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Research Article

(α, ψ)-Meir-Keeler Contraction Mappings in Generalized b-Metric Spaces

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We present a fixed point theorem for generalized (α, ψ)-Meir-Keeler type contractions in the setting of generalized b-metric spaces. The presented results improve, generalize, and unify many existing famous results in the corresponding literature.

1. Introduction and Preliminaries

The idea of a b-metric has been introduced in the papers [1, 2]. Very recently, this idea was extended in [3] to a generalized b-metric space in the following manner.

Definition 1. Let X be a nonempty set and $s \geq 1$ be a fixed constant. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized b-metric space (in brief, gbms) if and only if for $x, y, z \in X$ the conditions are satisfied:

(d_1) $d(x, y) = 0$ if and only if $x = y$.

(d_2) $d(x, y) = d(y, x)$.

(d_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A triple (X, d, s) is called a generalized b-metric space.

On the other hand, Meir and Keeler [4] have proved the following very general result on the existence of fixed points of Meir-Keeler contraction mappings in metric spaces.

Theorem 2 (see [4]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ satisfy the following condition:*

(d) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(fx, fy) < \varepsilon. \tag{1}$$

f has a unique fixed point ξ. Moreover, for any $x \in X$,

$$\lim_{n\to\infty} f^n x = \xi, \tag{2}$$

where $f^n x$ denotes the nth iteration of f at a point x.

This result has been generalized and extended in many directions; see [5–15]. Using some auxiliary functions, the main purpose of this paper is to extend and generalize this result on generalized b-metric spaces.

For the sake of explicitness, we recall some notations. The symbols $\mathbb{N}, \mathbb{R}$ denote the natural and real numbers, respectively. Furthermore, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{R}_0^+ := [0, \infty)$.

Berinde [16] characterized comparison functions to define the contraction mappings in the setting of b-metric spaces.

Definition 3. Let $s \geq 1$ be a real number. A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called a (c)-comparison function if

(1) ϕ is increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq a s^k \phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

Denote Ψ as the set of (c)-comparison functions. We will need the following essential properties in our further discussion.

Lemma 4 (see [16–18]). *For a (c)-comparison function $\varphi : [0,+\infty) \to [0,+\infty)$, the following statements hold:*

(1) *The series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in [0,+\infty)$.*

(2) *The function $b_s : [0,+\infty) \to [0,+\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.*

(3) *Each iterate φ^k of φ for $k \geq 1$ is also a (c)-comparison function.*

(4) *φ is continuous at 0.*

(5) *$\varphi(t) < t$ for any $t > 0$.*

Inspired by Popescu [19], we introduce the concept of generalized α-orbital admissible mappings.

Definition 5. Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty]$ be a function. We say that T is a generalized α-orbital admissible if

$$\begin{aligned} &\alpha(x,Tx) \geq 1 \Longrightarrow \alpha\left(Tx,T^2x\right) \geq 1, \\ &\alpha(x,Tx) < \infty \Longrightarrow \alpha\left(Tx,Tx^2\right) < \infty. \end{aligned} \tag{3}$$

Notice that each α-orbital admissible mapping [19] is generalized α-orbital admissible.

Based on the concept of generalized α-orbital admissibility, we are the first who establish a fixed point result for a Meir-Keeler type contraction in the setting of generalized b-metric spaces.

2. Main Results

We start with this definition.

Definition 6. For an arbitrary constant $s \geq 1$, let T be a self-mapping defined on a generalized b-metric space (X,d,s). Then T is called an (α,ψ)-Meir-Keeler contractive mapping if there exist two auxiliary mappings $\alpha : X \times X \to [0,\infty]$ and $\psi \in \Psi$ such that

$$\begin{gathered} \varepsilon \leq \psi(d(x,y)) < \varepsilon + \delta \\ \text{implies } \alpha(x,y)\, d(Tx,Ty) < \varepsilon, \quad \forall x,y \in X. \end{gathered} \tag{4}$$

Remark 7. For $x \neq y$ and $d(x,y) < \infty$ with $\alpha(x,y) < \infty$, from (4) we derive that

$$\alpha(x,y)\, d(Tx,Ty) < \psi(d(x,y)). \tag{5}$$

Our main result is as follows.

Theorem 8. *Let $s \geq 1$ be a fixed constant and (X,d,s) be a complete generalized b-metric space. Suppose that a self-mapping $T : X \to X$ is an (α,ψ)-Meir-Keeler type contraction. Assume also that*

(i) *T is generalized α-orbital admissible;*

(ii) *there exists $x \in X$ such that $1 \leq \alpha(x,Tx) < \infty$;*

(iii) *T is continuous.*

Then for such x, one of the following statements holds:

(A) *For every $n \in \mathbb{N}_0$,*

$$d\left(T^n x, T^{n+1}x\right) = \infty \tag{6}$$

$$\text{or } \alpha\left(T^n x, T^{n+1}x\right) = \infty. \tag{7}$$

(B) *There exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$ and $\alpha(T^k x, T^{k+1}x) < \infty$. In this case, there exists $u \in X$ such that $Tu = u$.*

Proof. On account of assumption (ii), there exists $x \in X$ such that $\alpha(x,Tx) \geq 1$. We suppose that case (A) is not satisfied. Consequently, we have to examine case (B). Consequently, there exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$ and $\alpha(T^k x, T^{k+1}x) < \infty$. If $T^k x = T^{k+1}x$, the proof is completed. Assume that $d(T^k x, T^{k+1}) > 0$. By property of ψ and Remark 7, we have

$$\begin{aligned} &\alpha\left(T^k x, T^{k+1}x\right) d\left(T^{k+1}x, T^{k+2}x\right) \\ &\quad < \psi\left(d\left(T^k x, T^{k+1}x\right)\right) < d\left(T^k x, T^{k+1}x\right) < \infty. \end{aligned} \tag{8}$$

Since T is a generalized α-orbital admissible mapping, by (ii), we derive that

$$1 \leq \alpha(x,Tx) < \infty \Longrightarrow 1 \leq \alpha\left(Tx,T^2x\right) < \infty. \tag{9}$$

Recursively, we obtain that

$$1 \leq \alpha\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N}_0. \tag{10}$$

Applying (10) in (8), we get

$$d\left(T^{k+1}x, T^{k+2}x\right) < \psi\left(d\left(T^k x, T^{k+1}x\right)\right) < \infty. \tag{11}$$

Thus

$$d\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Again, on account of (10) and (12) in (8), by induction, one gets

$$d\left(T^{k+n}x, T^{k+n+1}x\right) < \psi^n\left(d\left(T^k x, T^{k+1}x\right)\right). \tag{13}$$

Consequently, for $n, \nu \in \mathbb{N}_0$, by (13) we have

$$\begin{aligned} d\left(T^{k+n}x, T^{k+n+\nu}x\right) &\leq s d\left(T^{k+n}x, T^{k+n+1}x\right) + \cdots \\ &\quad + s^{\nu-1} d\left(T^{k+n+\nu-2}x, T^{k+n+\nu-1}x\right) \\ &\quad + s^{\nu} d\left(T^{k+n+\nu-1}x, T^{k+n+\nu}x\right) \\ &< s\psi^n\left(d\left(T^k x, T^{k+1}x\right)\right) + \cdots \\ &\quad + s^{\nu-1}\psi^{n+\nu-2}\left(d\left(T^k x, T^{k+1}x\right)\right) \\ &\quad + s^{\nu}\psi^{n+\nu-1}\left(d\left(T^k x, T^{k+1}x\right)\right) \end{aligned}$$

$$\leq s\sum_{m=0}^{\infty} s^m \psi^{n+m}\left(d\left(T^k x, T^{k+1}x\right)\right)$$
$$\leq s\sum_{m=0}^{\infty} s^m \psi^{m}\left(d\left(T^k x, T^{k+1}x\right)\right). \tag{14}$$

Finally,

$$d\left(T^{k+n}(x), T^{k+n+v}x\right) \leq s\sum_{m=0}^{\infty} s^m \psi^{m}\left(d\left(T^k x, T^{k+1}x\right)\right), \tag{15}$$

for all $n, v \in \mathbb{N}_0$. By (15) and the fact that $\psi \in \Psi$, it follows that $\{T^n x\}$ is a Cauchy sequence of elements of X.

Since X is complete, there exists $u \in X$ with

$$\lim_{n\to\infty} d\left(T^n x, u\right) = 0. \tag{16}$$

Since T is continuous, we get

$$u = \lim_{n\to\infty} T^{n+1}x = T\left(\lim_{n\to\infty} T^n x\right) = Tu, \tag{17}$$

and u is a fixed point of T, which ends the proof. □

Definition 9. Let $s \geq 1$ be a fixed constant. We say that a generalized b-metric space (X, d, s) is regular if $\{x_n\}$ is a sequence in X such that $1 \leq \alpha(x_n, x_{n+1}) < \infty$ for all n and $x_n \to x \in X$ as $n \to \infty$; then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1 \leq \alpha(x_{n(k)}, x) < \infty$ and $0 < d(x_{n(k)}, x) < \infty$ for all k.

Theorem 10. *Let $s \geq 1$ be a fixed constant and (X, d, s) be a complete generalized b-metric space. Suppose that a self-mapping $T : X \to X$ is an (α, ψ)-Meir-Keeler type contraction. Assume also that*

(i) *T is a generalized α-orbital admissible mapping;*

(ii) *there exists $x \in X$ such that $1 \leq \alpha(x, Tx) < \infty$;*

(iii) *(X, d, s) is regular.*

Then for such x, one of the following statements holds:

(A) *For every $n \in \mathbb{N}_0$,*

$$d\left(T^n x, T^{n+1}x\right) = \infty \tag{18}$$

$$\text{or } \alpha\left(T^n x, T^{n+1}x\right) = \infty. \tag{19}$$

(B) *There exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$ and $\alpha(T^k x, T^{k+1}x) < \infty$. In this case, there exists $u \in X$ such that $Tu = u$.*

Proof. In case (B), following the proof of Theorem 8, we know that the sequence $\{T^n(x)\}$ converges to some $u \in X$. By Definition 9 and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1 \leq \alpha(T^{n(k)}x, u) < \infty$ and $0 < d(T^{n(k)}x, u) < \infty$ for all k. Applying (5) for all k, we get that

$$\begin{aligned} d\left(T^{n(k)+1}x, Tu\right) &= d\left(T\left(T^{n(k)}x\right), Tu\right) \\ &\leq \alpha\left(T^{n(k)}x, u\right) d\left(T\left(T^{n(k)}x\right), Tu\right) \\ &< \psi\left(d\left(T^{n(k)}x, u\right)\right). \end{aligned} \tag{20}$$

Letting $k \to \infty$ in the above equality, we get $d(u, Tu) = 0$; that is, $u = Tu$. □

For the uniqueness of a fixed point of an (α, ψ)-Meir-Keeler type contraction mapping T in $Y := \{t \in X : d(T^k x, t) < \infty\}$, we shall consider the following condition:

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $1 \leq \alpha(x, y) < \infty$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 11. *By adding condition (U) to the hypotheses of Theorem 8 (resp., Theorem 10), T has at most one fixed point in $Y := \{t \in X : d(T^k x, t) < \infty\}$.*

Proof. Let T be an (α, ψ)-Meir-Keeler type contraction. Owing to Theorem 8 (resp., Theorem 10), T has a fixed point $u \in X$.

Now, we shall show that T has at most one fixed point in Y. We argue by contradiction. For this, assume that there exist two distinct fixed points u_1 and u_2 of T, where $u_1, u_2 \in Y$; that is,

$$d\left(T^k x, u_i\right) < \infty \quad \forall i = 1, 2. \tag{21}$$

We deduce

$$d\left(u_1, u_2\right) \leq sd\left(u_1, T^k x\right) + sd\left(T^k x, u_2\right) < \infty. \tag{22}$$

By condition (U), $1 \leq \alpha(u_1, u_2) < \infty$ and since $0 < d(u_1, u_2) < \infty$, in view of (5), one writes

$$\begin{aligned} d\left(u_1, u_2\right) &= d\left(Tu_1, Tu_2\right) \leq \alpha\left(u_1, u_2\right) d\left(Tu_1, Tu_2\right) \\ &< \psi\left(d\left(u_1, u_2\right)\right) < d\left(u_1, u_2\right), \end{aligned} \tag{23}$$

which is a contradiction, so $u_1 = u_2$. This completes the proof. □

3. Consequences

3.1. Meir-Keeler Contraction Mappings in gbms. In this section, we present our main result. By letting $\alpha(x, y) = 1$ and $\phi(t) = t/2s$, we get the following result.

Theorem 12. *Let (X, d, s) be a generalized complete b-metric space and $T : X \to X$ satisfy the following: given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \frac{\varepsilon}{2s}, \quad x, y \in X. \tag{24}$$

Let $x \in X$. Then one of the following alternatives holds:

(A) For every $n \in \mathbb{N}_0$ ($\mathbb{N}_0$ being the set of all nonnegative integers),

$$d\left(T^n x, T^{n+1}x\right) = \infty. \tag{25}$$

(B) There exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$.

In case (B), we assert the following:

(i) The sequence $\{T^m x\}$ is Cauchy in X.

(ii) There exists a point $u \in X$ such that $Tu = u$ and $\lim_{n\to\infty} d(T^n x, u) = 0$.

(iii) u is the unique fixed point of T in $B := \{t \in X : d(T^k x, t) < \infty\}$.

(iv) For every $t \in B$,

$$\lim_{n\to\infty} d\left(T^n t, u\right) = 0. \tag{26}$$

Remark 13. Unfortunately, if (X, d) is a metric space, we do not get the result of Meir-Keeler [4].

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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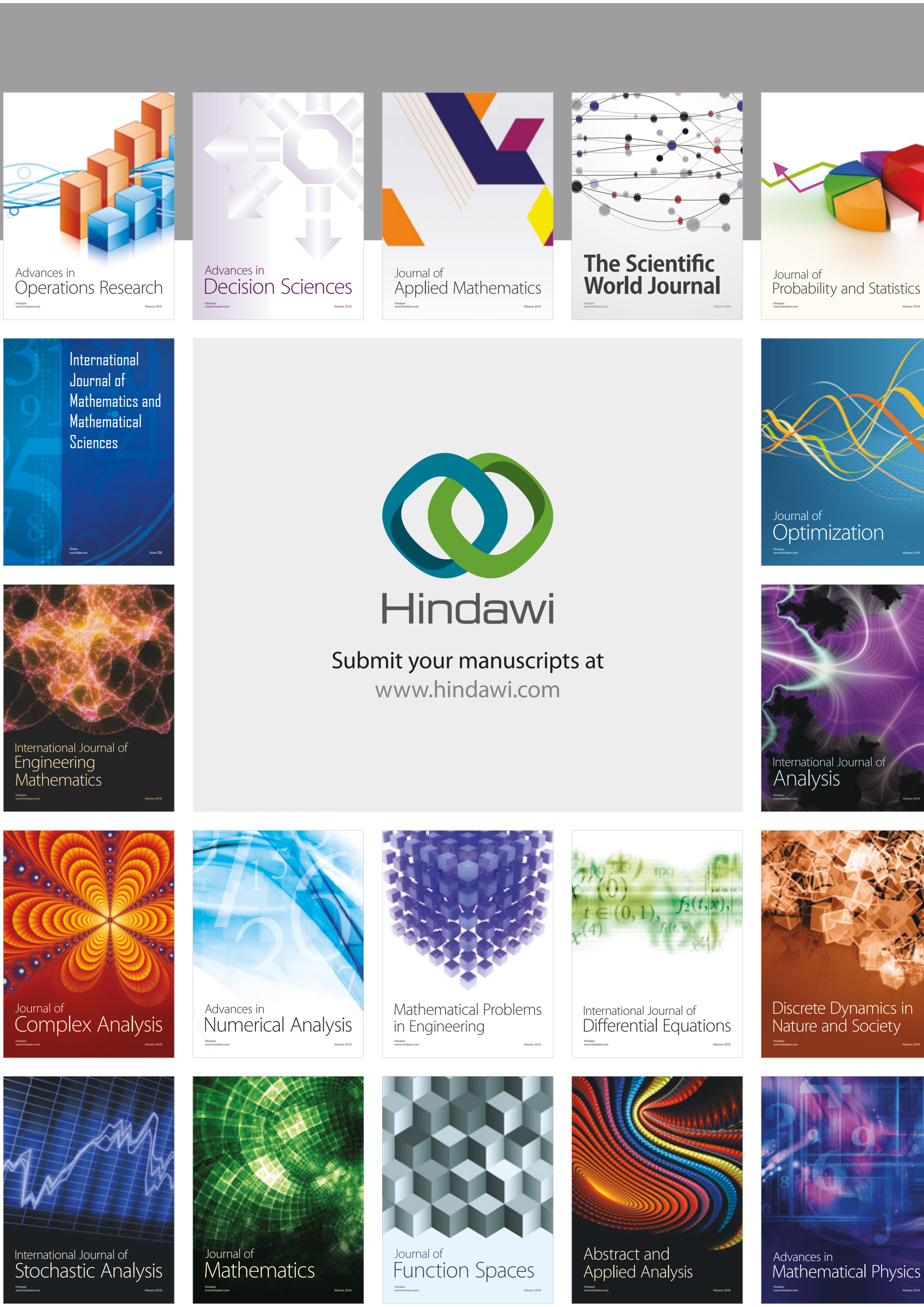
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