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Research Article

A Result on the Existence and Uniqueness of Stationary Solutions for a Bioconvective Flow Model

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In this note, we prove the existence and uniqueness of weak solutions for the boundary value problem modelling the stationary case of the bioconvective flow problem. The bioconvective model is a boundary value problem for a system of four equations: the nonlinear Stokes equation, the incompressibility equation, and two transport equations. The unknowns of the model are the velocity of the fluid, the pressure of the fluid, the local concentration of microorganisms, and the oxygen concentration. We derive some appropriate a priori estimates for the weak solution, which implies the existence, by application of Gossez theorem, and the uniqueness by standard methodology of comparison of two arbitrary solutions.

1. Introduction

Bioconvection is an important process in the biological treatment and in the life of some microorganisms. In a broad sense, biconvection originates from the concentration of upward swimming microorganisms in a culture fluid. It is well known that, under some physical assumptions, the process can be described by mathematical models which are called bioconvective flow models. The first model of this kind was derived by Moribe [1] and independently by Levandowsky et al. [2] (see also [3] for the mathematical analysis). In that model the unknowns are the velocity of the fluid, the pressure of the fluid, and the local concentration of microorganisms. More recently, Tuval et al. [4] have introduced a new bioconvective flow model considering an additional unknown variable, the oxygen concentration. Some advances in mathematical analysis and some numerical results for this new model are presented in [5] and [6], respectively.

In this paper, we are interested in the existence and uniqueness of solutions for the stationary problem associated with the bioconvective system given in [4] when the physical domain is a three-dimensional chamber [6] (a parallelepiped). Thus, the stationary bioconvective flow problem to be analyzed is formulated as follows. Given the external force $\mathbf{F}$, the source functions f_n, f_c, and the dimensionless function r, find the velocity of the fluid $\mathbf{u} = (u_1, u_2, u_3)^t$, the fluid pressure p, the local concentration of bacteria n, and the local concentration of oxygen c satisfying the boundary value problem:

$$-S_c\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + S_c\nabla p = \gamma S_c n\mathbf{g} + \mathbf{F}, \quad \text{in } \Omega \coloneqq \prod_{i=1}^{3}[0, L_i], \tag{1}$$

$$\operatorname{div}(\mathbf{u}) = 0, \quad \text{in } \Omega, \tag{2}$$

$$-\Delta n + (\mathbf{u}\cdot\nabla)\,n + \chi\operatorname{div}(nr(c)\nabla c) = f_n, \quad \text{in } \Omega, \tag{3}$$

$$-\delta\Delta c + (\mathbf{u}\cdot\nabla)\,c + \beta r(c)\,n = f_c, \quad \text{in } \Omega, \tag{4}$$

$$\nabla c\cdot\boldsymbol{\nu} = \nabla n\cdot\boldsymbol{\nu} = 0, \quad \mathbf{u} = 0, \quad \text{on } \partial\Omega_L\ (x_3 = 0), \tag{5}$$

$$\chi nr(c)\nabla c\cdot\boldsymbol{\nu} - \nabla n\cdot\boldsymbol{\nu} = 0, \quad \mathbf{u} = 0, \quad \text{on } \partial\Omega_U \coloneqq \partial\Omega - \partial\Omega_L. \tag{6}$$

Here $\boldsymbol{\nu}$ is the unit external normal to $\partial\Omega$; $\mathbf{g} = (0, 0, -g)$ is a gravitational field with constant acceleration g; and S_c, γ, α, δ, and β are some physical parameters defined as follows:

$$\begin{aligned} S_c &= \frac{\eta}{D_n \rho}, \\ \gamma &= \frac{V_b n_r (\rho_b - \rho) L^3}{\eta D_n}, \\ \chi &= \frac{\overline{\chi} c_{\mathrm{air}}}{D_n}, \\ \delta &= \frac{D_c}{D_n}, \\ \beta &= \frac{k n_r L^2}{c_{\mathrm{air}} D_n}, \end{aligned} \tag{7}$$

with η being the fluid viscosity, D_n the diffusion constant for bacteria, D_c the diffusion constant for oxygen, ρ the fluid density, ρ_b the bacterial density, $V_b > 0$ the bacterial volume, n_r a characteristic cell density, L a characteristic length, $\overline{\chi}$ the chemotactic sensitivity, c_{air} the oxygen concentration above the fluid, and k the oxygen consumption rate.

We consider the standard notation of the Lebesgue and Sobolev spaces which are used in the analysis of Navier-Stokes and related equations of fluid mechanics; see [7–11] for details and specific definitions. In particular, we use the following rather common spaces notation:

$$\begin{aligned} H^m(\Omega) &= W^{m,2}(\Omega), \\ \widetilde{H}^1(\Omega) &= \left\{ f \in H^1(\Omega) : \int_\Omega f \, d\mathbf{x} = 0 \right\}, \\ H_0^1(\Omega) &= \overline{\mathbf{C}_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}, \\ \mathbf{C}_{0,\sigma}^\infty(\Omega) &= \left\{ \mathbf{v} \in \left(C_0^\infty(\Omega)\right)^3 : \operatorname{div}(\mathbf{v}) = 0 \right\}, \\ \mathbf{V} &= \overline{\mathbf{C}_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{H_0^1(\Omega)}}, \end{aligned} \tag{8}$$

where $\overline{A}^{\|\cdot\|_B}$ denotes the completion of A in B. Also, we consider the notation for the applications $a_0 : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $b_0 : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, and $b : \mathbf{V} \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, which are defined as

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{v}) &= (\nabla \mathbf{u}, \nabla \mathbf{v}), \\ a(\phi, \psi) &= (\nabla \phi, \nabla \psi), \\ b_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}), \\ b(\mathbf{u}, \phi, \psi) &= (\mathbf{u} \cdot \nabla \phi, \psi), \end{aligned} \tag{9}$$

where $(\cdot, \cdot)$ is the standard inner product in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$. It is well known that a_0 and a are bilinear coercive forms and b_0 and b are well defined trilinear forms with the following properties:

$$\begin{aligned} b_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= -b_0(\mathbf{u}, \mathbf{w}, \mathbf{v}), \\ b(\mathbf{u}, \phi, \psi) &= -b(\mathbf{u}, \psi, \phi), \\ b_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) &= 0, \\ b(\mathbf{u}, \phi, \phi) &= 0, \end{aligned} \tag{10}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ and $\psi, \phi \in H^1(\Omega)$. Moreover, we need to introduce some notation related to some useful Sobolev inequalities and estimates for b and b_0. There exist $C_{\mathrm{poi}} > 0$, $C_{\mathrm{tr}} > 0$, and C_1 depending only on Ω such that

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} &\leq C_{\mathrm{poi}} \|\mathbf{u}\|_{\mathbf{V}}, \\ \|c\|_{L^2(\Omega)} &\leq C_{\mathrm{poi}} \|c\|_{\widetilde{H}^1(\Omega)}, \\ \|\varphi\|_{L^1(\partial\Omega)} &\leq C_{\mathrm{tr}} \|\varphi\|_{W^{1,1}(\Omega)}, \\ \left|b_0(\mathbf{u}, \mathbf{v}, \mathbf{w})\right| &\leq C_1 \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}, \\ |b(\mathbf{u}, c, n)| &\leq C_1 \|\mathbf{u}\|_{\mathbf{V}} \|c\|_{\widetilde{H}^1(\Omega)} \|n\|_{\widetilde{H}^1(\Omega)}, \end{aligned} \tag{11}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, $c, n \in \widetilde{H}^1(\Omega)$, and $\varphi \in W^{1,1}(\Omega)$. For details on Poincaré and trace inequalities, we refer to [8] and for the estimates of b_0 and b consult [11].

The main result of the paper is the existence and uniqueness of weak solutions for (1)–(6). Indeed, let us introduce some appropriate notation:

$$\begin{aligned} \Theta_1 &:= \frac{1 - C_{\mathrm{tr}}}{1 - C_{\mathrm{tr}} - 2\chi \|r\|_{L^1(\mathbb{R})} C_{\mathrm{tr}} C_{\mathrm{poi}}}, \\ \Theta_2 &:= \frac{1 - C_{\mathrm{tr}}}{1 - C_{\mathrm{tr}} - C_{\mathrm{tr}} C_{\mathrm{poi}}}, \end{aligned} \tag{12}$$

$$\begin{aligned} \Gamma_0 = {} & \frac{|\Omega| \Theta_1 C_{\mathrm{poi}}}{|\Omega| - \chi \beta \alpha_1 \|r\|_{L^\infty(\mathbb{R})}^2 C_{\mathrm{poi}}^2 \Theta_1 \Theta_2} \left[\frac{\chi \alpha_1 \|r\|_{L^\infty(\mathbb{R})}^2 \Theta_2}{\delta |\Omega|} \|f_c\|_{L^2(\Omega)} \right. \\ & \left. + \|f_n\|_{L^2(\Omega)} \right], \end{aligned} \tag{13}$$

$$\begin{aligned} \Gamma_1 &= \frac{\gamma S_c g C_{\mathrm{poi}}}{S_c - C_1 C_{\mathrm{poi}} \left(\gamma g \Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right)}, \\ \Gamma_2 &= \frac{1 - C_{\mathrm{tr}}}{1 - 2\|r\|_{L^1(\mathbb{R})} \left(1 - C_{\mathrm{tr}} + C_{\mathrm{tr}} C_{\mathrm{poi}}\right)}, \end{aligned} \tag{14}$$

$$\Gamma_3 = \frac{1 - C_{\mathrm{tr}}}{\delta \left(1 - C_{\mathrm{tr}} - C_{\mathrm{tr}} C_{\mathrm{poi}}\right) - (C_1)^3 \|r\|_{\mathrm{Lip}(\mathbb{R})} \Gamma_0}, \tag{15}$$

such that the result is precised as follows.

Theorem 1. *Let us consider that $f_c, f_b \in L^2(\Omega)$, $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\overline{n}$, the average of n on Ω, are given. Also consider notations (12)–(15). If we assume that the following assumptions,*

$$\begin{aligned} & r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \\ & 1 - C_{tr} > C_{tr} C_{poi} \max\left\{2\chi \|r\|_{L^1(\mathbb{R})}, 1\right\}, \\ & 1 > \chi \beta \overline{n} \|r\|_{L^\infty(\mathbb{R})}^2 C_{poi}^2 \Theta_1 \Theta_2, \end{aligned} \tag{16}$$

are satisfied, there is $(\mathbf{u}, p, n, c) \in \mathbf{V} \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ *satisfying (1)–(6). Moreover, if we consider that additionally* $r \in Lip(\mathbb{R})$ *and the following inequalities,*

$$S_c - C_1 C_{poi}\left(\gamma g \Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right) > 0,$$
$$\delta\left(1 - C_{tr} - C_{tr}C_{poi}\right) - (C_1)^3 \|r\|_{L^1(\mathbb{R})}\Gamma_0 > 0, \tag{17}$$
$$C_1 \|r\|_{Lip(\mathbb{R})}\Gamma_0 < 1,$$

$$\Pi = \Gamma_1\Gamma_2 \left\{ C_1\Gamma_0 + \frac{\|r\|_{L^\infty(\mathbb{R})} C\Gamma_3\Theta_2 C_{poi}}{\delta\left(1 - C_1\|r\|_{Lip(\mathbb{R})}\Gamma_0\right)} \left[\beta C_{poi}\|r\|_{L^\infty(\mathbb{R})}\Gamma_0 + \|f_c\|_{L^2(\Omega)}\right]\right\} < 1, \tag{18}$$

are satisfied, the weak solution is unique.

It should be noted that existence and uniqueness results are derived in [12, 13] for the bioconvection problem, when the concentration of oxygen is assumed to be constant. In the case of [12], the proof is based on the application of the Galerkin approximation and in [13] on the application of the Gossez theorem. Moreover, other related results are given in [3, 5]. In particular, in [5], a well detailed discussion of some particular models derived from (1)–(6) is given.

2. Proof of Theorem 1

2.1. Variational Formulation. By standard arguments, the variational formulation of (1)–(6) is given by

$$\begin{aligned}
&\text{Find } (\mathbf{u}, n, c) \in \mathbf{V} \times H^1(\Omega) \times H^1(\Omega) \text{ such that}\\
&S_c a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \gamma S_c(n\mathbf{g}, \mathbf{v}) + (\mathbf{F}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V},\\
&a(n, \phi) + b(\mathbf{u}, n, \phi) = \chi(nr(c)\nabla c, \nabla\phi) + (f_n, \phi), \quad \forall \phi \in H^1(\Omega),\\
&\delta a(c, \varphi) + b(\mathbf{u}, c, \varphi) = -\beta(r(c)n, \varphi) + \delta\int_{\partial\Omega_U} \nabla c \cdot \boldsymbol{\nu}\varphi\, dS + (f_c, \varphi), \quad \forall \varphi \in H^1(\Omega).
\end{aligned} \tag{19}$$

We notice that if $f_c = f_n = 0$ and $\mathbf{u}_0$ is a solution of (1)-(2) with $n = 0$, we have that $(\mathbf{u}_0, 0, 0)$ is a solution of (19). However, $(\mathbf{u}_0, 0, 0)$ does not describe the bioconvective flow problem and we need to study the variational problem when the total local concentration of bacteria and the total local concentration of oxygen are some given strictly positive constants, that is, $\int_\Omega n_{\boldsymbol{\alpha}} d\mathbf{x} = \alpha_1 > 0$ and $\int_\Omega c_{\boldsymbol{\alpha}} d\mathbf{x} = \alpha_2 > 0$. Thus, by considering the change of variable $\widehat{n}_{\boldsymbol{\alpha}} = n_{\boldsymbol{\alpha}} - \alpha_1|\Omega|^{-1}$ and $\widehat{c}_{\boldsymbol{\alpha}} = c_{\boldsymbol{\alpha}} - \alpha_2|\Omega|^{-1}$, we can rewrite (19) as follows:

$$\text{Given } \boldsymbol{\alpha} = (\alpha_2, \alpha_2) \in]0,1] \times]0,1] \text{ find } (\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}}) \in \mathbf{V} \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega): \tag{20}$$

$$S_c a_0(\mathbf{u}_{\boldsymbol{\alpha}}, \mathbf{v}) + b_0(\mathbf{u}_{\boldsymbol{\alpha}}, \mathbf{u}_{\boldsymbol{\alpha}}, \mathbf{v}) = \gamma S_c(\widehat{n}_{\boldsymbol{\alpha}}\mathbf{g}, \mathbf{v}) + (\mathbf{F}, \mathbf{v}), \tag{21}$$

$$a(\widehat{n}_{\boldsymbol{\alpha}}, \phi) + b(\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \phi) = \chi\left(\left(\widehat{n}_{\boldsymbol{\alpha}} + \frac{\alpha_1}{|\Omega|}\right) r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\phi\right) + (f_n, \phi), \tag{22}$$

$$\delta a(\widehat{c}_{\boldsymbol{\alpha}}, \varphi) + b(\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}}, \varphi) = -\beta\left(r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\left(\widehat{n}_{\boldsymbol{\alpha}} + \frac{\alpha_1}{|\Omega|}\right), \varphi\right) + \delta\int_{\partial\Omega_U} \nabla\widehat{c}_{\boldsymbol{\alpha}} \cdot \boldsymbol{\nu}\varphi\, dS + (f_c, \varphi), \tag{23}$$

$$\forall(\mathbf{v}, \phi, \varphi) \in \mathbf{V} \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega). \tag{24}$$

2.2. Some A Priori Estimates for $\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}$, *and* $\widehat{c}_{\boldsymbol{\alpha}}$

Proposition 2. *Consider that the assumptions for the existence result of Theorem 1 are satisfied. If we assume that* $(\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}})$ *is a solution of (20)–(24), then* $\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \le \Gamma_0$ *with* Γ_0 *defined on (13). Furthermore, the following estimates are valid:*

$$\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}} \le C_{poi}\left(\gamma g\Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right),$$
$$\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \le \frac{\Theta_2 C_{poi}}{\delta}\left[\beta C_{poi}\|r\|_{L^\infty(\mathbb{R})}\Gamma_0 + \|f_c\|_{L^2(\Omega)}\right]. \tag{25}$$

Proof. In order to prove the estimates, we select the test functions $(\mathbf{v}, \phi, \varphi) = (\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}})$ in (21)–(23). From (21) and (10), we deduce that

$$\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}} \le \gamma g C_{\mathrm{poi}}^2 \|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + (S_c)^{-1} C_{\mathrm{poi}}\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}. \tag{26}$$

Now, by the trace inequality and integration by parts, we have that

$$\begin{aligned}
\int_{\partial\Omega} |\nabla\widehat{n}_{\boldsymbol{\alpha}} \cdot \nu\widehat{n}_{\boldsymbol{\alpha}}|\, dS &\le C_{\mathrm{tr}}\|\widehat{n}_{\boldsymbol{\alpha}}\nabla\widehat{n}_{\boldsymbol{\alpha}} \cdot \nu\|_{W^{1,1}(\Omega)}\\
&\le C_{\mathrm{tr}}C_{\mathrm{poi}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 + C_{\mathrm{tr}}\int_{\partial\Omega} |\nabla\widehat{n}_{\boldsymbol{\alpha}} \cdot \nu\widehat{n}_{\boldsymbol{\alpha}}|\, dS,
\end{aligned} \tag{27}$$

which implies that

$$\int_{\partial\Omega} |\nabla\widehat{n}_{\boldsymbol{\alpha}} \cdot \nu\widehat{n}_{\boldsymbol{\alpha}}|\, dS \le \frac{C_{\mathrm{tr}}C_{\mathrm{poi}}}{1 - C_{\mathrm{tr}}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2. \tag{28}$$

Here, we have used the fact that $1 - C_{\mathrm{tr}} > 0$, as a consequence of the assumption (16). Then, by integration by parts we get the bound

$$\begin{aligned}
&\left(\widehat{n}_{\boldsymbol{\alpha}} r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad = \left(\nabla\left[\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right) dm\right], \nabla\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}^2}{2}\right)\right) \\
&\quad = -\left(\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right) dm, \Delta\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}}{2}\right)\right) \\
&\qquad + \int_{\partial\Omega}\left[\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right) dm\right]\nabla\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}^2}{2}\right)\cdot \nu\, dS \\
&\quad \leq 2\|r\|_{L^1(\mathbb{R})}\int_{\partial\Omega}\left|\widehat{n}_{\boldsymbol{\alpha}}\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\right| dS \\
&\quad \leq \frac{2\|r\|_{L^1(\mathbb{R})} C_{\mathrm{tr}} C_{\mathrm{poi}}}{1 - C_{\mathrm{tr}}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2.
\end{aligned} \tag{29}$$

From (22), using the properties (10) and the inequality (29), we have that

$$\begin{aligned}
\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 &= \chi\left(\widehat{n}_{\boldsymbol{\alpha}} r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad + \frac{\chi\alpha_1}{|\Omega|}\left(r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad + (f_n, \phi) \\
&\leq \frac{2\chi\|r\|_{L^1(\mathbb{R})} C_{\mathrm{tr}} C_{\mathrm{poi}}}{1 - C_{\mathrm{tr}}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 \\
&\quad + \frac{\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad + C_{\mathrm{poi}}\|f_n\|_{L^2(\Omega)}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)},
\end{aligned} \tag{30}$$

or equivalently, we get the following estimate for $\widehat{n}_{\boldsymbol{\alpha}}$:

$$\begin{aligned}
&\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad \leq \Theta_1\left[\frac{\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + C_{\mathrm{poi}}\|f_n\|_{L^2(\Omega)}\right],
\end{aligned} \tag{31}$$

with Θ_1 being defined in (12). Similarly, from (23) and (28) with $\widehat{c}_{\boldsymbol{\alpha}}$ instead of $\widehat{n}_{\boldsymbol{\alpha}}$, we deduce that

$$\begin{aligned}
&\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad \leq \frac{\Theta_2 C_{\mathrm{poi}}}{\delta}\left[\beta C_{\mathrm{poi}}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + \|f_c\|_{L^2(\Omega)}\right],
\end{aligned} \tag{32}$$

where Θ_2 is given in (12). Now, replacing the estimate (32) in (31) and applying (16), we deduce the existence of Γ_0 defined in (13) such that $\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \leq \Gamma_0$. We notice that the second and third relation in (16) imply that $\Theta_i > 1$, $i = 1, 2$, and $|\Omega| > \chi\beta\alpha_1\|r\|_{L^\infty(\mathbb{R})}^2 C_{\mathrm{poi}}^2\Theta_1\Theta_2$, respectively, that is, $\Gamma > 0$ under (16). Moreover, from (26) and (31), we deduce the estimates given in (25) for $\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}}$ and $\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}$, concluding the proof of the Proposition. □

2.3. Proof of Theorem 1. To prove the existence, we can apply the Gossez theorem [9, 14]. Let us first define the mapping $G : \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega) \to (\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$ by the relation

$$\begin{aligned}
&\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{v}, \phi, \varphi)\rangle\!\rangle = \lambda_1\left\{S_c a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{v})\right. \\
&\quad \left. - \gamma S_c(n\mathbf{g}, \mathbf{v}) - (\mathbf{F}, \mathbf{v})\right\} + \lambda_2\left\{a(n, \phi) + b(\mathbf{u}, n, \phi)\right. \\
&\quad \left. - \chi\left(\left(n + \frac{\alpha_1}{|\Omega|}\right) r\left(c + \frac{\alpha_2}{|\Omega|}\right)\nabla c, \nabla\phi\right) - (f_n, \phi)\right\} \\
&\quad + \lambda_3\left\{\delta a(c, \varphi) + b(\mathbf{u}, c, \varphi)\right. \\
&\quad + \beta\left(r\left(c + \frac{\alpha_2}{|\Omega|}\right)\left(n + \frac{\alpha_1}{|\Omega|}\right), \varphi\right) - \delta\int_{\partial\Omega_U}\nabla c \\
&\quad \left. \cdot\, \boldsymbol{\nu}\varphi\, dS - (f_c, \varphi)\right\}, \\
&\qquad \forall(\mathbf{u}, n, c), (\mathbf{v}, \phi, \varphi) \in \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega),
\end{aligned} \tag{33}$$

with $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denoting the duality pairing between $\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ and $(\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$ and λ_1, λ_2, and λ_3 are positive fixed constant. From (10), (12), and (29), we then have that

$$\begin{aligned}
&\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c)\rangle\!\rangle \geq \left\{\lambda_1 S_c\|\mathbf{u}\|_{\mathbf{V}}^2\right. \\
&\quad \left. - \lambda_1\gamma S_c g\left(C_{\mathrm{poi}}\right)^2\|n\|_{\widetilde{H}^1(\Omega)}\|\mathbf{u}\|_{\mathbf{V}} + \frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2\right\} \\
&\quad + \left\{\frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2\right. \\
&\quad - \frac{\lambda_2\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|c\|_{\widetilde{H}^1(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} \\
&\quad \left. + \frac{\lambda_3\delta}{2\Theta_2}\|c\|_{\widetilde{H}^1(\Omega)}^2\right\} + \left\{\frac{\lambda_3\delta}{2\Theta_2}\|c\|_{\widetilde{H}^1(\Omega)}^2\right. \\
&\quad - \lambda_3\beta\left(C_{\mathrm{poi}}\right)^2\|r\|_{L^\infty(\mathbb{R})}\|c\|_{\widetilde{H}^1(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} \\
&\quad \left. + \frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2\right\} - C_{\mathrm{poi}}\left\{\lambda_1\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}\|_{\mathbf{V}}\right. \\
&\quad \left. + \lambda_2\|f_n\|_{L^2(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} + \lambda_3\|f_c\|_{L^2(\Omega)}\|c\|_{\widetilde{H}^1(\Omega)}\right\} \\
&\quad := \Upsilon_1 + \Upsilon_2 - \Upsilon_3.
\end{aligned} \tag{34}$$

Now, selecting λ_1, λ_2, λ_3 and r such that

$$\lambda_1 < \frac{4\lambda_2}{3\Theta_1\gamma^2 g^2 S_c\left(C_{\mathrm{poi}}\right)^4},$$

$$\lambda_2 < \frac{4\delta|\Omega|^2\lambda_3}{6\Theta_1\Theta_2\left(\chi\alpha_1\|r\|_{L^\infty(\mathbb{R})}\right)^2},$$

$$\lambda_3 < \frac{4\delta\lambda_2}{6\Theta_1\Theta_2\left(\beta\left(C_{\text{poi}}\right)^2\|r\|_{L^\infty(\mathbb{R})}\right)^2}$$
$$r < \frac{\Upsilon_1+\Upsilon_2}{C_{\text{poi}}\left(\lambda_1\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}+\lambda_2\|f_n\|_{L^2(\Omega)}+\lambda_3\|f_c\|_{L^2(\Omega)}\right)}, \tag{35}$$

we can prove that $\langle\langle G(\mathbf{u},n,c),(\mathbf{u},n,c)\rangle\rangle$ is positive for all $(\mathbf{u},n,c) \in \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ such that $\|(\mathbf{u},n,c)\|_{\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)} = r$. Moreover, we notice that it is straightforward to deduce that G is continuous between the weak topologies of $\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ and $(\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$. Thus, there is $(\mathbf{u},n,c)\in\overline{B}_r(0)\subset\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ such that $\langle\langle G(\mathbf{u},n,c),(\mathbf{u},n,c)\rangle\rangle = 0$, concluding the proof of existence.

To prove the uniqueness we consider that there are two solutions $(\mathbf{u}^i,n^i,c^i)$, $i=1,2$, satisfying (21)–(23). Then, subtracting, selecting the test functions $(\mathbf{v},\phi,\varphi)=(\mathbf{u}^1-\mathbf{u}^2,n^1-n^2,c^1-c^2)$, using (10), (16), (17), and applying Proposition 2, we get

$$\left\|\mathbf{u}^1-\mathbf{u}^2\right\|_{\mathbf{V}} \le \Gamma_1\left\|n^1-n^2\right\|_{\widetilde{H}^1(\Omega)}, \tag{36}$$

$$\begin{aligned}\left\|n^1-n^2\right\|_{\widetilde{H}^1(\Omega)} &\le \Gamma_2\left[C_1\left\|\mathbf{u}^1-\mathbf{u}^2\right\|_{\mathbf{V}}\left\|n^1\right\|_{\widetilde{H}^1(\Omega)}\right.\\&\quad\left.+\|r\|_{L^\infty(\mathbb{R})}\left\|c^1-c^2\right\|_{\widetilde{H}^1(\Omega)}\right],\end{aligned} \tag{37}$$

$$\begin{aligned}\left\|c^1-c^2\right\|_{\widetilde{H}^1(\Omega)} &\le C_1\Gamma_3\left[\left\|\mathbf{u}^1-\mathbf{u}^2\right\|_{\mathbf{V}}\left\|c^2\right\|_{\widetilde{H}^1(\Omega)}\right.\\&\quad\left.+\left(C_1\right)^2\|r\|_{\text{Lip}(\mathbb{R})}\left\|n^1\right\|_{\widetilde{H}^1(\Omega)}\left\|c^1-c^2\right\|_{\widetilde{H}^1(\Omega)}\right],\end{aligned} \tag{38}$$

with Γ_i being defined in (13)–(15). From (38), Proposition 2, and the first inequality in (18), we have that

$$\begin{aligned}&\left\|c^1-c^2\right\|_{\widetilde{H}^1(\Omega)}\\&\quad\le\frac{C_1\Gamma_3\Theta_2 C_{\text{poi}}}{\delta\left(1-\left(C_1\right)^2\|r\|_{\text{Lip}(\mathbb{R})}\Gamma_0\right)}\left[\beta C_{\text{poi}}\|r\|_{L^\infty(\mathbb{R})}\Gamma_0\right.\\&\quad\left.+\left\|f_c\right\|_{L^2(\Omega)}\right]\left\|\mathbf{u}^1-\mathbf{u}^2\right\|_{\mathbf{V}}.\end{aligned} \tag{39}$$

Then, replacing (39) in (37), using Proposition 2 to estimate $\|n^1\|_{\widetilde{H}^1(\Omega)}$, we obtain the bound $\|n^1-n^2\|_{\widetilde{H}^1(\Omega)}\le\Pi(\Gamma_1)^{-1}\|\mathbf{u}^1-\mathbf{u}^2\|_{\mathbf{V}}$ with Π being defined in (18). Now, using this estimate in (36), we get that $\|\mathbf{u}^1-\mathbf{u}^2\|_{\mathbf{V}}\le\Pi\|\mathbf{u}^1-\mathbf{u}^2\|_{\mathbf{V}}$. Thus using the fact that $\Pi\le 1$ we deduce that $\mathbf{u}^1=\mathbf{u}^2$ on $\mathbf{V}$, which also implies that $n^1=n^2$ and $c^1=c^2$ on $\widetilde{H}^1(\Omega)$, concluding the uniqueness proof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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