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Research Article

Real Interpolation of Small Lebesgue Spaces in a Critical Case

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We establish an interpolation formula for small Lebesgue spaces in a critical case.

1. Introduction

The small Lebesgue spaces were introduced by A. Fiorenza in [1], and they have found applications, for example, in boundary value problems [2, 3], Besov embeddings [4, 5], and dimension-free Sobolev embeddings [6]. The reader is referred to a recent survey paper [7] for more details.

Let Ω be a bounded domain in $\mathbb{R}^n$ with measure 1, and let $1 < p < \infty$ and $\alpha > 0$. The small Lebesgue space $L^{(p,\alpha} = L^{(p,\alpha}(\Omega)$ is formed by all those real-valued Lebesgue measurable functions f on Ω, for which the norm

$$\|f\|_{L^{(p,\alpha}} = \int_0^1 (1 - \ln t)^{-\alpha/p+\alpha-1} \left(\int_0^t f^* (u)^p \, du \right)^{1/p} \frac{dt}{t} \quad (1)$$

is finite; see [8]. Here f^* denotes the nonincreasing rearrangement of f (see, for instance, [9]). For $\alpha = 1$, the spaces $L^{(p,\alpha}$ coincide with the classical small Lebesgue spaces $L^{(p}$ (see [10, Corollary 3.3]).

Let $0 < \theta < 1$, $1 < q < \infty$, $1 < p < r < \infty$, and $\alpha > 0$. The real interpolation spaces $(L^{(p,\alpha}, L^{(r,\alpha})_{\theta,q}$ have been recently characterized in [11, Theorem 3.4] as the well-known Lorentz-Zygmund spaces. Here $(\cdot,\cdot)_{\theta,q}$ denotes the classical real interpolation method (see Section 2). The critical case $p = r$ has not been considered in [11], and the goal of the present paper is to fill this gap. More precisely, we will characterize the interpolation spaces $(L^{(p,\alpha}, L^{(p,\beta})_{\theta,q}$ under an appropriate condition on α and β ($\beta > 0$). Our approach is essentially different from that of [11] and allows the parameter q to lie also in the interval $(0, 1]$. The key ingredient of our method is a general Holmstedt-type estimate from a recent paper [12].

The plan of the paper is simple. We collect the necessary background, along with certain weighted Hardy-type inequalities, in Section 2. The main result is contained in Section 3.

2. Preliminaries

Throughout the paper, we will write $A \lesssim B$ or $B \gtrsim A$ for two nonnegative quantities A and B to mean that $A \leq cB$ for some positive constant c which is independent of appropriate parameters involved in A and B. We put $A \approx B$, if $A \lesssim B$ and $A \gtrsim B$. Moreover, we will write $X \hookrightarrow Y$ for two quasi-normed spaces X and Y to mean that X is continuously embedded in Y.

Let (A_0, A_1) be a compatible couple of quasi-normed spaces; that is, we assume that both A_0 and A_1 are continuously embedded in the same Hausdorff topological vector space. Peetre's K-functional $K(t, f) = K(t, f; A_0, A_1)$ is defined, for each $f \in A_0 + A_1$ and $t > 0$, by

$$K(t, f) = \inf \left\{ \|f_0\|_{A_0} + t \|f_1\|_{A_1} : f_0 \in A_0, f_1 \in A_1, f = f_0 + f_1 \right\}. \quad (2)$$

In what follows, we always assume that the couple (A_0, A_1) is ordered in the sense that $A_1 \hookrightarrow A_0$.

Let $0 < q \leq \infty$, $0 \leq \theta \leq 1$, and $\gamma \in \mathbb{R}$. The real interpolation space $\overline{A}_{\theta,q;\gamma} = (A_0, A_1)_{\theta,q;\gamma}$ consists of all $f \in A_0$ for which the quasi-norm

$$\|f\|_{\overline{A}_{\theta,q;\gamma}} = \left(\int_0^1 t^{-\theta q} (1-\ln t)^{q\gamma} K^q(t,f)\,\frac{dt}{t}\right)^{1/q} \quad (3)$$

is finite (with usual modification when $q=\infty$). When $\gamma = 0$ and $0<\theta<1$, we put $\overline{A}_{\theta,q} = \overline{A}_{\theta,q;\gamma}$. Since $A_1 \hookrightarrow A_0$, we have $K(t,f) \approx \|f\|_{A_0}$, $t>1$. Combining this with the fact that, as a function of t, $K(t,f)/t$ is nonincreasing, we can easily check that

$$\|f\|_{\overline{A}_{\theta,q}} \approx \left(\int_0^\infty t^{-\theta q} K^q(t,f)\,\frac{dt}{t}\right)^{1/q}. \quad (4)$$

Thus, the spaces $\overline{A}_{\theta,q}$ are in fact the classical Lions-Peetre spaces (see [9, 13, 14]).

In the sequel, we will work with the limiting spaces $\overline{A}_{0,q;\gamma}$ and the classical spaces $\overline{A}_{\theta,q}$. It is not hard to verify that the limiting spaces $\overline{A}_{0,q;\gamma}$ are intermediate, without any condition on γ and q, for the couple (A_0, A_1), that is,

$$A_1 \hookrightarrow \overline{A}_{0,q;\gamma} \hookrightarrow A_0. \quad (5)$$

However, in order to exclude the trivial case $\overline{A}_{0,q;\gamma} = A_0$, we have to work under the assumption $\gamma \geq -1/q$ (if $q<\infty$) or $\gamma > 0$ (if $q=\infty$). This observation immediately follows from the elementary fact that, as a function of t, $K(t,f)$ is nondecreasing.

We close this section with the following weighted Hardy-type inequalities which will be needed in order to prove our main result in the next section.

Lemma 1 ([15, Lemma 3.2]). *Let $1\leq s<\infty$, and assume that w and ϕ are nonnegative functions on $(0,\infty)$. Put*

$$v(t) = (w(t))^{1-s}\left(\phi(t)\int_t^\infty w(u)\,du\right)^s. \quad (6)$$

Then

$$\int_0^\infty \left(\int_0^t \phi(u)\,h(u)\,du\right)^s w(t)\,dt \lesssim \int_0^\infty h^s(t)\,v(t)\,dt \quad (7)$$

holds for all nonnegative functions h on $(0,\infty)$.

By an obvious change of variable, we get the following variant of the previous result.

Lemma 2. *Let $1\leq s<\infty$, and assume that w and ϕ are nonnegative functions on $(0,\infty)$. Put*

$$v_0(t) = (w(t))^{1-s}\left(\phi(t)\int_0^t w(u)\,du\right)^s. \quad (8)$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)\,h(u)\,du\right)^s w(t)\,dt \lesssim \int_0^\infty h^s(t)\,v_0(t)\,dt \quad (9)$$

holds for all nonnegative functions h on $(0,\infty)$.

Lemma 3 ([16, Theorem 3.3 (b)]). *Let $0<s<1$. Assume w and v are positive functions on $(0,1)$, and ψ is a positive function on $(0,1)\times(0,1)$. Then*

$$\int_0^1 \left(\int_0^1 \psi(t,u)\,g(u)\,du\right)^s w(t)\,dt \lesssim \int_0^1 g^s(t)\,v(t)\,dt \quad (10)$$

holds for all positive and nondecreasing functions g on $(0,1)$ if and only if

$$\int_0^1 \left(\int_x^1 \psi(t,u)\,du\right)^s w(t)\,dt \lesssim \int_x^1 v(t)\,dt \quad (11)$$

holds for all $0<x<1$.

3. Interpolation Formula

First we compute the K-functional for the couple $(L^{(p,\alpha}, L^{(p,\beta})$, $0<\alpha<\beta$. To this end, we need the following Holmstedt-type estimate.

Theorem 4. *Let $-1<\gamma_0<\gamma_1$, and let $f\in A_0$. Then*

$$\begin{aligned} &K\left((1-\ln t)^{\gamma_0-\gamma_1}, f; \overline{A}_{0,1;\gamma_0}, \overline{A}_{0,1;\gamma_1}\right) \\ &\quad \approx I(t,f) + (1-\ln t)^{\gamma_0-\gamma_1} J(t,f), \quad 0<t<1, \end{aligned} \quad (12)$$

where

$$I(t,f) = \int_0^t (1-\ln u)^{\gamma_0} K(u,f)\,\frac{du}{u}, \quad (13)$$

and

$$J(t,f) = \int_t^1 (1-\ln u)^{\gamma_1} K(u,f)\,\frac{du}{u}. \quad (14)$$

Proof. Let $j=0,1$. Put

$$g_j(t) = t\left(1+\int_t^1 (1-\ln u)^{\gamma_j}\,\frac{du}{u}\right), \quad 0<t<1, \quad (15)$$

and

$$h_j(t) = \int_0^t u(1-\ln u)^{\gamma_j}\,\frac{du}{u}, \quad 0<t<1. \quad (16)$$

According to case 4 in [12, Theorem 4], we have

$$\begin{aligned} &K\left(\frac{g_0(t)}{g_1(t)}, f; \overline{A}_{0,1;\gamma_0}, \overline{A}_{0,1;\gamma_1}\right) \\ &\quad \approx I(t,f) + (1-\ln t)^{\gamma_0-\gamma_1} J(t,f), \end{aligned} \quad (17)$$

provided that

$$\int_0^t \frac{u(1-\ln u)^{\gamma_0}}{g_1(u)+h_1(u)}\,\frac{du}{u} \lesssim \frac{g_0(t)}{g_1(t)}, \quad 0<t<1, \quad (18)$$

and

$$1+\int_t^1 \frac{u(1-\ln u)^{\gamma_1}}{g_0(u)+h_0(u)}\,\frac{du}{u} \lesssim \frac{g_1(t)}{g_0(t)}, \quad 0<t<1. \quad (19)$$

Since $\gamma_j>-1$, then $g_j(t)\approx t(1-\ln t)^{\gamma_j+1}$. Moreover, note that $h_j \lesssim g_j$. Finally, the condition $\gamma_0<\gamma_1$ implies that both (18) and (19) hold. The proof is complete. □

Remark 5. If $\gamma_0 < \gamma_1$, then we have $\overline{A}_{0,1;\gamma_1} \hookrightarrow \overline{A}_{0,1;\gamma_0}$. Thus, in this case, we trivially have

$$K\left(t, f; \overline{A}_{0,1;\gamma_0}, \overline{A}_{0,1;\gamma_1}\right) \approx \|f\|_{\overline{A}_{0,1;\gamma_0}}, \quad t > 1. \tag{20}$$

Corollary 6. *Let $1 < p < \infty$, $0 < \alpha < \beta$, and let $f \in L^p(\Omega)$. Then, for all $0 < t < 1$, we have*

$$\begin{aligned} K&\left((1-\ln t)^{(1-1/p)(\alpha-\beta)}, f; L^{(p,\alpha}, L^{(p,\beta}\right) \\ &\approx I_1(t,f) + (1-\ln t)^{(1-1/p)(\alpha-\beta)} J_1(t,f), \end{aligned} \tag{21}$$

where

$$\begin{aligned} I_1&(t,f) \\ &= \int_0^t (1-\ln u)^{-\alpha/p+\alpha-1} \left(\int_0^u f^*(\tau)^p\, d\tau\right)^{1/p} \frac{du}{u}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} J_1&(t,f) \\ &= \int_t^1 (1-\ln u)^{-\beta/p+\beta-1} \left(\int_0^u f^*(\tau)^p\, d\tau\right)^{1/p} \frac{du}{u}. \end{aligned} \tag{23}$$

Proof. First observe that $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ as Ω has finite measure. We put $L^p = L^p(\Omega)$ and $L^\infty = L^\infty(\Omega)$. Since (see [13, Theorem 5.2.1])

$$K\left(t^{1/p}, f; L^p, L^\infty\right) \approx \left(\int_0^t f^*(u)^p\, du\right)^{1/p}, \tag{24}$$

it follows immediately that $L^{(p,\eta} = (L^p, L^\infty)_{0,1;-\eta/p+\eta-1}$, where $\eta > 0$. Thus, the proof simply follows by applying Theorem 4 to $A_0 = L^p$, $A_1 = L^\infty$, $\gamma_0 = -\alpha/p + \alpha - 1$, and $\gamma_1 = -\beta/p + \beta - 1$. □

Following [4], we define small Lebesgue spaces in a slightly general way as follows. Let $1 \le p < \infty$, $0 < q \le \infty$, and $b > -1/q$. The small Lebesgue space $L^{(p,b,q} = L^{(p,b,q}(\Omega)$ is formed by all those real-valued Lebesgue measurable functions f on Ω, for which the quasi-norm

$$\begin{aligned} \|f&\|_{L^{(p,b,q}} \\ &= \left(\int_0^1 (1-\ln t)^{qb} \left(\int_0^t f^*(u)^p\, du\right)^{q/p} \frac{dt}{t}\right)^{1/q} \end{aligned} \tag{25}$$

is finite. The spaces $L^{(p,b,q}$ are a particular case of more general $G\Gamma(p,m,w)$ spaces introduced and studied in [17–19]. Note that if $1 < p < \infty$, $q = 1$, and $b = -\alpha/p + \alpha - 1$ with $\alpha > 0$, then we recover the spaces $L^{(p,\alpha}$.

The following interpolation formula is the main contribution of this paper.

Theorem 7. *Let $1 < p < \infty$, $0 < \alpha < \beta$, $0 < \theta < 1$, and $0 < q \le \infty$. Then*

$$\left(L^{(p,\alpha}, L^{(p,\beta}\right)_{\theta,q} = L^{(p,b,q}, \tag{26}$$

where $b = (1-1/p)((1-\theta)\alpha + \theta\beta) - 1/q$.

Proof. First note that $L^{(p,\beta} \hookrightarrow L^{(p,\alpha}$ as $\alpha < \beta$. Put $X = (L^{(p,\alpha}, L^{(p,\beta})_{\theta,q}$ and $Y = L^{(p,b,q}$. Let $f \in L^p$. Assume that $0 < q < \infty$. By making an appropriate change of variable and using Corollary 6, we arrive at

$$\|f\|_X^q \approx C_1 + C_2, \tag{27}$$

where

$$\begin{aligned} C_1 = \int_0^1 &(1-\ln t)^{\theta q(1-1/p)(\beta-\alpha)-1} \\ &\cdot \left(\int_0^t (1-\ln u)^{-\alpha/p+\alpha-1} F(u) \frac{du}{u}\right)^q \frac{dt}{t}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} C_2 = \int_0^1 &(1-\ln t)^{(1-\theta)q(1-1/p)(\alpha-\beta)-1} \\ &\cdot \left(\int_t^1 (1-\ln u)^{-\beta/p+\beta-1} F(u) \frac{du}{u}\right)^q \frac{dt}{t}, \end{aligned} \tag{29}$$

with

$$F(t) = \left(\int_0^t f^*(u)^p\, du\right)^{1/p}. \tag{30}$$

Thus, the proof will be complete, in the case $0 < q < \infty$, if we show that $C_1 \lesssim \|f\|_Y^q$ and $C_2 \approx \|f\|_Y^q$. Since F is nondecreasing, we get

$$\begin{aligned} C_2 \ge \int_0^1 &(1-\ln t)^{(1-\theta)q(1-1/p)(\alpha-\beta)-1} F^q(t) \\ &\cdot \left(\int_t^1 (1-\ln u)^{-\beta/p+\beta-1} \frac{du}{u}\right)^q \frac{dt}{t}, \end{aligned} \tag{31}$$

or

$$\begin{aligned} C_2 \gtrsim \int_0^1 &(1-\ln t)^{(1-\theta)q(1-1/p)(\alpha-\beta)-1} F^q(t) \\ &\cdot \left((1-\ln t)^{-\beta/p+\beta} - 1\right)^q \frac{dt}{t}, \end{aligned} \tag{32}$$

whence we obtain $C_2 \gtrsim \|f\|_Y^q$ as $(1-\ln t)^{-\beta/p+\beta} - 1$ and $(1-\ln t)^{-\beta/p+\beta}$ are asymptotically the same as $t \longrightarrow 0^+$.

Next we establish the estimates $C_1 \lesssim \|f\|_Y^q$ and $C_2 \lesssim \|f\|_Y^q$. To this end, we distinguish two cases: $q \ge 1$ and $q < 1$. Assume first that $q \ge 1$. Then, $C_1 \lesssim \|f\|_Y^q$ follows from Lemma 1, applied with $s = q$, $h = F$, $\phi(t) = t^{-1}(1-\ln u)^{-\alpha/p+\alpha-1}$, and $w(t) = t^{-1}(1-\ln t)^{\theta q(1-1/p)(\beta-\alpha)-1}\chi_{(0,1)}(t)$. Similarly, $C_2 \lesssim \|f\|_Y^q$ follows from Lemma 2. Next assume that $q < 1$. Since F is nondecreasing, we can apply Lemma 3 with $s = q$, $g = F$, $w(t) = t^{-1}(1-\ln t)^{\theta q(1-1/p)(\beta-\alpha)-1}$, $\psi(t,u) = u^{-1}(1-\ln u)^{-\alpha/p+\alpha-1}\chi_{(0,t)}(u)$, and $v(t) = t^{-1}(1-\ln t)^{q(1-1/p)((1-\theta)\alpha+\theta\beta)-1}$. Note that (10) turns into $C_1 \lesssim \|f\|_Y^q$. Thus, we have to verify (11) for all $0 < x < 1$. Observe that (11) holds trivially for all $1/2 \le x < 1$, and thus we may assume that $0 < x < 1/2$. Let us put $\eta = -\alpha/p + \alpha$ and $\delta = \theta q(1 - 1/p)(\beta - \alpha)$. Now we have

$$\begin{aligned} &\int_0^1 \left(\int_x^1 \psi(t,u)\,du \right)^s w(t)\,dt \\ &\quad = \int_x^1 \left(\int_x^t (1-\ln u)^{\eta-1} \frac{du}{u} \right)^q (1-\ln t)^{\delta-1} \frac{dt}{t} \\ &\quad \le \left(\int_x^1 (1-\ln u)^{\eta-1} \frac{du}{u} \right)^q \int_x^1 (1-\ln t)^{\delta-1} \frac{dt}{t} \\ &\quad \approx (1-\ln x)^{q\eta+\delta} \approx \int_x^1 v(t)\,dt \end{aligned} \tag{33}$$

which proves the validity of (11). Hence, the estimate $C_1 \lesssim \|f\|_Y^q$ follows from (10). Similarly, we can obtain $C_2 \lesssim \|f\|_Y^q$ from Lemma 3. This completes the proof in the case $0 < q < \infty$. Next we assume that $q = \infty$. This time we have

$$\|f\|_X \approx D_1 + D_2, \tag{34}$$

where

$$\begin{aligned} D_1 = &\sup_{0<t\le 1} (1-\ln t)^{\theta(1-1/p)(\beta-\alpha)} \\ &\cdot \int_0^t (1-\ln u)^{-\alpha/p+\alpha-1} F(u) \frac{du}{u}, \end{aligned} \tag{35}$$

and

$$\begin{aligned} D_2 = &\sup_{0<t\le 1} (1-\ln t)^{(1-\theta)(1-1/p)(\alpha-\beta)} \\ &\cdot \int_t^1 (1-\ln u)^{-\beta/p+\beta-1} F(u) \frac{du}{u}. \end{aligned} \tag{36}$$

In order to estimate D_1, we note that

$$\begin{aligned} D_1 \le \|f\|_Y &\sup_{0<t\le 1} (1-\ln t)^{\theta(1-1/p)(\beta-\alpha)} \\ &\cdot \int_0^t (1-\ln u)^{\theta(1-1/p)(\alpha-\beta)-1} \frac{du}{u}, \end{aligned} \tag{37}$$

whence we get $D_1 \lesssim \|f\|_Y$. Similarly, we can show that $D_2 \lesssim \|f\|_Y$. On the other hand, again using the fact that F is nondecreasing, we get $D_2 \gtrsim \|f\|_Y$. Altogether, we have $D_1 + D_2 \approx \|f\|_Y$ which gives $\|f\|_X \approx \|f\|_Y$ in view of (34). The proof of the theorem is complete. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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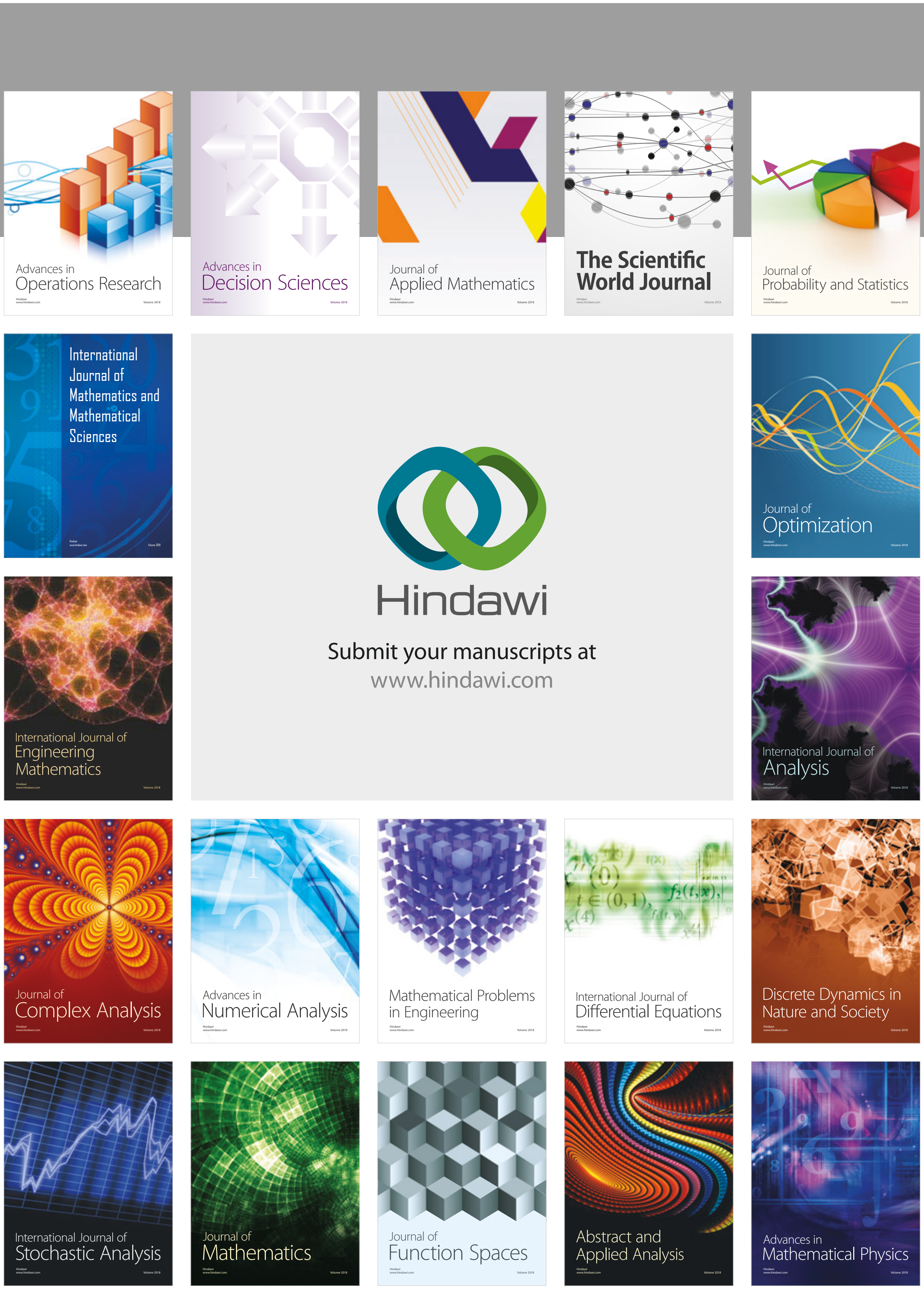

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