Research Article

Existence Results for Impulsive Fractional $q$-Difference Equation with Antiperiodic Boundary Conditions

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In this paper, we investigate the impulsive fractional $q$-difference equation with antiperiodic conditions. The existence and uniqueness results of solutions are established via the theorem of nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle. Two examples are given to illustrate our results.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of solutions for the following impulsive fractional $q$-difference equation with antiperiodic boundary conditions

\[
\begin{align*}
&cD^\alpha_q u(t) = f(t, u(t), Tu(t), Su(t)), \\
&t \in J = [0, 1] \setminus \{t_1, t_2, \ldots, t_m\}, \\
&\Delta u|_{t=t_k} = I_k (u(t_k)), \\
&\Delta D^\beta_q u|_{t=t_k} = I_k^* (u(t_k)), \\
&k = 1, 2, \ldots, m, \\
&u(0) = -u(1), \\
&cD^\beta_q u(0) = -cD^\beta_q u(1),
\end{align*}
\]

where $q \in (0, 1)$, $1 < \alpha \leq 2$, $0 < \beta < 1$, $\alpha - \beta - 1 > 0$, $J = [0, 1]$, $D^\alpha_q$ is $q$-derivative, $cD^\alpha_q$, and $cD^\beta_q$ denote the Caputo $q$-derivative of orders $\alpha$ and $\beta$, respectively. $f \in C(J \times \mathbb{R} \times \mathbb{R})$, $I_k, I_k^*$ are linear operators defined by

\[
Tu(t) = \int_0^t k(t, s) u(s) \, d_q s, \\
Su(t) = \int_0^1 h(t, s) u(s) \, d_q s,
\]

where $k \in C(D, \mathbb{R}), h \in C(J \times J, \mathbb{R}), D = \{(t, s) \in J \times J : t \geq s\}$, $\Delta u|_{t=t_k} = u(t_k) - u(t_k)$, and $\Delta D^\beta_q u|_{t=t_k} = D^\beta_q u(t_k) - D^\beta_q u(t_k)$ represent the right and left limits of $u(t)$ at $t = t_k$. $\Delta D^\beta_q u|_{t=t_k}$ has a similar meaning.

Fractional $q$-difference calculus plays a very important role in modern applied mathematics due to their deep physical background and has been studied extensively [1–4]. Impulsive differential equations are important in both theory and applications. Considerable effort has been devoted to differential equations with or without impulse, for example, [5–21]. In recent years, impulsive fractional difference and differential equations with antiperiodic conditions have received much attention; see [22–27] and the references therein. Zhang and Wang [24] have applied cone contraction fixed point theorem to establish the existence of solutions to nonlinear fractional differential equation with impulses and antiperiodic boundary conditions

\[
cD^\alpha u(t) = f(t, u(t)), \\
1 < \alpha \leq 2, \ t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \ J = [0, T], \\
\Delta u|_{t=t_k} = I_k (u(t_k)),
\]
\[ \Delta u^k_{t=t_k} = I^*_k(u(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ u(0) = -u(T), \]
\[ u'(0) = -u'(T), \]
\[ (3) \]

where \( \delta^c D^\alpha \) is the Caputo fractional derivative, \( f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}), I_k, I^*_k \in C(\mathbb{R}, \mathbb{R}) \). By using Banach fixed point theorem, Schaefer fixed point theorem, and nonlinear alternative of Leray-Schauder type theorem, some existence results of solutions for problem (3) are obtained in [25]. Ahmad et al. [28] studied existence of solutions for the following impulsive fractional antiperiodic boundary value problem (BVP for short) of impulsive fractional \( q \)-difference equation
\[ \delta^c D^\alpha q x(t) = f(t, x(t)), \quad t \in I_k \subseteq [0, T], \quad t \neq t_k, \]
\[ \Delta x|_{t=t_k} = x(t^*_k) - x(t_k) = \phi_k(t_{k-1})^{-\beta_{q_k}}}x(t_k)), \quad k = 1, 2, \ldots, m, \]
\[ t_k D^\alpha q x(t_k) - t_{k-1} D^\alpha q x(t_k) = \phi_k(t_{k-1})^{\beta_{q_k}}}x(t_k)), \quad k = 1, 2, \ldots, m, \]
\[ x(0) = -x(T), \]
\[ D^\alpha q x(0) = -x(T), \]
\[ (4) \]

where \( \delta^c D^\alpha q \) denotes the Caputo \( q \)-fractional derivative of order \( q_k \) on \( I_k, 1 < q_k \leq 2, 0 < q_k < 1, f \in C(\mathbb{R} \times \mathbb{R}), \phi_k, \phi_k^* \in C(\mathbb{R}, \mathbb{R}), k = 1, 2, \ldots, m, t_{k-1} \delta^c D^\alpha q \) and \( t_{k-1} \delta^c D^\alpha q \) denote the Riemann-Liouville \( q_k \)-integral of orders \( \beta_{q_k} \) and \( \gamma_{q_k} \), respectively.

In this paper we are concerned with the existence and uniqueness of solutions for impulsive fractional \( q \)-difference equation antiperiodic BVP. By applying the theorem of nonlinear alternative of Leray-Schauder type and Banach contraction mapping principle, we show the existence and uniqueness of solutions for the BVP (1). Some ideas of this paper are from [29, 30].

2. Preliminaries and Lemmas

For \( q \in (0, 1) \), let
\[ [a]_q = \frac{1 - q^a}{1 - q}, \]
\[ (a; q)_\infty = \prod_{n=0}^{\infty} \left( 1 - a q^n \right), \]
\[ (a; q)_\alpha = \frac{(a; q)_\infty}{(a q^{\alpha}; q)_\infty}, \]
\[ (a, \alpha \in \mathbb{R}). \]
\[ (5) \]

We define the \( q \)-analogue of the power function \((a - b)^n\) with \( n \in \mathbb{N}_q \) is
\[ (a-b)^0 = 1, \]
\[ (a-b)^n = \prod_{k=0}^{n-1} (a - bq_k), \]
\[ n \in \mathbb{N}, \ a, b \in \mathbb{R}, \]
and, for \( \alpha \in \mathbb{R}, \)
\[ (a-b)^{(a)} = a^{\sum_{n=0}^{\infty} \frac{a-bq^n}{(a-bq)^{\alpha}}}. \]
\[ (7) \]
The \( q \)-derivative of \( f \) is defined by
\[ \left(D^\alpha q f \right)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \]
\[ (8) \]
and \( q \)-derivative of higher order by
\[ \left(D^\alpha q f \right)(x) = D^\alpha q \left(D^\alpha q f \right)(x), \]
\[ n \in \mathbb{N}. \]
The \( q \)-integral of \( f \) is defined by
\[ \left(I^\alpha q f \right)(x) = \int_{0}^{x} f(t) q^t \frac{dt}{(1-q)x}, \]
\[ (9) \]
\[ (10) \]

**Lemma 1** (see [31]). (1) If \( f \) is \( q \)-integral on the interval \([0, x], \) then \[ \left[I^\alpha q f \right](t)dt \leq \frac{1}{\alpha} \left[I^\alpha q f \right](0), \]
\[ (2) \] If \( f \) and \( g \) are \( q \)-integral on the interval \([0, x], \) \( f(t) \leq g(t) \) for all \( t \in [0, x], \) then \[ \left[I^\alpha q f \right](t)dt \leq \left[I^\alpha q g \right](0). \]

**Definition 2** (see [2]). Let \( \alpha \geq 0 \) and \( f \) be a function defined on \([0, b], \) the fractional \( q \)-integral of the Riemann-Liouville type is defined by \( \left(I^\alpha q f \right)(x) = f(x) \) and
\[ \left(I^\alpha q f \right)(x) = \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{x} \frac{(x - qt)^{(\alpha-1)}}{(1-q)^{\alpha-1}} f(t) dt, \]
\[ \alpha > 0, \ x \in [0, b]. \]
\[ (11) \]

**Definition 3** (see [3]). The fractional \( q \)-derivative of the Caputo type of order \( \alpha \geq 0 \) is defined by
\[ \left(\delta^c D^\alpha q f \right)(x) = \left(I^\alpha q f \right)(t)\Gamma_{q}(\alpha) \left(I^\alpha q \right)(f(t)), \]
\[ \alpha \geq 0, \]
\[ (12) \]
where \( [\alpha] \) is the smallest integer greater than or equal to \( \alpha. \) If \( f(x) = x^{\beta-1}, \ \beta > 0, \) then \( \delta^c D^\alpha q f(x) = (\Gamma \left(I_{q}^{\alpha} \left(I_{q}^{\beta-\alpha} f \right) \left(t\right) dt, \right) \]
\[ \alpha > 0, \ x \in [0, b]. \]
\[ (13) \]

**Lemma 4** (see [2, 3]). Let \( \alpha, \beta \geq 0 \) and \( f \) be a function defined on \([0, b], \) the following formulas hold:
\[ (1) \left(I^\alpha q \left(I^\beta q f \right)(x) = \left(I^\alpha q \left(I^\beta q f \right)(x); \right) \right) \]
\[ (2) \left(D^\alpha q \left(I^\beta q f \right)(x) = f(x). \right) \]
Lemma 5 (see [3]). Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. Then

\[
\left( I_q^a \, D_q^\beta f \right)(x) = f(x) - \sum_{k=0}^{[\sigma]-1} \left( D_q^k f \right)(a) \frac{a^k}{\Gamma_q (k + 1)} x^k, \quad \sigma \frac{a}{x} \in \mathbb{N},
\]

If $\alpha \geq m \geq \beta$, then $\varepsilon D_q^\beta I_q^a f(x) = I_q^m \varepsilon D_q^\beta f(x) = I_q^a \varepsilon D_q^\beta f(x)$.

Lemma 6 (see [3]). For $\beta \in \mathbb{R}^+$, $\lambda \in (-1, +\infty)$ and $0 \leq a < t \leq b$,

\[
I_q^\beta ((t-a)^\lambda) = \frac{\Gamma_q (\lambda + 1)}{\Gamma_q (\beta + \lambda + 1)} (t-a)^{\beta + \lambda}. \quad (14)
\]

In particular, when $\lambda = 0$ and $a = 0$, using $q$-integration by part,

\[
\left( I_q^\beta 1 \right)(t) = \frac{1}{\Gamma_q (q\beta)} \int_0^t (t-s)^{(1-q\beta)} d_q s = \frac{1}{\Gamma_q (\beta + 1)} t^{\beta}. \quad (15)
\]

Lemma 7 (see [32] (nonlinear alternative of Leray-Schauder type)). Let $X$ be a Banach space, $U$ be a bounded open subset of $X$ with $0 \in U$, and $P : \overline{U} \rightarrow X$ be a completely continuous operator. Then, either there exists $x \in \partial U$ such that $x = \lambda P x$ for $\lambda \in (0, 1)$ or there exists a fixed point $x^* \in \overline{U}$.

Let $P C(J, \mathbb{R}) = \{ u : u \text{ is a map from } J \text{ into } \mathbb{R} \text{ such that } u(t) \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k \text{ and its right limit at } t = t_k \text{ exists for } k = 1, \ldots, m \};$ then $P C(J, \mathbb{R})$ is a Banach space with the norm $\| u \|_{P C} = \sup \{ |u(t)| : t \in J \}$.

Lemma 8. For $h \in P C(J, \mathbb{R})$, the solution of impulsive BVP,

\[
\varepsilon D_q^\beta u(t) = h(t), \quad t \in J', \quad \Delta u_{|t=t_k} = I_k (u(t_k)),
\]

\[
\Delta D_q u_{|t=t_k} = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, m,
\]

\[
u (0) = -u (1),
\]

\[
\varepsilon D_q^\beta u(0) = -\varepsilon D_q^\beta u(1),
\]

is given by

\[
u (t) = \begin{cases} 
\frac{1}{\Gamma_q (\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s) d_q s - \frac{1}{2 \Gamma_q (\alpha)} \int_0^1 (1-q\alpha)^{(\alpha-1)} h(s) d_q s - \frac{1}{2} \sum_{i=1}^{m} I_q (u(t_i)) \\
+ \frac{\Gamma_q (2-\beta)}{\Gamma_q (\alpha)} \left( \frac{1}{2} - t \right) \int_0^1 (1-q\alpha)^{(\alpha-1)} h(s) d_q s + \frac{1}{2} \sum_{i=1}^{m} I_q (u(t_i)) - \sum_{i=1}^{m} I_q (u(t_i)) \\
- \sum_{i=k+1}^{m} I_q (u(t_i)) + \sum_{j=1}^{m} (t_i - t) \Gamma_q^* (u(t_i)), \quad t \in [0, t_k), \quad k = 1, \ldots, m - 1,
\end{cases}
\]

\[
\frac{1}{\Gamma_q (\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s) d_q s - \frac{1}{2 \Gamma_q (\alpha)} \int_0^1 (1-q\alpha)^{(\alpha-1)} h(s) d_q s - \frac{1}{2} \sum_{i=1}^{m} I_q (u(t_i)) \\
+ \frac{\Gamma_q (2-\beta)}{\Gamma_q (\alpha)} \left( \frac{1}{2} - t \right) \int_0^1 (1-q\alpha)^{(\alpha-1)} h(s) d_q s + \frac{1}{2} \sum_{i=1}^{m} I_q (u(t_i)) - \sum_{i=1}^{m} I_q (u(t_i)) \\
- \sum_{i=k+1}^{m} I_q (u(t_i)) + \sum_{j=1}^{m} (t_i - t) \Gamma_q^* (u(t_i)), \quad t \in [t_k, t_{k+1}), \quad k = 1, \ldots, m - 1,
\]

\[
\frac{1}{\Gamma_q (\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s) d_q s + \frac{1}{2} \sum_{i=1}^{m} I_q (u(t_i)) - \sum_{i=1}^{m} I_q (u(t_i)) + \sum_{j=1}^{m} (t_i - t) \Gamma_q^* (u(t_i)), \quad t \in [t_m, 1] .
\]

Proof. In view of Definitions 2 and 3 and Lemma 5, for $t \in J_k = [t_k, t_{k+1}], k = 0, 1, 2, \ldots, m$, we have

\[
u (t) = I_q^a h(t) + d_k + e_k t
\]

\[
= \frac{1}{\Gamma_q (\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s) d_q s + d_k + e_k t, \quad (18)
\]

and

\[
(D_q u)(t) = \frac{1}{\Gamma_q (\alpha - 1)} \int_0^t (t-s)^{(\alpha-2)} h(s) d_q s + e_k. \quad (19)
\]

It follows from Definition 3, Lemma 5, and (18) that

\[
\varepsilon D_q^\beta u(t) = \frac{1}{\Gamma_q (\alpha - \beta)} \int_0^t (t-s)^{(\alpha-\beta-1)} h(s) d_q s
\]

\[
+ e_k \frac{t^{1-\beta}}{\Gamma_q (2 - \beta)}, \quad t \in J_k.
\]

Applying $\varepsilon D_q^\beta u(0) = -\varepsilon D_q^\beta u(1)$ in (20), we obtain

\[
\epsilon_m = -\frac{\Gamma_q (2 - \beta)}{\Gamma_q (\alpha - \beta)} \int_0^1 (1-q\alpha)^{(\alpha-\beta-1)} h(s) d_q s. \quad (21)
\]
Applying (21) and (22), we have
\[ d_k - d_{k-1} + (e_k - e_{k-1}) t_k = I_k(u(t_k^*)), \]
\[ e_k - e_{k-1} = I_k^*(u(t_k^*)), \]
for \( k = 1, 2, \ldots, m. \)

Note boundary conditions
\[ \Delta u |_{t=4} = I_k(u(t_k^*)), \]

Therefore, for \( t \in J_k, k = 0, 1, 2, \ldots, m - 1, \)
\[ u(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q s)^{(\alpha-1)} h(s) d_q s \]
\[ - \frac{1}{2 \Gamma_q(\alpha)} \int_0^1 (1 - q s)^{(\alpha-1)} h(s) d_q s + \frac{\Gamma_q'(2 - \beta)}{2 \Gamma_q(\alpha - \beta)} \int_0^1 (1 - q s)^{(\alpha-\beta-1)} h(s) d_q s + \frac{1}{2} \sum_{i=1}^m I_i(u(t_i^*)) \]
\[ - \frac{1}{2} \sum_{i=1}^m t_i I_i^*(u(t_i^*)). \]
and, for \( t \in J_m, \)
\[
u(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) \, ds
- \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s) \, ds
\]
\[
+ \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left( \frac{1}{2} - t \right) \int_0^1 (1-qs)^{(\alpha-\beta-1)} h(s) \, ds
+ \frac{1}{2} \sum_{i=1}^m l_i(u(t_i^*)) + \sum_{i=1}^m t_i I_i^*(u(t_i^*)) \]
\[
- \sum_{i=1}^m l_i(u(t_i^*)) + \sum_{i=1}^m (t_i - t) I_i^*(u(t_i^*))
\]
\[
\quad t \in J_k, \quad k = 0, 1, \ldots, m.
\]

**3. Main Results**

Define an operator \( A : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) by
\[
A(u)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s, u(s), Tu(s), Su(s)) \, ds
- \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} f(s, u(s), Tu(s), Su(s)) \, ds
\]
\[
+ \frac{\Gamma_q(2-\beta)}{\Gamma_q(\alpha-\beta)} \left( \frac{1}{2} - t \right) \int_0^1 (1-qs)^{(\alpha-\beta-1)} f(s, u(s), Tu(s), Su(s)) \, ds
+ \frac{1}{2} \sum_{i=1}^m l_i(u(t_i^*)) + \sum_{i=1}^m t_i I_i^*(u(t_i^*))
\]
\[
- \sum_{i=1}^m l_i(u(t_i^*)) + \sum_{i=1}^m (t_i - t) I_i^*(u(t_i^*))
\]
\[
\quad t \in J_k, \quad k = 0, 1, \ldots, m.
\]

**Theorem 9.** Assume that

\((H_1)\) There exist nonnegative functions \( L_j(t) \in C(J) \) \( (j = 1, 2, 3) \) such that
\[
|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)|
\leq L_1(t) |u_1 - u_2| + L_2(t) |v_1 - v_2|
+ L_3(t) |w_1 - w_2|
\]
for \( t \in J, u_i, v_i, w_i \in \mathbb{R}, i = 1, 2. \)

\((H_2)\) There exist positive numbers \( N \) and \( N^* \) such that
\[
|I_k(u) - I_k(v)| \leq N |u - v|
\]
\[
|I_k^*(u) - I_k^*(v)| \leq N^* |u - v|
\]
for \( u, v \in \mathbb{R}, k = 1, 2, \ldots, m. \)

\((H_3)\)
\[
X = \frac{3(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0)}{2\Gamma_q(\alpha + 1)} + \frac{\Gamma_q(2-\beta)(\overline{L_1} + \overline{L_2}k_0 + \overline{L_3}h_0)}{2\Gamma_q(\alpha - \beta + 1)} + \frac{3}{2} mN
\]
\[
+ \frac{5}{2} mN^* < 1,
\]

where \( \overline{L_j} = \max\{L_j(t) : t \in J\}, \quad i = 1, 2, 3, \quad k_0 = \max\{|k(t, s)| : (t, s) \in D\}, \quad h_0 = \max\{|h(t, s)| : (t, s) \in D\}. \)

Then BVP (1) has a unique solution.

**Proof.** For \( u, v \in PC(J, \mathbb{R}) \) and \( t \in J, \) we have
\[
|\max_{I_k}(A(u)(t) - (Av)(t))|
\leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} |f(s, u(s), Tu(s), Su(s))|
\]
\[
- f(s, v(s), Tv(s), Sv(s))| \, ds + \frac{1}{2}
\]
\[
\cdot \int_0^1 (1-qs)^{(\alpha-1)} |f(s, u(s), Tu(s), Su(s))| \, ds
- f(s, v(s), Tv(s), Sv(s))| \, ds + \frac{1}{2}
\]
\[
\cdot \sum_{i=1}^m |I_i(u(t_i^*)) - I_i(v(t_i^*))| + \sum_{i=1}^m |I_i^*(u(t_i^*)) - I_i^*(v(t_i^*))|
\]
\[
\leq L_1(t) |u_1 - u_2| + L_2(t) |v_1 - v_2|
+ L_3(t) |w_1 - w_2|
\]
\[
\left( L_1(s) |u(s) - v(s)| \right)
\]
\[
|\max_{I_k}(A(u)(t) - (Av)(t))|
\leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} |f(s, u(s), Tu(s), Su(s))|
\]
\[
- f(s, v(s), Tv(s), Sv(s))| \, ds + \frac{1}{2}
\]
\[
\cdot \int_0^1 (1-qs)^{(\alpha-1)} |f(s, u(s), Tu(s), Su(s))| \, ds
- f(s, v(s), Tv(s), Sv(s))| \, ds + \frac{1}{2}
\]
\[
\cdot \sum_{i=1}^m |I_i(u(t_i^*)) - I_i(v(t_i^*))| + \sum_{i=1}^m |I_i^*(u(t_i^*)) - I_i^*(v(t_i^*))|
\]
\[
\leq L_1(t) |u_1 - u_2| + L_2(t) |v_1 - v_2|
+ L_3(t) |w_1 - w_2|
\]
\[
\begin{align*}
&+ L_2 (s) \left[ \int_0^t k(s, \tau) (u(\tau) - v(\tau)) \, d_q \tau \right] \\
&+ L_3 (s) \left[ \int_0^t h(s, \tau) (u(\tau) - v(\tau)) \, d_q \tau \right] \\
&+ \frac{1}{2 \Gamma_q (\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \left( L_1 (s) |u(s) - v(s)| \right) \\
&+ L_2 (s) \left[ \int_0^t k(s, \tau) (u(\tau) - v(\tau)) \, d_q \tau \right] \\
&+ L_3 (s) \left[ \int_0^t h(s, \tau) (u(\tau) - v(\tau)) \, d_q \tau \right] \\
&+ \frac{\Gamma_q (2 - \beta)}{2 \Gamma_q (\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} \\
&\cdot \left( L_1 (s) |u(s) - v(s)| \right) \\
&+ \frac{1}{2 \Gamma_q (\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} \left( L_1 (s) |u(s) - v(s)| \right)
\end{align*}
\]

\[
\begin{align*}
&+ L_1 (s) \left[ \int_t^1 |L_1 (s) |u(s) - v(s)| \right] \\
&+ \frac{\Gamma_q (2 - \beta)}{2 \Gamma_q (\alpha - \beta)} \int_0^1 (1 - qs)^{(\alpha-\beta-1)} \\
&\cdot \left( L_1 (s) |u(s) - v(s)| \right)
\end{align*}
\]

and then \( \|Au - Av\|_{PC} \leq \chi \|u - v\|_{PC} \), and hence \( A \) is a contraction operator. It follows from Banach contraction mapping principle that BVP (1) has a unique solution.

**Theorem 10.** Assume the following:

\( (H_4) \) There exist continuous and nondecreasing function \( g : [0, +\infty) \rightarrow (0, +\infty) \) and \( a(t) \in C[0,1] \) such that

\[
|f(t, u, v, w)| \leq a(t) g(\max\{|u|, |v|, |w|\})
\]

\( t \in [0,1] \), \( u, v, w \in \mathbb{R} \).

\( (H_5) \) There exist continuous and nondecreasing functions \( \varphi, \psi : [0, +\infty) \rightarrow (0, +\infty) \) such that

\[
|I_k^* (u)| \leq \varphi(|u|),
\]

\[
|I_k^{\ast} (u)| \leq \psi(|u|),
\]

\( u \in \mathbb{R}, \ k = 1, \ldots, m. \)

\( (H_6) \) There exists constant \( M > 0 \) such that

\[
M > a' \left( \max\{|M, k_0 M, h_0 M\} \right)
\]

\[
\cdot \left( \frac{3}{2 \Gamma_q (\alpha + 1)} + \frac{\Gamma_q (2 - \beta)}{2 \Gamma_q (\alpha - \beta + 1)} \right) + \frac{3 m}{2} \varphi (M)
\]

\[
+ \frac{5}{2} m \psi (M),
\]

where \( a' = \max\{a(t) : t \in [0,1]\} \).

Then BVP (1) has at least one solution.

**Proof.** The continuity of \( f, I_k, I_k^* \) implies that operator \( A : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) is continuous. Let \( B \subset PC(J, \mathbb{R}) \) be bounded; then there exist positive constants \( P_1, P_2, \) and \( P_3 \) such that \( |f(t, u(t), Tu(t), Su(t))| \leq P_1 \), \( |I_k (u(t))| \leq P_2 \), and \( |I_k^{\ast} (u(t))| \leq P_3 \) for all \( t \in J, u \in B, \ i = 1, 2, \ldots, m \). Thus, we have

\[
|(Au) (t)| \leq \int_0^t \left( \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q (\alpha)} \right) P_i d_q s
\]

\[
+ \frac{1}{2} \int_0^1 (1 - qs)^{(\alpha-1)} \left( L_1 + L_2 k_0 + L_3 h_0 \right) d_q s
\]
\[ (Au)(t_2) - (Au)(t_1) = \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (\tau - qs)^{(\alpha-1)} f(s, u(s), Tu(s), Su(s)) \, ds \]
\[ \cdot f(s, u(s), Tu(s), Su(s)) \, ds + \frac{1}{\Gamma_q(\alpha)} (t_1 - t_2)  \]
\[ \cdot f(s, u(s), Tu(s), Su(s)) \, ds + \frac{\Gamma_q(2\beta)}{\Gamma_q(\alpha - \beta)} (t_1 - t_2) \int_0^1 (1 - qs)^{(\alpha-\beta-1)} \, ds \]
\[ \cdot f(s, u(s), Tu(s), Su(s)) \, ds \]
\[ + \sum_{i=k+1}^m f_i(u(t_i))(t_1 - t_2) + \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} \left| (\tau - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)} \right| \, ds \]
\[ \cdot f(s, u(s), Tu(s), Su(s)) \, ds + \frac{1}{\Gamma_q(\alpha)} (t_1 - t_2) \int_0^1 (1 - qs)^{(\alpha-\beta-1)} \, ds \]
\[ \cdot f(s, u(s), Tu(s), Su(s)) \, ds + \frac{\Gamma_q(2\beta)}{\Gamma_q(\alpha - \beta)} P_1 |r_2 - r_1| + \frac{1}{\Gamma_q(\alpha + 1)} + P_1 \]
\[ \cdot \frac{\Gamma_q(2\beta)}{\Gamma_q(\alpha - \beta + 1)} |r_2 - r_1| + mP_3 |r_2 - r_1|, \]

Consequently, operator \( A \) is uniformly bounded on \( B \).

On the other hand, for \( t_k \leq t_1 < t_2 \leq t_{k+1}, \ u \in B \), we have

\[ \|u\|_{PC} \leq a' g (\max \{ \|u\|_{PC}, k_0 \|u\|_{PC} \}), \]

\[ h_0 \|u\|_{PC} d_s + \frac{1}{2} \]

\[ \cdot \psi (\|u\|_{PC}) \sum_{i=1}^m t_i + m \psi (\|u\|_{PC}) + \psi (\|u\|_{PC}) \sum_{i=1}^m t_i \]

\[ + m \psi (\|u\|_{PC}) \leq a' g (\max \{ \|u\|_{PC}, k_0 \|u\|_{PC} \}), \]

\[ h_0 \|u\|_{PC} d_s + \frac{1}{2} m \psi (\|u\|_{PC}) + \frac{1}{2} \]

\[ \cdot \varphi (\|u\|_{PC}) + \frac{5}{2} m \psi (\|u\|_{PC}), \]

and hence

\[ \|u\|_{PC} \leq a' g (\max \{ \|u\|_{PC}, k_0 \|u\|_{PC}, h_0 \|u\|_{PC} \}) \]

\[ \cdot \left( \frac{3}{2\Gamma_q(\alpha + 1)} + \frac{\Gamma_q(2\beta)}{2\Gamma_q(\alpha - \beta + 1)} \right) + \frac{3m}{2} \]

\[ \cdot \varphi (\|u\|_{PC}) + \frac{5}{2} m \psi (\|u\|_{PC}). \]

Let \( U = \{ u \in PC(J, \mathbb{R}) : \|u\| < M \}; \) then operator \( A : \mathbb{U} \rightarrow PC(J, \mathbb{R}) \) is completely continuous. By \((H_2),\) one has \( u \neq \lambda Au \) for any \( \lambda \in (0, 1) \) and \( u \in \partial U \). By Lemma 7, BVP (1) has at least one solution. \( \square \)

### 4. Examples

**Example 1.** Consider the BVP

\[ \frac{\epsilon}{3} D_{1/2}^{3/2} u(t) = \frac{u(t)}{100} + \frac{1}{50 + t^2} \int_0^t u(s) e^{(t-s)/2} \, ds + \frac{e^{-t}}{80} \int_0^1 u(s) e^{-2(t-s)/5} \, ds, \]

\[ t \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}, \]

where \( \epsilon = 1 \). Hence by \( PC \)-type Arzela-Ascoli Theorem ([33]), operator \( A : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) is completely continuous.
\[ \Delta u|_{t=1/2} = \frac{|u(1/2)|}{10 + |u(1/2)|}, \]
\[ \Delta D_{1/2}u|_{t=1/2} = \frac{|u(1/2)|}{20 + |u(1/2)|}, \]
\[ u(0) = -u(1), \]
\[ cD_{1/2}^{1/4}u(0) = -cD_{1/2}^{1/4}u(1). \] (42)

Let
\[ f(t, u, v, w) = \frac{u + 1}{100} + \frac{1}{50 + t^2} v + \frac{e^{-t}}{80} w, \]
\[ (Tu)(t) = \int_0^t e^{-(t+s)} u(s) \, ds, \]
\[ (Su)(t) = \int_0^1 \frac{u(s)}{2 + t + s} \, ds. \] (43)

By direct computation, \( k_0 = \max\{|e^{-(t+s)} : 0 \leq s \leq t \leq 1| = 1, \)
\( h_0 = \max\{1/(2 + t + s) : 0 \leq s, t \leq 1| = 1/2. \)
For any \( u_1, u_2, v_1, v_2, \omega_1, \omega_2 \in \mathbb{R} \) and \( t \in J, \) we have
\[ |f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \]
\[ \leq \frac{1}{100} |u_1 - u_2| + \frac{1}{50} |v_1 - v_2| + \frac{1}{80} |w_1 - w_2|, \]
\[ |L_k(u) - L_k(v)| \leq \frac{1}{10} |u - v|, \]
\[ |L_k^*(u) - L_k^*(v)| \leq \frac{1}{20} |u - v|. \] (44)

Let \( L_1(t) = 1/100, L_2(t) = 1/50, L_3(t) = 1/80, N = 1/10, \) and \( N^* = 1/20; \) then
\[ \chi = \frac{3}{2\Gamma_{1/2}(5/2)} \left( \Gamma_{1/2}(7/4) + \frac{\Gamma_{1/2}(7/4)}{2\Gamma_{1/2}(9/4)} + \frac{2N + 5N^*}{2} \right) \]
\[ \approx 0.307 < 1. \]

Then, \((H_1)-(H_4)\) hold. It follows from Theorem 9 that BVP (42) has a unique solution.

**Example 2.** Consider the BVP
\[ cD_{1/2}^{3/2} u(t) = \frac{t^2}{60} (\sin u(t) + \int_0^t \cos (u(s)) \, ds), \]
\[ + \int_0^1 \frac{1}{(u(s))^2 + t^2 + 1} \, ds, \quad t \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}, \]
\[ \Delta u|_{t=1/2} = \frac{|u(1/2)|}{20 + |u(1/2)|}, \]
\[ \Delta D_{1/2}^{3/4} u|_{t=1/2} = \frac{|u(1/2)|}{30 + |u(1/2)|}, \]
\[ u(0) = -u(1), \]
\[ cD_{1/2}^{3/4} u(0) = -cD_{1/2}^{3/4} u(1). \] (49)

Let
\[ f(t, u, v, w) = \frac{t^2}{60} (\sin u + v + w), \]
\[ (Tu)(t) = \int_0^t \cos (u(s)) \, ds, \]
\[ (Su)(t) = \int_0^1 \frac{d_q s}{(u(s))^2 + t^2 + 1}; \]
then
\[ |f(t, u(t), Tu(t), Su(t))| \leq \frac{1}{60} (1 + t + 1) \]
\[ \leq 6(t + 1). \] (48)

Let \( g(r) = 6, a(t) = t + 1; \) then \( a'_1 = \max\{|t + 1 : t \in [0, 1]| = 2. \)
Choose \( \phi(u) = 1, \psi(u) = 1; \) we have
\[ 2 \times 6 \times \left( \frac{3}{2\Gamma_{1/2}(5/2)} + \frac{\Gamma_{1/2}(7/4)}{2\Gamma_{1/2}(9/4)} + \frac{3}{2} + \frac{5}{2} \right) \]
\[ \approx 19.87. \]

Let \( M = 20; \) then condition \((H_5)\) holds. Therefore, by Theorem 10, BVP (46) has at least one solution.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.

**Authors’ Contributions**

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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