \beta \left| e_1(t) \right|, \\
{}_0^C D_t^{\mu} \widetilde{\theta}^2(t) &= \alpha \left| e_2(t) \right| \varphi^2(\mathbf{y}), \\
{}_0^C D_t^{\mu} \widetilde{\varepsilon}^{*2}(\mathbf{y}) &= \beta \left| e_2(t) \right|, \\
{}_0^C D_t^{\mu} \widetilde{\theta}^3(t) &= \alpha \left| e_3(t) \right| \varphi^3(\mathbf{y}), \\
{}_0^C D_t^{\mu} \widetilde{\varepsilon}^{*3}(\mathbf{y}) &= \beta \left| e_3(t) \right|.
\end{aligned} \tag{50}$$

The simulation results of the proposed adaptive control approach are shown in Figure 3, where subgraph (a) denotes the tracking error trajectory and subgraph (b) denotes the control trajectory. Define the initial conditions of the approximation errors as $\widehat{\varepsilon}^{*1}(0) = 0, \widehat{\varepsilon}^{*2}(0) = 0, \widehat{\varepsilon}^{*3}(0) = 0$. In reducing the computation of the numerical simulation, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are replaced by $\mathbf{e}(t)$. Four fuzzy sets are defined for the tracking errors $e_1(t), e_2(t), e_3(t)$ over the interval $[-3, 3]$ with the Gaussian membership functions, where the first parameters are 1.1 and the second parameters are $-3, -1, 1, 3$, respectively. Comparing the conventional control method with the proposed method, we can see that the proposed approach can synchronize two chaotic plants to desired high accuracy and improve the performance as shown in Figure 3.

5. Conclusions

In this paper, synchronization of different fractional-order chaotic or hyperchaotic systems with unknown disturbances



(a) (b)

Figure 3: (a) Synchronization error and (b) controller.

and parametric uncertainties is addressed with adaptive fuzzy control algorithm based on T-S fuzzy models. The distinctive features of the proposed control approach are that T-S fuzzy logic systems are introduced to approximate the unknown disturbances and to model the unknown controlled systems; both adaptive fuzzy controller and fractional adaptation laws are developed based on combined fractional Lyapunov stability theory and parallel distributed compensation technique. It is shown that the proposed control method can guarantee that all the signals in the closed-loop system remain bounded and the synchronization error converges towards an arbitrary small neighbourhood of the origin asymptotically. A simulation example is used for verifying the effectiveness of the proposed control strategy. Further works would focus on chaos synchronization control of different uncertain fractional-order chaotic systems with time delay and input saturation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors do not have a direct financial relation with any commercial identity mentioned in their paper that might lead to conflicts of interest for any of the authors.

Acknowledgments

This work is supported by the Natural Science Foundation of Anhui Province of China under Grant 1808085MF181.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[2] X. Hao, H. Wang, L. Liu, and Y. Cui, "Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and *p*-Laplacian operator," *Boundary Value Problems*, vol. 2017, no. 1, Article ID 182, 2017.

[3] X. Zhang, L. Liu, Y. Wu, and Y. Cui, "New Result on the Critical Exponent for Solution of an Ordinary Fractional Differential Problem," *Journal of Function Spaces*, vol. 2017, Article ID 3976469, 4 pages, 2017.

[4] R. Xu and Y. Zhang, "Generalized Gronwall fractional summation inequalities and their applications," *Journal of Inequalities and Applications*, vol. 2015, no. 1, p. 242, 2015.

[5] X. Zhang, C. Mao, L. Liu, and Y. Wu, "Exact iterative solution for an abstract fractional dynamic system model for bioprocess," *Qualitative Theory of Dynamical Systems*, vol. 16, no. 1, pp. 205–222, 2017.

[6] Q. Feng and F. Meng, "Traveling wave solutions for fractional partial differential equations arising in mathematical physics by an improved fractional Jacobi elliptic equation method," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 10, pp. 3676–3686, 2017.

[7] L. Zhang and Z. Zheng, "Lyapunov type inequalities for the Riemann-Liouville fractional differential equations of higher order," *Advances in Difference Equations*, vol. 2017, no. 1, p. 270, 2017.

[8] R. Xu and F. Meng, "Some new weakly singular integral inequalities and their applications to fractional differential equations," *Journal of Inequalities and Applications*, vol. 2016, no. 1, p. 78, 2016.

[9] J. Wang, Y. Yuan, and S. Zhao, "Fractional factorial split-plot designs with two- and four-level factors containing clear effects," *Communications in Statistics—Theory and Methods*, vol. 44, no. 4, pp. 671–682, 2015.

[10] H. Liu, S. Li, J. D. Cao, A. G. Alsaedi, and F. E. Alsaadi, "Adaptive fuzzy prescribed performance controller design for a class of uncertain fractional-order nonlinear systems with external disturbances," *Neurocomputing*, vol. 219, pp. 422–430, 2017.

[11] H. Liu, S. Li, G. Li, and H. Wang, "Adaptive Controller Design for a Class of Uncertain Fractional-Order Nonlinear Systems: An Adaptive Fuzzy Approach," *International Journal of Fuzzy Systems*, pp. 1–14, 2017.

[12] J.-S. Gao, G. Song, and L.-W. Deng, "Finite-time sliding mode synchronization control of chaotic systems with uncertain parameters," *Kongzhi yu Juece/Control and Decision*, vol. 32, no. 1, pp. 149–156, 2017.

[13] H. Liu, Y. Pan, S. Li, and Y. Chen, "Synchronization for fractional-order neural networks with full/under-actuation using fractional-order sliding mode control," *International Journal of Machine Learning and Cybernetics*, pp. 1–14, 2017.

[14] S. Das and V. K. Yadav, "Stability analysis, chaos control of fractional order Vallis and El-Nino systems and their synchronization," *IEEE/CAA Journal of Automatica Sinica*, vol. 4, no. 1, pp. 114–124, 2017.

[15] Y. Pan, M. J. Er, T. Sun, B. Xu, and H. Yu, "Adaptive fuzzy pd control with stable h tracking guarantee," *Neurocomputing*, vol. 237, pp. 71–78, 2017.

[16] F. Qiao, Q. Zhu, and B. Zhang, *Fuzzy Sliding Mode Control and Observation of Complex Dynamic Systems and Applications*, Bingjing Institute of Technology Press, 2013.

[17] H. Wu, "Liouville-type theorem for a nonlinear degenerate parabolic system of inequalities," *Mathematical Notes*, vol. 103, no. 1-2, pp. 155–163, 2018.

[18] L. Liu, F. Sun, X. Zhang, and Y. Wu, "Bifurcation analysis for a singular differential system with two parameters via to topological degree theory," *Nonlinear Analysis: Modelling and Control*, vol. 22, no. 1, pp. 31–50, 2017.

[19] Y. Sun, L. Liu, and Y. Wu, "The existence and uniqueness of positive monotone solutions for a class of nonlinear Schrödinger equations on infinite domains," *Journal of Computational and Applied Mathematics*, vol. 321, pp. 478–486, 2017.

[20] R. Xu and X. Ma, "Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications," *Journal of Inequalities and Applications*, vol. 2017, article no. 187, 2017.

[21] X. Peng, Y. Shang, and X. Zheng, "Lower bounds for the blow-up time to a nonlinear viscoelastic wave equation with strong damping," *Applied Mathematics Letters*, vol. 76, pp. 66–73, 2018.

[22] Y. Sun and F. Meng, "Reachable Set Estimation for a Class of Nonlinear Time-Varying Systems," *Complexity*, vol. 2017, Article ID 5876371, 6 pages, 2017.

[23] D. Feng, M. Sun, and X. Wang, "A family of conjugate gradient methods for large-scale nonlinear equations," *Journal of Inequalities and Applications*, vol. 236, 2017.

[24] X. Lin and Z. Zhao, "Existence and uniqueness of symmetric positive solutions of 2n-order nonlinear singular boundary value problems," *Applied Mathematics Letters*, vol. 26, no. 7, pp. 692–698, 2013.

[25] H. Liu, S. Li, H. Wang, and Y. Sun, "Adaptive fuzzy control for a class of unknown fractional-order neural networks subject to input nonlinearities and dead-zones," *Information Sciences*, vol. 454-455, pp. 30–45, 2018.

[26] Y. Pan and H. Yu, "Biomimetic hybrid feedback feedforward neural-network learning control," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 6, pp. 1481–1487, 2017.

[27] Y. Pan and H. Yu, "Composite learning robot control with guaranteed parameter convergence," *Automatica*, vol. 89, pp. 398–406, 2018.

[28] K. Sun, Y. Li, and S. Tong, "Fuzzy Adaptive Output Feedback Optimal Control Design for Strict-Feedback Nonlinear Systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 1, pp. 33–44, 2017.

[29] Y. Li and S. Tong, "Command-filtered-based fuzzy adaptive control design for MIMO-switched nonstrict-feedback nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 3, pp. 668–681, 2017.

[30] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 15, no. 1, pp. 116–124, 1985.

[31] T. A. Johansen, R. Shorten, and R. Murray-Smith, "On the interpretation and identification of dynamic Takagi-Sugeno fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 3, pp. 297–313, 2000.

[32] H. Ying, "General SISO Takagi-Sugeno fuzzy systems with linear rule consequent are universal approximators," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 4, pp. 582–587, 1998.

[33] A. E. Matouk, "Chaos synchronization of a fractional-order modified van der Pol-Duffing system via new linear control, backstepping control and Takagi-Sugeno fuzzy approaches," *Complexity*, vol. 21, no. S1, pp. 116–124, 2016.

[34] Z. Qi-Shui, B. Jing-Fu, Y. Yong-Bin, and L. Xiao-Feng, "Impulsive Control for Fractional-Order Chaotic Systems," *Chinese Physics Letters*, vol. 25, no. 8, pp. 2812–2815, 2008.

[35] A. Bouzeriba, A. Boulkroune, and T. Bouden, "Projective synchronization of two different fractional-order chaotic systems via adaptive fuzzy control," *Neural Computing and Applications*, vol. 27, no. 5, pp. 1349–1360, 2016.

[36] B. Mao and Q. Li, "Chaos synchronization between different fractional order systems with uncertain parameters," *Periodical of Ocean University of China*, vol. 47, no. 7, pp. 149–152, 2017.

[37] S. G. Cao, N. W. Rees, and G. Feng, "Quadratic stability analysis and design of continuous-time fuzzy control systems," *International Journal of Systems Science*, vol. 27, no. 2, pp. 193–203, 1996.

[38] I. Petráš, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Springer Science & Business Media, 2011.

[39] T. Shen, J. Xin, and J. Huang, "Time-space fractional stochastic Ginzburg-Landau equation driven by gaussian white noise," *Stochastic Analysis and Applications*, vol. 36, no. 1, pp. 103–113, 2018.

[40] Y. Wang, L. Liu, and Y. Wu, "Existence and uniqueness of a positive solution to singular fractional differential equations," *Boundary Value Problems*, vol. 2012, no. 1, p. 81, 2012.

[41] M. Li and J. Wang, "Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations," *Applied Mathematics and Computation*, vol. 324, pp. 254–265, 2018.

[42] J. Zhang, Z. Lou, Y. Ji, and W. Shao, "Ground state of Kirchhoff type fractional Schrödinger equations with critical growth," *Journal of Mathematical Analysis and Applications*, vol. 462, no. 1, pp. 57–83, 2018.

[43] L. Guo, L. Liu, and Y. Wu, "Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions," *Nonlinear Analysis, Modelling and Control*, vol. 21, no. 5, pp. 635–650, 2016.

[44] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters," *Advances in Difference Equations*, vol. 2014, no. 1, p. 268, 2014.

[45] Y. Yuan and S.-l. Zhao, "Mixed two- and eight-level fractional factorial split-plot designs containing clear effects," *Acta Mathematicae Applicatae Sinica*, vol. 32, no. 4, pp. 995–1004, 2016.

[46] J. Wu, X. Zhang, L. Liu, and Y. Wu, "Positive solution of singular fractional differential system with nonlocal boundary conditions," *Advances in Difference Equations*, vol. 2014, no. 1, p. 323, 2014.

[47] X. Zhang, L. Liu, and Y. Wu, "Variational structure and multiple solutions for a fractional advection-dispersion equation," *Computers & Mathematics with Applications*, vol. 68, no. 12, pp. 1794–1805, 2014.

[48] L. Wang, *Adaptive Fuzzy Systems and Control: Design and Stability Analysis*, Englewood Cliffs, 1994.

[49] L. Merazka and A. Boulkroune, "Adaptive fuzzy state-feedback control for a class of multivariable nonlinear systems," *IEEE*, pp. 925–931, 2016.

[50] A. Boulkroune, M. Tadjine, M. M'Saad, and M. Farza, "Fuzzy adaptive controller for MIMO nonlinear systems with known and unknown control direction," *Fuzzy Sets and Systems*, vol. 161, no. 6, pp. 797–820, 2010.

[51] H. Liu, Y. Pan, S. Li, and Y. Chen, "Adaptive Fuzzy Backstepping Control of Fractional-Order Nonlinear Systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 8, pp. 2209–2217, 2017.

[52] Y. Li and S. Tong, "Adaptive Neural Networks Prescribed Performance Control Design for Switched Interconnected Uncertain Nonlinear Systems," *IEEE Transactions on Neural Networks and Learning Systems*, pp. 1–10, 2017.

[53] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice Hall PTR, Upper Saddle River, 2001.

[54] N. Aguila-Camacho, M. A. Duarte-Mermoud, and J. A. Gallegos, "Lyapunov functions for fractional order systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 9, pp. 2951–2957, 2014.

[55] Y. Li, Y. Chen, and I. Podlubny, "Mittag-Leffler stability of fractional order nonlinear dynamic systems," *Automatica*, vol. 45, no. 8, pp. 1965–1969, 2009.