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Research Article

Existence Results for Impulsive Fractional q-Difference Equation with Antiperiodic Boundary Conditions

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In this paper, we investigate the impulsive fractional q-difference equation with antiperiodic conditions. The existence and uniqueness results of solutions are established via the theorem of nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle. Two examples are given to illustrate our results.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of solutions for the following impulsive fractional q-difference equation with antiperiodic boundary conditions

$$\begin{aligned} {}^{c}D_q^{\alpha}u(t) &= f(t,u(t),Tu(t),Su(t)), \\ &\qquad t\in J'=J\setminus\{t_1,t_2,\dots,t_m\}, \\ \Delta u|_{t=t_k} &= I_k(u(t_k^-)), \\ \Delta D_q u\big|_{t=t_k} &= I_k^*(u(t_k^-)), \\ &\qquad k=1,2,\dots,m, \\ u(0) &= -u(1), \\ {}^{c}D_q^{\beta}u(0) &= -{}^{c}D_q^{\beta}u(1), \end{aligned} \tag{1}$$

where $q\in(0,1)$, $1<\alpha\le 2$, $0<\beta<1$, $\alpha-\beta-1>0$, $J=[0,1]$, D_q is q-derivative, ${}^{c}D_q^{\alpha}$, and ${}^{c}D_q^{\beta}$ denote the Caputo q-derivative of orders α and β, respectively. $f\in C(J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $I_k, I_k^*\in C(\mathbb{R},\mathbb{R})$ $(k=1,2,\dots,m)$, $\mathbb{R}$ is the set of all real numbers, and $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. T and S are linear operators defined by

$$\begin{aligned} Tu(t) &= \int_0^t k(t,s)\,u(s)\,d_qs, \\ Su(t) &= \int_0^1 h(t,s)\,u(s)\,d_qs, \\ &\qquad t\in J, \end{aligned} \tag{2}$$

where $k\in C(D,\mathbb{R})$, $h\in C(J\times J,\mathbb{R})$, $D=\{(t,s)\in J\times J: t\ge s\}$. $\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t=t_k$, $\Delta D_q u|_{t=t_k}=D_q u(t_k^+)-D_q u(t_k^-)$ has a similar meaning.

Fractional q-difference calculus plays a very important role in modern applied mathematics due to their deep physical background and has been studied extensively [1–4]. Impulsive differential equations are important in both theory and applications. Considerable effort has been devoted to differential equations with or without impulse, for example, [5–21]. In recent years, impulsive fractional difference and differential equations with antiperiodic conditions have received much attention; see [22–27] and the references therein. Zhang and Wang [24] have applied cone contraction fixed point theorem to establish the existence of solutions to nonlinear fractional differential equation with impulses and antiperiodic boundary conditions

$$\begin{aligned} {}^{c}D^{\alpha}u(t) &= f(t,u(t)), \\ &\qquad 1<\alpha\le 2,\ t\in J\setminus\{t_1,t_2,\dots,t_m\},\ J=[0,T], \\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \end{aligned}$$

$$\begin{aligned} &\Delta u'\big|_{t=t_k} = I_k^*\left(u\left(t_k\right)\right), \\ &\qquad\qquad k = 1, 2, \ldots, p, \\ &u(0) = -u(T), \\ &u'(0) = -u'(T), \end{aligned} \tag{3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$. By using Banach fixed point theorem, Schaefer fixed point theorem, and nonlinear alternative of Leray-Schauder type theorem, some existence results of solutions for problem (3) are obtained in [25]. Ahmad et al. [28] studied existence of solutions for the following antiperiodic boundary value problem (BVP for short) of impulsive fractional q-difference equation

$$\begin{aligned} &{}_{t_k}^{c}D_{q_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J_k \subseteq [0, T], \ t \neq t_k, \\ &\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k\right) = \varphi_k\left({}_{t_{k-1}}I_{q_{k-1}}^{\beta_{k-1}} x\left(t_k\right)\right), \\ &\qquad\qquad k = 1, 2, \ldots, m, \\ &{}_{t_k}D_{q_k} x\left(t_k^+\right) - {}_{t_{k-1}}D_{q_{k-1}} x\left(t_k\right) = \varphi_k^*\left({}_{t_{k-1}}I_{q_{k-1}}^{\gamma_{k-1}} x\left(t_k\right)\right), \\ &\qquad\qquad k = 1, 2, \ldots, m, \\ &x(0) = -x(T), \\ &{}_0D_{q0} x(0) = -{}_{t_m}D_{q_m} x(T), \end{aligned} \tag{4}$$

where ${}_{t_k}^{c}D_{q_k}^{\alpha_k}$ denotes the Caputo q_k-fractional derivative of order α_k on J_k, $1 < \alpha_k \leq 2$, $0 < q_k < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\varphi_k, \varphi_k^* \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots, m$. ${}_{t_k}I_{q_k}^{\beta_k}$ and ${}_{t_k}I_{q_k}^{\gamma_k}$ denote the Riemann-Liouville q_k-integral of orders β_k and γ_k, respectively.

In this paper we are concerned with the existence and uniqueness of solutions for impulsive fractional q-difference equation antiperiodic BVP. By applying the theorem of nonlinear alternative of Leray-Schauder type and Banach contraction mapping principle, we show the existence and uniqueness of solutions for the BVP (1). Some ideas of this paper are from [29, 30].

2. Preliminaries and Lemmas

For $q \in (0, 1)$, let

$$\begin{aligned} [a]_q &= \frac{1 - q^a}{1 - q}, \\ (a; q)_\infty &= \prod_{i=0}^{\infty}\left(1 - aq^i\right), \\ (a; q)_\alpha &= \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}, \\ &\qquad (a, \alpha \in \mathbb{R}). \end{aligned} \tag{5}$$

We define the q-analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$\begin{aligned} (a - b)^0 &= 1, \\ (a - b)^n &= \prod_{k=0}^{n-1}\left(a - bq^k\right), \\ &\qquad n \in \mathbb{N}, \ a, b \in \mathbb{R}, \end{aligned} \tag{6}$$

and, for $\alpha \in \mathbb{R}$,

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}. \tag{7}$$

The q-derivative of f is defined by

$$\begin{aligned} \left(D_q f\right)(x) &= \frac{f(x) - f(qx)}{(1 - q)x}, \\ \left(D_q f\right)(0) &= \lim_{x \longrightarrow 0}\left(D_q f\right)(x) \end{aligned} \tag{8}$$

and q-derivative of higher order by

$$\begin{aligned} \left(D_q^0 f\right)(x) &= f(x), \\ \left(D_q^n f\right)(x) &= D_q\left(D_q^{n-1} f\right)(x), \quad n \in \mathbb{N}. \end{aligned} \tag{9}$$

The q-integral of f is defined by

$$\begin{aligned} \left(I_q f\right) &= \int_0^x f(t)\, d_q t = x(1 - q)\sum_{n=0}^{\infty} f\left(xq^n\right) q^n, \\ &\qquad x \in [0, b]. \end{aligned} \tag{10}$$

Lemma 1 (see [31]). *(1) If $|f|$ is q-integral on the interval $[0, x]$, then $|\int_0^x f(t) d_q t| \leq \int_0^x |f(t)| d_q t$.*

(2) If f and g are q-integral on the interval $[0, x]$, $f(t) \leq g(t)$ for all $t \in [0, x]$, then $\int_0^x f(t) d_q t \leq \int_0^x g(t) d_q t$.

Definition 2 (see [2]). Let $\alpha \geq 0$ and f be a function defined on $[0, b]$. The fractional q-integral of the Riemann-Liouville type is defined by $(I_q^0 f)(x) = f(x)$ and

$$\begin{aligned} \left(I_q^\alpha f\right)(x) &= \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t)\, d_q t, \\ &\qquad \alpha > 0, \ x \in [0, b]. \end{aligned} \tag{11}$$

Definition 3 (see [3]). The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$\left({}^cD_q^\alpha f\right)(x) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(x), \quad \alpha \geq 0, \tag{12}$$

where $[\alpha]$ is the smallest integer greater than or equal to α. If $f(x) = x^{\beta-1}$, $\beta > 0$, then ${}^cD_q^\alpha f(x) = (\Gamma_q(\beta)/\Gamma_q(\beta-\alpha)) x^{\beta-\alpha-1}$.

Lemma 4 (see [2, 3]). *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, b]$. The following formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$;

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 5 (see [3]). *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. Then*

$$\left(I_q^{\alpha}\, {}^cD_q^{\alpha} f\right)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{\left(D_q^k f\right)(a)}{\Gamma_q(k+1)} x^k \left(\frac{a}{x} : q\right)_k. \quad (13)$$

If $\alpha \geq m \geq \beta$, then ${}^cD_q^{\beta} I_q^{\alpha} f(x) = I_q^{m-\beta} I_q^{\alpha-m} f(x) = I_q^{\alpha-\beta} f(x)$.

Lemma 6 (see [3]). *For $\beta \in \mathbb{R}^+$, $\lambda \in (-1, +\infty)$ and $0 \leq a < t \leq b$,*

$$I_q^{\beta}\left((t-a)^{(\lambda)}\right) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)} (t-a)^{(\beta+\lambda)}. \quad (14)$$

In particular, when $\lambda = 0$ and $a = 0$, using q-integration by part,

$$\left(I_q^{\beta} 1\right)(t) = \frac{1}{\Gamma_{q(\beta)}} \int_0^t (t-qs)^{(\beta-1)} d_q s = \frac{1}{\Gamma_q(\beta+1)} t^{\beta}. \quad (15)$$

Lemma 7 (see [32] (nonlinear alternative of Leray-Schauder type)). *Let X be a Banach space, U be a bounded open subset of X with $0 \in U$, and $P : \overline{U} \longrightarrow X$ be a completely continuous operator. Then, either there exists $x \in \partial U$ such that $x = \lambda P x$ for $\lambda \in (0,1)$ or there exists a fixed point $x^* \in \overline{U}$.*

Let $PC(J, \mathbb{R}) = \{u : u$ ia a map from J into $\mathbb{R}$ such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and its right limit at $t = t_k$ exists for $k = 1, \ldots, m\}$; then $PC(J, \mathbb{R})$ is a Banach space with the norm $\|u\|_{PC} = \sup\{|u(t)| : t \in J\}$.

Lemma 8. *For $h \in PC(J, \mathbb{R})$, the solution of impulsive BVP,*

$$\begin{aligned} {}^cD_q^{\alpha} u(t) &= h(t), \quad t \in J', \\ \Delta u|_{t=t_k} &= I_k\left(u\left(t_k^-\right)\right), \\ \Delta D_q u\big|_{t=t_k} &= I_k^*\left(u\left(t_k^-\right)\right), \\ & \qquad k = 1, 2, \ldots, m, \\ u(0) &= -u(1), \\ {}^cD_q^{\beta} u(0) &= -{}^cD_q^{\beta} u(1), \end{aligned} \quad (16)$$

is given by

$$u(t) = \begin{cases} \dfrac{1}{\Gamma_q(\alpha)} \displaystyle\int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s - \dfrac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s - \dfrac{1}{2} \sum_{i=1}^m I_i\left(u\left(t_i^-\right)\right) \\ \quad + \dfrac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\dfrac{1}{2} - t\right) \displaystyle\int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \dfrac{1}{2} \sum_{i=1}^m t_i I_i^*\left(u\left(t_i^-\right)\right) - \sum_{i=1}^m t I_i^*\left(u\left(t_i^-\right)\right), \quad t \in [0, t_1), \\ \dfrac{1}{\Gamma_q(\alpha)} \displaystyle\int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s - \dfrac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ \quad + \dfrac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\dfrac{1}{2} - t\right) \displaystyle\int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \dfrac{1}{2} \sum_{i=1}^m I_i\left(u\left(t_i^-\right)\right) - \dfrac{1}{2} \sum_{i=1}^m t_i I_i^*\left(u\left(t_i^-\right)\right) \\ \quad - \displaystyle\sum_{i=k+1}^m I_i\left(u\left(t_i^-\right)\right) + \sum_{i=k+1}^m \left(t_i - t\right) I_i^*\left(u\left(t_i^-\right)\right), \qquad t \in \left[t_k, t_{k+1}\right), \ k = 1, \ldots, m-1, \\ \dfrac{1}{\Gamma_q(\alpha)} \displaystyle\int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s - \dfrac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ \quad + \dfrac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\dfrac{1}{2} - t\right) \displaystyle\int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \dfrac{1}{2} \sum_{i=1}^m I_i\left(u\left(t_i^-\right)\right) - \dfrac{1}{2} \sum_{i=1}^m t_i I_i^*\left(u\left(t_i^-\right)\right), \quad t \in [t_m, 1]. \end{cases} \quad (17)$$

Proof. In view of Definitions 2 and 3 and Lemma 5, for $t \in J_k = [t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, we have

$$\begin{aligned} u(t) &= I_q^{\alpha} h(t) + d_k + e_k t \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s + d_k + e_k t, \end{aligned} \quad (18)$$

and

$$\left(D_q u\right)(t) = \frac{1}{\Gamma_q(\alpha-1)} \int_0^t (t-qs)^{(\alpha-2)} h(s)\, d_q s + e_k. \quad (19)$$

It follows from Definition 3, Lemma 5, and (18) that

$$\begin{aligned} {}^cD_q^{\beta} u(t) &= \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad + e_k \frac{t^{1-\beta}}{\Gamma_q(2-\beta)}, \quad t \in J_k. \end{aligned} \quad (20)$$

Applying ${}^cD_q^{\beta} u(0) = -{}^cD_q^{\beta} u(1)$ in (20), we obtain

$$e_m = -\frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s. \quad (21)$$

Note boundary conditions $\Delta u|_{t=t_k} = I_k(u(t_k^-))$ and $\Delta D_q u|_{t=t_k} = I_k^*(u(t_k^-))$; we get

$$d_k - d_{k-1} + (e_k - e_{k-1}) t_k = I_k(u(t_k^-)),$$
$$e_k - e_{k-1} = I_k^*(u(t_k^-)), \quad k = 1, 2, \ldots, m. \tag{22}$$

Applying (21) and (22), we have

$$\begin{aligned} e_k &= e_m - \sum_{i=k+1}^{m} I_i^*(u(t_i^-)) \\ &= -\frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad - \sum_{i=k+1}^{m} I_i^*(u(t_i^-)). \end{aligned} \tag{23}$$

Thanks to $u(0) = -u(1)$, it is derived that

$$d_0 = -\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s - d_m - e_m, \tag{24}$$

and

$$\begin{aligned} d_m + d_0 &= -\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s - e_m \\ &= -\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s. \end{aligned} \tag{25}$$

By (22), we get

$$d_m - d_0 = \sum_{i=1}^{m} I_i(u(t_i^-)) - \sum_{i=1}^{m} t_i I_i^*(u(t_i^-)). \tag{26}$$

Combining (25) and (26), we have

$$\begin{aligned} d_m &= \frac{1}{2}\Bigg[-\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad + \sum_{i=1}^{m} I_i(u(t_i^-)) - \sum_{i=1}^{m} t_i I_i^*(u(t_i^-)) \Bigg], \end{aligned}$$

$$\begin{aligned} d_k &= d_m - \sum_{i=k+1}^{m} I_i(u(t_i^-)) + \sum_{i=k+1}^{m} t_i I_i^*(u(t_i^-)) \\ &= -\frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta)} \\ &\quad \cdot \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \frac{1}{2}\sum_{i=1}^{m} I_i(u(t_i^-)) \\ &\quad - \frac{1}{2}\sum_{i=1}^{m} t_i I_i^*(u(t_i^-)) - \sum_{i=k+1}^{m} I_i(u(t_i^-)) \\ &\quad + \sum_{i=k+1}^{m} t_i I_i^*(u(t_i^-)). \end{aligned} \tag{27}$$

Therefore, for $t \in J_k$, $k = 0, 1, 2, \ldots, m-1$,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad - \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} I_i(u(t_i^-)) - \frac{1}{2}\sum_{i=1}^{m} t_i I_i^*(u(t_i^-)) \\ &\quad - \sum_{i=k+1}^{m} I_i(u(t_i^-)) + \sum_{i=k+1}^{m} t_i I_i^*(u(t_i^-)) \\ &\quad - \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} t \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad - t \sum_{i=k+1}^{m} I_i^*(u(t_i^-)) \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad - \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\frac{1}{2} - t\right) \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} I_i(u(t_i^-)) - \frac{1}{2}\sum_{i=1}^{m} t_i I_i^*(u(t_i^-)) \\ &\quad - \sum_{i=k+1}^{m} I_i(u(t_i^-)) + \sum_{i=k+1}^{m} (t_i - t) I_i^*(u(t_i^-)), \end{aligned} \tag{28}$$

and, for $t \in J_m$,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad - \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\frac{1}{2} - t\right) \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \\ &\quad + \frac{1}{2}\sum_{i=1}^m I_i(u(t_i^-)) - \frac{1}{2}\sum_{i=1}^m t_i I_i^*(u(t_i^-)). \end{aligned} \tag{29}$$

□

3. Main Results

Define an operator $A : PC(J,\mathbb{R}) \longrightarrow PC(J,\mathbb{R})$ by

$$\begin{aligned} Au(t) &= \frac{1}{\Gamma_q(\alpha)} \\ &\quad \cdot \int_0^t (t-qs)^{(\alpha-1)} f(s,u(s),Tu(s),Su(s))\, d_q s \\ &\quad - \frac{1}{2\Gamma_q(\alpha)} \\ &\quad \cdot \int_0^1 (1-qs)^{(\alpha-1)} f(s,u(s),Tu(s),Su(s))\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left(\frac{1}{2} - t\right) \\ &\quad \cdot \int_0^1 (1-qs)^{(\alpha-\beta-1)} f(s,u(s),Tu(s),Su(s))\, d_q s \\ &\quad + \frac{1}{2}\sum_{i=1}^m I_i(u(t_i^-)) - \frac{1}{2}\sum_{i=1}^m t_i I_i^*(u(t_i^-)) \\ &\quad - \sum_{i=k+1}^m I_i(u(t_i^-)) + \sum_{i=k+1}^m (t_i - t) I_i^*(u(t_i^-)) \\ &\qquad\qquad t \in J_k,\ k = 0,1,\ldots,m. \end{aligned} \tag{30}$$

Theorem 9. *Assume that*

(H_1) *There exist nonnegative functions* $L_j(t) \in C(J)$ $(j = 1,2,3)$ *such that*

$$\begin{aligned} &|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)| \\ &\quad \le L_1(t)|u_1-u_2| + L_2(t)|v_1-v_2| \\ &\qquad + L_3(t)|w_1-w_2| \end{aligned} \tag{31}$$

for $t \in J$, $u_i, v_i, \omega_i \in \mathbb{R}$, $i = 1,2$.

(H_2) *There exist positive numbers* N *and* N^* *such that*

$$\begin{aligned} &|I_k(u) - I_k(v)| \le N|u-v|, \\ &|I_k^*(u) - I_k^*(v)| \le N^*|u-v|, \\ &\qquad u,v \in \mathbb{R},\ k = 1,2,\ldots,m. \end{aligned} \tag{32}$$

(H_3)

$$\begin{aligned} \chi &= \frac{3(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0)}{2\Gamma_q(\alpha+1)} \\ &\quad + \frac{\Gamma_q(2-\beta)(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0)}{2\Gamma_q(\alpha-\beta+1)} + \frac{3}{2}mN \\ &\quad + \frac{5}{2}mN^* < 1, \end{aligned} \tag{33}$$

where $\overline{L_i} = \max\{L_i(t) : t \in J\}$, $i = 1,2,3$, $k_0 = \max\{|k(t,s)| : (t,s) \in D\}$, $h_0 = \max\{|h(t,s)| : (t,s) \in D_0\}$.

Then BVP (1) has a unique solution.

Proof. For $u,\ v \in PC(J,\mathbb{R})$ and $t \in J$, we have

$$\begin{aligned} &|(Au)(t) - (Av)(t)| \\ &\quad \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),Tu(s),Su(s)) \\ &\quad - f(s,v(s),Tv(s),Sv(s))|\, d_q s + \frac{1}{2} \\ &\quad \cdot \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),Tu(s),Su(s)) \\ &\quad - f(s,v(s),Tv(s),Sv(s))|\, d_q s \\ &\quad + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} \\ &\quad \cdot |f(s,u(s),Tu(s),Su(s)) \\ &\quad - f(s,v(s),Tv(s),Sv(s))|\, d_q s + \frac{1}{2} \\ &\quad \cdot \sum_{i=1}^m |I(u(t_i^-)) - I_i(v(t_i^-))|_i + \frac{1}{2}\sum_{i=1}^m t_i |I_i^*(u(t_i^-)) \\ &\quad - I_i^*(v(t_i^-))| + \sum_{i=1}^m |I_i(u(t_i^-)) - I_i(v(t_i^-))| \\ &\quad + \sum_{i=1}^m t_i |I_i^*(u(t_i^-)) - I_i^*(v(t_i^-))| + t\sum_{i=1}^m |I_i^*(u(t_i^-)) \\ &\quad - I_i^*(v(t_i^-))| \le \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \\ &\quad \cdot \Big(L_1(s)|u(s) - v(s)| \end{aligned}$$

$$
\begin{aligned}
&+ L_2(s)\left|\int_0^s k(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right| \\
&+ L_3(s)\left|\int_0^1 h(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right|\Bigg)\,d_qs \\
&+ \frac{1}{2\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}\Bigg(L_1(s)\,|u(s)-v(s)| \\
&+ L_2(s)\left|\int_0^s k(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right| \\
&+ L_3(s)\left|\int_0^1 h(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right|\Bigg)\,d_qs \\
&+ \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta)}\int_0^1 (1-qs)^{(\alpha-\beta-1)} \\
&\cdot\Bigg(L_1(s)\,|u(s)-v(s)| \\
&+ L_2(s)\left|\int_0^s k(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right| \\
&+ L_3(s)\left|\int_0^1 h(s,\tau)(u(\tau)-v(\tau))\,d_q\tau\right|\Bigg)\,d_qs \\
&+ \frac{3}{2}\sum_{i=1}^m \left|I_i(u(t_i^+)) - I_i(v(t_i^-))\right| + \sum_{i=1}^m\left(\frac{3}{2}t_i + t\right) \\
&\cdot\left|I_i^*(u(t_i^+)) - I_i^*(v(t_i^-))\right| \le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t \\
&- qs)^{(\alpha-1)}\left(\overline{L_1}\,\|u-v\|_{PC} + \overline{L_2}k_0\,\|u-v\|_{PC}\right. \\
&\left.+ \overline{L_3}h_0\,\|u-v\|_{PC}\right)d_qs + \frac{1}{2\Gamma_q(\alpha)}\int_0^1 (1 \\
&- qs)^{(\alpha-1)}\left(\overline{L_1}\,\|u-v\|_{PC} + \overline{L_2}k_0\,\|u-v\|_{PC}\right. \\
&\left.+ \overline{L_3}h_0\,\|u-v\|_{PC}\right)d_qs + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta)}\int_0^1 (1 \\
&- qs)^{(\alpha-\beta-1)}\left(\overline{L_1}\,\|u-v\|_{PC} + \overline{L_2}k_0\,\|u-v\|_{PC}\right. \\
&\left.+ \overline{L_3}h_0\,\|u-v\|_{PC}\right)d_qs + \frac{3}{2}mN\,\|u-v\|_{PC} \\
&+ \frac{5}{2}mN^*\,\|u-v\|_{PC} \le \left(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0\right) \\
&\cdot\left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{2\Gamma_q(\alpha+1)} + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta+1)}\right)\|u \\
&- v\|_{PC} + \left(\frac{3}{2}mN + \frac{5}{2}mN^*\right)\|u-v\|_{PC} \\
&= \Bigg\{\frac{3\left(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0\right)}{2\Gamma_q(\alpha+1)} \\
&+ \frac{\Gamma_q(2-\beta)\left(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0\right)}{2\Gamma_q(\alpha-\beta+1)} + \frac{3}{2}mN + \frac{5}{2} \\
&\cdot mN^*\Bigg\}\,\|u-v\|_{PC} = \chi\,\|u-v\|_{PC},
\end{aligned}
\tag{34}
$$

and then $\|Au - Av\|_{PC} \le \chi\|u - v\|_{PC}$, and hence A is a contraction operator. It follows from Banach contraction mapping principle that BVP (1) has a unique solution. □

Theorem 10. *Assume the following:*

(H_4) *There exist continuous and nondecreasing function* $g : [0,+\infty) \longrightarrow (0,+\infty)$ *and* $a(t) \in C[0,1]$ *such that*

$$
\begin{aligned}
&\left|f(t,u,v,w)\right| \le a(t)\,g(\max\{|u|,|v|,|w|\}), \\
&\qquad t \in [0,1],\ u,v,w \in \mathbb{R}.
\end{aligned}
\tag{35}
$$

(H_5) *There exist continuous and nondecreasing functions* $\varphi,\ \psi : [0,+\infty) \longrightarrow (0,+\infty)$ *such that*

$$
\begin{aligned}
&\left|I_k(u)\right| \le \varphi(|u|), \\
&\left|I_k^*(u)\right| \le \psi(|u|), \\
&\qquad u \in \mathbb{R},\ k = 1,\ldots,m.
\end{aligned}
\tag{36}
$$

(H_6) *There exists constant* $M > 0$ *such that*

$$
\begin{aligned}
&M > a'g(\max\{M, k_0M, h_0M\}) \\
&\quad\cdot\left(\frac{3}{2\Gamma_q(\alpha+1)} + \frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta+1)}\right) + \frac{3m}{2}\varphi(M) \\
&\quad + \frac{5}{2}m\psi(M),
\end{aligned}
\tag{37}
$$

where $a' = \max\{a(t) : t \in [0,1]\}$.

Then BVP (1) has at least one solution.

Proof. The continuity of f, I_k, I_k^* implies that operator $A : PC(J,\mathbb{R}) \longrightarrow PC(J,\mathbb{R})$ is continuous. Let $B \subset PC(J,\mathbb{R})$ be bounded; then there exist positive constants P_1, P_2, and P_3 such that $|f(t,u(t),Tu(t),Su(t))| \le P_1$, $|I_i(u(t_i^-))| \le P_2$, and $|I_i^*(u(t_i^-))| \le P_3$ for all $t \in J$, $u \in B$, $i = 1,2,\ldots,m$. Thus, we have

$$
\begin{aligned}
|(Au)(t)| \le{}& \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}P_1\,d_qs \\
&+ \frac{1}{2}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}P_1\,d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Gamma_q(2-\beta)\int_0^1\frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}P_1d_qs\\
&+\frac{m}{2}P_2+\frac{P_3}{2}\sum_{i=1}^{m}t_i+mP_2+P_3\sum_{i=1}^{m}t_i+mP_3\\
\le{}&\frac{3P_1}{2\Gamma_q(\alpha+1)}+\frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta+1)}P_1+\frac{3m}{2}P_2\\
&+\frac{5}{2}mP_3.
\end{aligned}
\tag{38}
$$

Consequently, operator A is uniformly bounded on B.

On the other hand, for $t_k \le \tau_1 < \tau_2 \le t_{k+1}$, $u \in B$, we have

$$
\begin{aligned}
&|(Au)(\tau_2)-(Au)(\tau_1)| = \Bigg|\frac{1}{\Gamma_q(\alpha)}\int_0^{\tau_2}(\tau_2-qs)^{(\alpha-1)}\\
&\cdot f(s,u(s),Tu(s),Su(s))\,d_qs-\frac{1}{\Gamma_q(\alpha)}\\
&\cdot\int_0^{\tau_1}(\tau_1-qs)^{(\alpha-1)}\\
&\cdot f(s,u(s),Tu(s),Su(s))\,d_qs\\
&+\frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)}(\tau_1-\tau_2)\int_0^1(1-qs)^{(\alpha-\beta-1)}\\
&\cdot f(s,u(s),Tu(s),Su(s))\,d_qs\\
&+\sum_{i=k+1}^{m}I_i^*(u(t_i^-))(\tau_1-\tau_2)\Bigg| \le \frac{1}{\Gamma_q(\alpha)}\\
&\cdot\int_0^{\tau_1}\left|(\tau_2-qs)^{(\alpha-1)}-(\tau_1-qs)^{(\alpha-1)}\right|\\
&\cdot\left|f(s,u(s),Tu(s),Su(s))\right|d_qs+\frac{1}{\Gamma_q(\alpha)}\\
&\cdot\int_{\tau_1}^{\tau_2}\left|(\tau_2-qs)^{(\alpha-1)}f(s,u(s),Tu(s),Su(s))\right|d_qs\\
&+\frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)}P_1|\tau_2-\tau_1|\int_0^1(1-qs)^{(\alpha-\beta-1)}d_qs\\
&+mP_3|\tau_2-\tau_1| \le P_1\frac{\tau_2^\alpha-\tau_1^\alpha}{\Gamma_q(\alpha+1)}+P_1\\
&\cdot\frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta+1)}|\tau_2-\tau_1|+mP_3|\tau_2-\tau_1|,
\end{aligned}
\tag{39}
$$

which tends to zero as $\tau_2 \longrightarrow \tau_1$; then A is equicontinuous on J_k. Hence by PC-type Arzela-Ascoli Theorem ([33]), operator $A : PC(J,\mathbb{R}) \longrightarrow PC(J,\mathbb{R})$ is completely continuous.

Let $u \in PC(J,\mathbb{R})$ be such that $u(t) = \lambda(Au)(t)$ for some $\lambda \in (0,1)$; then

$$
\begin{aligned}
&|u(t)| = |\lambda(Au)(t)| \le |(Au)(t)|\\
&\le\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}a'g(\max\{\|u\|_{PC},k_0\|u\|_{PC},\\
&h_0\|u\|_{PC}\})\,d_qs+\frac{1}{2}\\
&\cdot\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}a'g(\max\{\|u\|_{PC},k_0\|u\|_{PC},\\
&h_0\|u\|_{PC}\})\,d_qs+\frac{1}{2}\Gamma_q(2-\beta)\\
&\cdot\int_0^1\frac{(1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}a'g(\max\{\|u\|_{PC},k_0\|u\|_{PC},\\
&h_0\|u\|_{PC}\})\,d_qs+\frac{1}{2}m\varphi(\|u\|_{PC})+\frac{1}{2}\\
&\cdot\psi(\|u\|_{PC})\sum_{i=1}^{m}t_i+m\varphi(\|u\|_{PC})+\psi(\|u\|_{PC})\sum_{i=1}^{m}t_i\\
&+m\psi(\|u\|_{PC}) \le a'g(\max\{\|u\|_{PC},k_0\|u\|_{PC},\\
&h_0\|u\|_{PC}\})\left(\frac{3}{2\Gamma_q(\alpha+1)}+\frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta+1)}\right)+\frac{3m}{2}\\
&\cdot\varphi(\|u\|_{PC})+\frac{5}{2}m\psi(\|u\|_{PC}),
\end{aligned}
\tag{40}
$$

and hence

$$
\begin{aligned}
\|u\|_{PC} \le{}& a'g(\max\{\|u\|_{PC},k_0\|u\|_{PC},h_0\|u\|_{PC}\})\\
&\cdot\left(\frac{3}{2\Gamma_q(\alpha+1)}+\frac{\Gamma_q(2-\beta)}{2\Gamma_q(\alpha-\beta+1)}\right)+\frac{3m}{2}\\
&\cdot\varphi(\|u\|_{PC})+\frac{5}{2}m\psi(\|u\|_{PC}).
\end{aligned}
\tag{41}
$$

Let $U = \{u \in PC(J,\mathbb{R}) : \|u\| < M\}$; then operator $A : \overline{U} \longrightarrow PC(J,\mathbb{R})$ is completely continuous. By (H_6), one has $u \ne \lambda Au$ for any $\lambda \in (0,1)$ and $u \in \partial U$. By Lemma 7, BVP (1) has at least one solution. □

4. Examples

Example 1. Consider the BVP

$$
\begin{aligned}
{}^cD_{1/2}^{3/2}u(t) = {}&\frac{u(t)+1}{100}+\frac{1}{50+t^2}\int_0^t\frac{u(s)}{e^{(t+s)}}d_qs\\
&+\frac{e^{-t}}{80}\int_0^1\frac{u(s)}{2+t+s}d_qs,\\
&t\in[0,1]\setminus\left\{\frac{1}{2}\right\},
\end{aligned}
$$

$$\begin{aligned}
\Delta u|_{t=1/2} &= \frac{|u(1/2)|}{10+|u(1/2)|},\\
\Delta D_{1/2}u|_{t=1/2} &= \frac{|u(1/2)|}{20+|u(1/2)|},\\
u(0) &= -u(1),\\
{}^cD_{1/2}^{1/4}u(0) &= -{}^cD_{1/2}^{1/4}u(1).
\end{aligned} \tag{42}$$

Let

$$\begin{aligned}
f(t,u,v,w) &= \frac{u+1}{100}+\frac{1}{50+t^2}v+\frac{e^{-t}}{80}w,\\
(Tu)(t) &= \int_0^t e^{-(t+s)}u(s)\,d_qs,\\
(Su)(t) &= \int_0^1 \frac{u(s)}{2+t+s}d_qs.
\end{aligned} \tag{43}$$

By direct computation, $k_0 = \max\{e^{-(t+s)} : 0 \le s \le t \le 1\} = 1$, $h_0 = \max\{1/(2+t+s) : 0 \le s,t \le 1\} = 1/2$. For any $u_1, u_2, v_1, v_2, \omega_1, \omega_2 \in \mathbb{R}$ and $t \in J$, we have

$$\begin{aligned}
&|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|\\
&\quad\le \frac{1}{100}|u_1-u_2|+\frac{1}{50}|v_1-v_2|+\frac{1}{80}|w_1-w_2|,\\
&|I_k(u)-I_k(v)| \le \frac{1}{10}|u-v|,\\
&|I_k^*(u)-I_k^*(v)| \le \frac{1}{20}|u-v|.
\end{aligned} \tag{44}$$

Let $L_1(t) = 1/100$, $L_2(t) = 1/50$, $L_3(t) = 1/80$, $N = 1/10$, and $N^* = 1/20$; then

$$\begin{aligned}
\chi &= \frac{3\left(\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0\right)}{2\Gamma_{1/2}(5/2)}\\
&\quad+\frac{\Gamma_{1/2}(7/4)\left(\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0\right)}{2\Gamma_{1/2}(9/4)}+\frac{3}{2}N+\frac{5}{2}N^*\\
&\approx 0.307 < 1.
\end{aligned} \tag{45}$$

Then, (H_1)-(H_3) hold. It follows from Theorem 9 that BVP (42) has a unique solution.

Example 2. Consider the BVP

$$\begin{aligned}
{}^cD_q^{3/2}u(t) &= \frac{t^2}{60}\left(\sin u(t)+\int_0^t \cos(u(s))\,d_qs\right.\\
&\quad\left.+\int_0^1 \frac{1}{(u(s))^2+t^2+1}d_qs\right), \quad t\in[0,1]\setminus\left\{\frac{1}{2}\right\},\\
\Delta u|_{t=1/2} &= \frac{|u(1/2)|}{20+|u(1/2)|},\\
\Delta D_qu|_{t=1/2} &= \frac{|u(1/2)|}{30+|u(1/2)|},\\
u(0) &= -u(1),\\
{}^cD_q^{1/4}u(0) &= -{}^cD_q^{1/4}u(1).
\end{aligned} \tag{46}$$

Let

$$\begin{aligned}
f(t,u,v,w) &= \frac{t^2}{60}(\sin u+v+w),\\
(Tu)(t) &= \int_0^t \cos(u(s))\,d_qs,\\
(Su)(t) &= \int_0^1 \frac{d_qs}{(u(s))^2+t^2+1};
\end{aligned} \tag{47}$$

then

$$\begin{aligned}
|f(t,u(t),Tu(t),Su(t))| &\le \frac{1}{60}(1+t+1)\\
&\le 6(t+1).
\end{aligned} \tag{48}$$

Let $g(r) = 6$, $a(t) = t+1$; then $a' = \max\{t+1 : t\in[0,1]\} = 2$. Choose $\varphi(u) = 1$ and $\psi(u) = 1$; we have

$$\begin{aligned}
&2\times 6\times\left(\frac{3}{2\Gamma_{1/2}(5/2)}+\frac{\Gamma_{1/2}(7/4)}{2\Gamma_{1/2}(9/4)}\right)+\frac{3}{2}+\frac{5}{2}\\
&\quad\approx 19.87.
\end{aligned} \tag{49}$$

Let $M = 20$; then condition (H_6) holds. Therefore, by Theorem 10, BVP (46) has at least one solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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