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Research Article

Essential Norm of Difference of Composition Operators from Weighted Bergman Spaces to Bloch-Type Spaces

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We compute upper and lower bounds for essential norm of difference of composition operators acting from weighted Bergman spaces to Bloch-type spaces.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the set of all holomorphic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$, the composition operator C_φ is a linear operator defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}). \tag{1}$$

For a general reference on composition operator we refer to [1]. To understand the topological structures of spaces of composition operators, many people have studied difference of composition operators on different spaces of holomorphic functions; see, for example, [2–26] and references therein. Recently, Zhu and Yang [26] characterized boundedness and compactness of the differences of two composition operators from weighted Bergman spaces to Bloch spaces. Motivated by these results, in this paper we compute essential norm of difference of composition operators from weighted Bergman spaces to Bloch-type spaces.

Throughout this paper, constants are denoted by C; they are positive and not necessary the same in each occurrence. The notation $A \lesssim B$ means that A is less than or equal to a constant times B and $D \gtrsim E$ means that D is greater than or equal to a constant times E. When $A \lesssim B$ and $A \gtrsim B$, then we write $A \asymp B$.

For $0 < \alpha < \infty$ and $0 < p < \infty$, the weighted Bergman space $\mathcal{A}^p_\alpha$ is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{A}^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha dA(z) < \infty, \tag{2}$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$. For any $z \in \mathbb{D}$, the following point-evaluation estimate holds:

$$|f(z)| \le \frac{\|f\|_{\mathcal{A}^p_\alpha}}{\left(1 - |z|^2\right)^{(2+\alpha)/p}}, \quad f \in \mathcal{A}^p_\alpha. \tag{3}$$

The following equivalent norm in the weighted Bergman spaces is well-known. Let $0 < p < \infty$, $n \in \mathbb{N}$, and $f \in H(\mathbb{D})$. Then $f \in \mathcal{A}^p_\alpha$ if and only if, for all $n \in \mathbb{N}$, $f^{(n)}(z)(1 - |z|^2)^n \in L^p((1 - |z|^2)^\alpha dA_\alpha)$, and

$$\begin{aligned} \|f\|_{\mathcal{A}^p_\alpha} \asymp & \sum_{k=0}^{n-1} \left|f^{(k)}(0)\right| \\ & + \left(\int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p \left(1 - |z|^2\right)^{pn+\alpha} dA(z)\right)^{1/p}. \end{aligned} \tag{4}$$

Thus, if $f \in \mathcal{A}^p_\alpha$, then $f^{(n)} \in \mathcal{A}^p_{pn+\alpha}$ and $\|f^{(n)}\|_{\mathcal{A}^p_{pn+\alpha}} \le C\|f\|_{\mathcal{A}^p_\alpha}$. Therefore, by (3) applied to the function $f^{(n)}$, for every z in $\mathbb{D}$, we have

$$\left|f^{(n)}(z)\right| \leq \frac{\left\|f^{(n)}\right\|_{\mathscr{A}^p_{pn+\alpha}}}{\left(1-|z|^2\right)^{(2+pn+\alpha)/p}} \lesssim \frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-|z|^2\right)^{(2+\alpha)/p+n}}. \quad (5)$$

Moreover, by Lemma 2 in [26], for any $z, w \in \mathbb{D}$, there is a constant $C > 0$ such that

$$\begin{aligned} &\left|\left(1-|z|^2\right)^{1+(2+\alpha)/p} f'(z) - \left(1-|w|^2\right)^{1+(2+\alpha)/p} f'(w)\right| \\ &\quad \leq C\|f\|_{\mathscr{A}^p_\alpha} \rho(z,w) \end{aligned} \quad (6)$$

for all $f \in \mathscr{A}^p_\alpha$. For more about weighted Bergman spaces, we refer the reader to [27, 28].

Let μ be a positive continuous function on $\mathbb{D}$, which we call a *weight*. A weight μ is called *typical* if it is radial; that is, $\mu(z) = \mu(|z|)$, for each $z \in \mathbb{D}$ and $\mu(|z|)$, decreasingly converges to 0 as $|z| \to 1$. Throughout this paper by a weight, we shall mean a typical weight. For a weight μ, the *weighted Bloch-type space* $\mathscr{B}_\mu$ on $\mathbb{D}$ is a Banach space of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)\left|f'(z)\right| < \infty. \quad (7)$$

The space $\mathscr{B}_\mu$ is a Banach space with the norm

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)\left|f'(z)\right| < \infty. \quad (8)$$

In particular, the space $\mathscr{B}_{1-|z|^2}$ is the classical Bloch space and we denote it by $\mathscr{B}$.

2. Main Results

In this section, we compute upper and lower bounds for the essential norm of difference of composition operators mapping the weighted Bergman space into Bloch-type spaces.

Given Banach spaces X and Y, we recall that the essential norm of a bounded linear map $T : X \to Y$ is defined as

$$\|T\|_{e,X\to Y} = \{\|T-K\| : K : X \to Y \text{ is compact}\}; \quad (9)$$

that is, the essential norm of an operator $T : X \to Y$ is the distance of T from the set of all compact operators from X to Y. Clearly, if $T : X \to Y$ is compact, then $\|T\|_{e,X\to Y} = 0$.

For $\zeta \in \mathbb{D}$, let η_ζ be the conformal automorphism of $\mathbb{D}$ that interchanges 0 and ζ:

$$\eta_\zeta(z) = \frac{\zeta - z}{1-\bar{\zeta}z}, \quad z \in \mathbb{D}. \quad (10)$$

The pseudo-hyperbolic distance between ζ and z is given by

$$\rho(\zeta, z) = \left|\eta_\zeta(z)\right| = \left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|. \quad (11)$$

For μ, a typical weight, and $\varphi, \psi \in S(\mathbb{D})$, let $\mathfrak{D}^\varphi_\mu$ and $\mathfrak{D}^\psi_\mu$ be two functions defined as

$$\begin{aligned} \mathfrak{D}^\varphi_\mu(z) &= \frac{\mu(z)\varphi'(z)}{\left(1-|\varphi(z)|^2\right)^{1+(\alpha+2)/p}}, \\ \mathfrak{D}^\psi_\mu(z) &= \frac{\mu(z)\psi'(z)}{\left(1-|\psi(z)|^2\right)^{1+(\alpha+2)/p}}. \end{aligned} \quad (12)$$

Theorem 1. *Let $\alpha \in (1,\infty)$, $p \in (0,\infty)$, $\varphi, \psi \in S(\mathbb{D})$ such that $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ and let μ be a weight function such that $C_\varphi, C_\psi : \mathscr{A}^p_\alpha \to \mathscr{B}_\mu$ is bounded. Then*

$$\begin{aligned} &\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}^p_\alpha\to\mathscr{B}_\mu} \\ &\quad \asymp \limsup_{|\varphi(z)|\to 1} \left|\mathfrak{D}^\varphi_\mu(z)\right| \rho(\varphi(z), \psi(z)) \\ &\qquad + \limsup_{|\psi(z)|\to 1} \left|\mathfrak{D}^\psi_\mu(z)\right| \rho(\varphi(z), \psi(z)) \\ &\qquad + \limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1} \left|\mathfrak{D}^\varphi_\mu(z) - \mathfrak{D}^\psi_\mu(z)\right|. \end{aligned} \quad (13)$$

Proof. Let $(z_j)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$ and

$$\begin{aligned} &\lim_{j\to\infty} \left|\mathfrak{D}^\varphi_\mu(z_j)\right| \rho(\varphi(z_j), \psi(z_j)) \\ &\quad = \limsup_{|\varphi(z)|\to 1} \left|\mathfrak{D}^\varphi_\mu(z)\right| \rho(\varphi(z), \psi(z)). \end{aligned} \quad (14)$$

For each $k \in \mathbb{N}$, let $f_{j,k,\varphi}$ be defined as

$$f_{j,k,\varphi}(z) = \frac{1-\left|\varphi(z_j)\right|^2}{\left(1-\overline{\varphi(z_j)}z\right)^{k+(2+\alpha)/p}}. \quad (15)$$

Now consider the following functions:

$$\begin{aligned} g_{j,\varphi}(z) &= \frac{p}{(2+\alpha+p)\overline{\varphi(z_j)}} f_{j,1,\varphi}(z), \\ h_{j,\varphi}(z) &= \frac{p}{(2+\alpha+2p)\overline{\varphi(z_j)}} f_{j,2,\varphi}(z) \\ &\quad - \frac{p^2}{(2+\alpha+2p)\left(\overline{\varphi(z_j)}\right)^2} f_{j,1,\varphi}(z). \end{aligned} \quad (16)$$

Then it is easy to see that $g_{j,\varphi}$ and $h_{j,\varphi}$ are norm bounded sequences in $\mathscr{A}^p_\alpha$. Moreover, both sequences $(g_{j,\varphi})$ and $(h_{j,\varphi})$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Let $K : \mathscr{A}^p_\alpha \to \mathscr{B}_\mu$ be any compact operator. Then

$$\begin{aligned} &\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}^p_\alpha\to\mathscr{B}_\mu} \\ &\quad \geq C\limsup_{j\to\infty} \left\|\left(C_\varphi - C_\psi - K\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} \\ &\quad \geq C\limsup_{j\to\infty} \left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} \\ &\qquad - \limsup_{j\to\infty} \left\|K g_{j,\varphi}\right\|_{\mathscr{B}_\mu}, \end{aligned}$$

$$\begin{aligned} &\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}^p_\alpha\to\mathscr{B}_\mu} \\ &\quad \geq C\limsup_{j\to\infty} \left\|\left(C_\varphi - C_\psi - K\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} \end{aligned}$$

$$\geq C \limsup_{j\to\infty} \left\|\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} - \limsup_{j\to\infty} \left\|K h_{j,\varphi}\right\|_{\mathscr{B}_\mu}. \tag{17}$$

Again

$$\begin{aligned}\left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(z) \left|\left(\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right)' \cdot (z)\right| \\ &\geq \left|\frac{\mu(z_j)\left(1-\left|\varphi(z_j)\right|^2\right)\varphi'(z_j)}{\left(1-\left|\varphi(z_j)\right|^2\right)^{2+(2+\alpha)/p}} - \frac{\mu(z_j)\left(1-\left|\varphi(z_j)\right|^2\right)\psi'(z_j)}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{2+(2+\alpha)/p}}\right| \geq \left|\mathfrak{D}_\mu^\varphi(z_j)\right| \\ &\quad - \left|\mathfrak{D}_\mu^\psi(z_j)\right| \frac{\left(1-\left|\varphi(z_j)\right|^2\right)\left(1-\left|\psi(z_j)\right|^2\right)^{1+(2+\alpha)/p}}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{2+(2+\alpha)/p}}\right|,\end{aligned} \tag{18}$$

$$\begin{aligned}\left\|\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(z) \left|\left(\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right)'(z)\right| \\ &\geq \left|\frac{\mu(z_j)\left(1-\left|\varphi(z_j)\right|^2\right)\left|\psi'(z_j)\right|}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{2+(2+\alpha)/p}}\right| \cdot \left|\frac{\varphi(z_j)-\psi(z_j)}{1-\overline{\varphi(z_j)}\psi(z_j)}\right| = \left|\mathfrak{D}_\mu^\psi(z_j)\right. \\ &\quad \cdot \left.\frac{\left(1-\left|\varphi(z_j)\right|^2\right)\left(1-\left|\varphi(z_j)\right|^2\right)^{1+(2+\alpha)/p}}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{2+(2+\alpha)/p}}\right| \\ &\quad \cdot \rho\left(\varphi(z_j), \psi(z_j)\right).\end{aligned} \tag{19}$$

Multiplying (18) by $\rho(\varphi(z_j), \psi(z_j))$ and then adding it to (19), we get

$$\begin{aligned}&\left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} + \left\|\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} \\ &\quad \geq \left\|\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} + \left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} \rho\left(\varphi(z_j), \psi(z_j)\right) \\ &\quad \geq \left|\mathfrak{D}_\mu^\varphi(z_j)\right| \rho\left(\varphi(z_j), \psi(z_j)\right).\end{aligned} \tag{20}$$

Similarly, by considering a sequence $(z_j)_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\psi(z_j)| \to 1$ as $j \to \infty$ and

$$\begin{aligned}&\lim_{j\to\infty} \left|\mathfrak{D}_\mu^\psi(z_j)\right| \rho\left(\varphi(z_j), \psi(z_j)\right) \\ &\quad = \limsup_{|\psi(z)|\to 1} \left|\mathfrak{D}_\mu^\psi(z)\right| \rho\left(\varphi(z), \psi(z)\right),\end{aligned} \tag{21}$$

we can prove that

$$\begin{aligned}&\left\|\left(C_\varphi - C_\psi\right) g_{j,\psi}\right\|_{\mathscr{B}_\mu} + \left\|\left(C_\varphi - C_\psi\right) h_{j,\psi}\right\|_{\mathscr{B}_\mu} \\ &\quad \gtrsim \left|\mathfrak{D}_\mu^\psi(z_j)\right| \rho\left(\varphi(z_j), \psi(z_j)\right).\end{aligned} \tag{22}$$

Combining (17) and (20) and using the fact that $\|Kg_{j,\varphi}\|_{\mathscr{B}_\mu} \to 0$ and $\|Kh_{j,\varphi}\|_{\mathscr{B}_\mu} \to 0$ as $j \to \infty$, we have

$$\begin{aligned}&\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}_\alpha^p \to \mathscr{B}_\mu} \\ &\quad \gtrsim \limsup_{|\varphi(z)|\to 1} \left|\mathfrak{D}_\mu^\varphi(z)\right| \rho\left(\varphi(z), \psi(z)\right).\end{aligned} \tag{23}$$

Combining (17) and (22) and using the fact that $\|Kg_{j,\psi}\|_{\mathscr{B}_\mu} \to 0$ and $\|Kh_{j,\psi}\|_{\mathscr{B}_\mu} \to 0$ as $j \to \infty$, we have

$$\begin{aligned}&\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}_\alpha^p \to \mathscr{B}_\mu} \\ &\quad \gtrsim \limsup_{|\psi(z)|\to 1} \left|\mathfrak{D}_\mu^\psi(z_j)\right| \rho\left(\varphi(z), \psi(z)\right).\end{aligned} \tag{24}$$

Again, let $(z_j)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \wedge |\psi(z_j)| \to 1$ as $j \to \infty$ and

$$\begin{aligned}&\lim_{j\to\infty} \left|\mathfrak{D}_\mu^\varphi(z_j) - \mathfrak{D}_\mu^\psi(z_j)\right| \\ &\quad = \limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1} \left|\mathfrak{D}_\mu^\varphi(z) - \mathfrak{D}_\mu^\psi(z)\right|.\end{aligned} \tag{25}$$

Then from (18) we have

$$\begin{aligned}\left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(z) \cdot \left|\left(\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right)'(z)\right| \\ &\geq \left|\frac{\mu(z)\left(1-\left|\varphi(z_j)\right|^2\right)\varphi'(z_j)}{\left(1-\left|\varphi(z_j)\right|^2\right)^{2+(2+\alpha)/p}}\right|\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mu(z)\left(1-\left|\varphi(z_j)\right|^2\right)\psi'(z_j)}{\left(1-\overline{\varphi(z_j)}\psi(z_j)\right)^{2+(2+\alpha)/p}}\Bigg| \geq \left|\mathfrak{D}_\mu^\varphi(z_j)\right. \\
&\left.- \mathfrak{D}_\mu^\psi(z_j)\right| - \left|\mathfrak{D}_\mu^\psi(z_j)\right| \times \Bigg|1 \\
&- \frac{\left(1-\left|\varphi(z_j)\right|^2\right)\left(1-\left|\psi(z_j)\right|^2\right)^{1+(2+\alpha)/p}}{\left(1-\left|\varphi(z_j)\right|^2\right)^{2+(2+\alpha)/p}}\Bigg| \\
&\geq \left|\mathfrak{D}_\mu^\varphi(z_j) - \mathfrak{D}_\mu^\psi(z_j)\right| - \left|\mathfrak{D}_\mu^\psi(z_j)\right| \\
&\cdot \rho\left(\varphi(z_j), \psi(z_j)\right).
\end{aligned} \tag{26}
$$

Using (20) and (26), we have

$$
\begin{aligned}
&\left\|\left(C_\varphi - C_\psi\right) g_{j,\varphi}\right\|_{\mathscr{B}_\mu} + \left\|\left(C_\varphi - C_\psi\right) h_{j,\varphi}\right\|_{\mathscr{B}_\mu} \\
&\quad \geq \left|\mathfrak{D}_\mu^\varphi(z_j) - \mathfrak{D}_\mu^\psi(z_j)\right|.
\end{aligned} \tag{27}
$$

Combining (17) and (27) and using the fact that $\|Kg_j\|_{\mathscr{B}_\mu} \to 0$ and $\|Kh_j\|_{\mathscr{B}_\mu} \to 0$ as $j \to \infty$, we have

$$
\begin{aligned}
&\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}_\alpha^p \to \mathscr{B}_\mu} \\
&\quad \gtrsim \lim_{|\varphi(z)|\wedge|\psi(z)|\to 1} \left|\mathfrak{D}_\mu^\varphi(z_j) - \mathfrak{D}_\mu^\psi(z_j)\right|.
\end{aligned} \tag{28}
$$

Combining (23), (24), and (28), we get the lower bound as

$$
\begin{aligned}
&\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}_\alpha^p \to \mathscr{B}_\mu} \\
&\quad \gtrsim \limsup_{|\varphi(z)|\to 1} \mu(z)\left|\mathfrak{D}_\mu^\varphi(z)\right| \rho\left(\varphi(z), \psi(z)\right) \\
&\qquad + \limsup_{|\psi(z)|\to 1} \mu(z)\left|\mathfrak{D}_\mu^\psi(z)\right| \rho\left(\varphi(z), \psi(z)\right) \\
&\qquad + \limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1} \mu(z)\left|\mathfrak{D}_\mu^\varphi(z) - \mathfrak{D}_\mu^\psi(z)\right|.
\end{aligned} \tag{29}
$$

Let $\tau_k(z) = (k/(k+1))z$. Then $\|\tau_k\|_\infty < 1$. Let

$$
L_k f(z) = C_{\tau_k} f(z) = f\left(\frac{k}{k+1} z\right). \tag{30}
$$

Then, we have the fact that $L_k : \mathscr{A}_\alpha^p \to \mathscr{A}_\alpha^p$ is compact. Since $C_\varphi - C_\psi : \mathscr{A}_\alpha^p \to \mathscr{B}_\mu$ is bounded, so $(C_\varphi - C_\psi)L_k : \mathscr{A}_\alpha^p \to \mathscr{B}_\mu$ is compact. Thus

$$
\begin{aligned}
\left\|C_\varphi - C_\psi\right\|_{e,\mathscr{A}_\alpha^p \to \mathscr{B}_\mu} &\leq \left\|\left(C_\varphi - C_\psi\right) - \left(C_\varphi - C_\psi\right) L_k\right\| \\
&\leq \sup_{\|f\|_{\mathscr{A}_\alpha^p} \leq 1} \left\|C_\varphi - C_\psi (I - L_k) f\right\|_{\mathscr{B}_\mu} \\
&\leq \sup_{\|f\|_{\mathscr{A}_\alpha^p} \leq 1} \Bigg[\left|\left(C_\varphi - C_\psi\right)(I - L_k) f(0)\right| \\
&\quad + \sup_{z\in\mathbb{D}} \mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right|\Bigg],
\end{aligned} \tag{31}
$$

where I is the identity operator on $\mathscr{A}_\alpha^p$. Let $f \in \mathscr{A}_\alpha^p$ be such that $\|f\|_{\mathscr{A}_\alpha^p} \leq 1$. For any $r \in (0,1)$, we can write

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right| \\
&\quad = \sup_{|\varphi(z)|\wedge|\psi(z)|\leq r} \mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right| \\
&\qquad + \sup_{|\varphi(z)|>r} \mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right| \\
&\qquad + \sup_{|\psi(z)|>r} \mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right|.
\end{aligned} \tag{32}
$$

If $|\varphi(z)| \wedge |\psi(z)| \leq r$, then we have

$$
\begin{aligned}
&\left|\left(\left(C_\varphi - C_\psi\right)(I - L_k) f\right)'(z)\right| = \Bigg|f'(\varphi(z))\varphi'(z) \\
&\quad - f'(\psi(z))\psi'(z) - \frac{k}{k+1}\varphi'(z) f'\left(\frac{k}{k+1}\varphi(z)\right) \\
&\quad + \frac{k}{k+1}\psi'(z) f'\left(\frac{k}{k+1}\psi'(z)\right)\Bigg| \leq \left|\varphi'(z)\right| \\
&\quad \cdot \left|f'(\varphi(z)) - f'\left(\frac{k}{k+1}\varphi(z)\right)\right| \\
&\quad + \frac{1}{k+1}\left|f'\left(\frac{k}{k+1}\varphi(z)\right)\right| + \left|\psi'(z)\right|\Bigg|f'(\psi(z)) \\
&\quad - f'\left(\frac{k}{k+1}\psi(z)\right)\Bigg| + \frac{1}{k+1}\left|f'\left(\frac{k}{k+1}\psi(z)\right)\right|.
\end{aligned} \tag{33}
$$

Let $|\varphi(z)| \wedge |\psi(z)| \leq r$, and then $|\varphi(z)| \leq r$ and $|\psi(z)| \leq r$. Let $w \in \mathbb{D}$ be such that $|w| < r$. Denote the straight line segment from $kw/(k+1)$ to w by $[kw/(k+1), w]$. Then the segment

$[kw/(k+1), w] \subset D(0,r)$, where $D(0,r) = \{z : |z| \le r\}$. Thus by (5), we have

$$\begin{aligned}\left|f'(w) - f'\left(\frac{k}{k+1}w\right)\right| &= \left|\int_{[kw/(k+1),w]} f''(\zeta)\,d\zeta\right| \\ &\le \frac{|w|}{k+1}\sup_{\zeta\in D(0,r)}\left|f''(\zeta)\right| \\ &\lesssim \frac{|w|}{k+1}\sup_{\zeta\in D(0,r)}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-|\zeta|^2\right)^{2+(2+\alpha)/p}} \\ &\lesssim \frac{|w|}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{2+(2+\alpha)/p}}.\end{aligned} \tag{34}$$

Similarly, we can show that

$$\left|f(w) - f\left(\frac{k}{k+1}w\right)\right| \lesssim \frac{|w|}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{1+(2+\alpha)/p}}. \tag{35}$$

Since $C_\eta : \mathscr{A}^p_\alpha \to \mathscr{B}_\mu$ is bounded, where $\eta = \varphi$ or ψ, so

$$\left\|C_\eta f\right\|_{\mathscr{B}_\mu} \lesssim \|f\|_{\mathscr{A}^p_\alpha}. \tag{36}$$

By taking $f(z) = z$ in (36), we have

$$\begin{aligned}M_1 &= \sup_{z\in\mathbb{D}}\mu(z)\left|\varphi'(z)\right| < \infty, \\ M_2 &= \sup_{z\in\mathbb{D}}\mu(z)\left|\psi'(z)\right| < \infty.\end{aligned} \tag{37}$$

Combining (33), (34), and (37), we have

$$\begin{aligned}&\sup_{\|f\|_{\mathscr{A}^p_\alpha}\le 1}\sup_{|\varphi(z)|\le r}\mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I-L_k)f\right)'(z)\right| \\ &\quad\le \mu(z)\frac{\left|\varphi'(z)\varphi(z)\right|}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{2+(2+\alpha)/p}} \\ &\qquad + \frac{1}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{1+(2+\alpha)/p}} \\ &\quad\le \mu(z)\frac{\left|\psi'(z)\psi(z)\right|}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{2+(2+\alpha)/p}} \\ &\qquad + \frac{1}{k+1}\frac{\|f\|_{\mathscr{A}^p_\alpha}}{\left(1-r^2\right)^{1+(2+\alpha)/p}} \\ &\quad\le \frac{M_1}{k+1}\frac{1}{\left(1-r^2\right)^{2+(2+\alpha)/p}} + \frac{1}{k+1}\frac{1}{\left(1-r^2\right)^{1+(2+\alpha)/p}} \\ &\quad\longrightarrow 0 \quad \text{as } k\longrightarrow\infty.\end{aligned} \tag{38}$$

Using (35) with $w = \varphi(0)$ and $w = \psi(0)$, we have

$$\begin{aligned}&\left|\left(C_\varphi - C_\psi\right)(I-L_k)f(0)\right| \\ &\quad= \left|f(\varphi(0)) - f\left(\frac{k}{k+1}\varphi(0)\right)\right| \\ &\qquad + \left|f(\psi(0)) - f\left(\frac{k}{k+1}\psi(0)\right)\right| \\ &\quad\le \frac{|\psi(0)| + |\varphi(0)|}{k+1}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\left(1-r^2\right)^{1+(2+\alpha)/p}} \\ &\quad\le \frac{|\psi(0)| + |\varphi(0)|}{k+1}\frac{1}{\left(1-r^2\right)^{1+(2+\alpha)/p}} \longrightarrow 0 \\ &\qquad\qquad \text{as } k\longrightarrow\infty.\end{aligned} \tag{39}$$

Combining (38) and (39), we have

$$\begin{aligned}&\sup_{\|f\|_{\mathscr{A}^p_\alpha}\le 1}\left[\left|\left(C_\varphi - C_\psi\right)(I-L_k)f(0)\right|\right. \\ &\qquad + \sup_{\|f\|_{\mathscr{A}^p_\alpha}\le 1}\sup_{|\varphi(z)|\wedge|\psi(z)|\le r}\mu(z) \\ &\qquad \left.\cdot\left|\left(\left(C_\varphi - C_\psi\right)(I-L_k)f\right)'(z)\right|\right] \longrightarrow 0\end{aligned} \tag{40}$$

as $k\to\infty$. Finally, we have

$$\begin{aligned}&\mu(z)\left|\left(\left(C_\varphi - C_\psi\right)(I-L_k)f\right)'(z)\right| = \mu(z)\left|\varphi'(z)\right. \\ &\quad\cdot\left(f'(\varphi(z)) - \frac{k}{k+1}f'\left(\frac{k}{k+1}\varphi(z)\right)\right) - \psi'(z) \\ &\quad\left.\cdot\left(f'(\psi(z)) - \frac{k}{k+1}f'\left(\frac{k}{k+1}\psi(z)\right)\right)\right| \\ &\quad\le \left|\mathfrak{D}^\varphi_\mu(z)\right|\left|\left(1-|\varphi(z)|^2\right)^{1+(2+\alpha)/p}\right. \\ &\quad\cdot\left(f'(\varphi(z)) - \frac{k}{k+1}f'\left(\frac{k}{k+1}\varphi(z)\right)\right) \\ &\quad\left.-\left(1-|\psi(z)|^2\right)^{1+(2+\alpha)/p}\right. \\ &\quad\left.\cdot\left(f'(\psi(z)) - \frac{k}{k+1}f'\left(\frac{k}{k+1}\psi(z)\right)\right)\right| + \left|\mathfrak{D}^\varphi_\mu(z)\right. \\ &\quad\left.-\mathfrak{D}^\psi_\mu(z)\right|\left(1-|\psi(z)|^2\right)^{1+(2+\alpha)/p}\times\left|f'(\psi(z))\right. \\ &\quad\left.-\frac{k}{k+1}f'\left(\frac{k}{k+1}\psi(z)\right)\right|.\end{aligned} \tag{41}$$

For every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, we have

$$\lim_{k\to\infty}\left(f'(z) - \frac{k}{k+1}f'\left(\frac{k}{k+1}z\right)\right) = 0 \tag{42}$$

and by Banach-Steinhaus theorem; it converges to zero uniformly on compact subsets of $\mathbb{D}$, so we have

$$\lim_{k\to\infty}\sup_{\|f\|_{\mathscr{A}^p_\alpha}\le 1}\sup_{|z|\le r}\left|f'(z)-\frac{k}{k+1}f'\left(\frac{k}{k+1}z\right)\right|=0. \tag{43}$$

Also, for the boundedness of $C_\varphi, C_\psi : \mathscr{A}^p_\alpha \to \mathscr{B}_\mu$, we have the facts that $\sup_{z\in\mathbb{D}}\mu(z)|\varphi'(z)| < \infty$ and $\sup_{z\in\mathbb{D}}\mu(z)|\psi'(z)| < \infty$. Using these facts and (41), we see that for each $r \in (0,1)$ and $|\psi(z)| \le r$ the right hand side of (41) is dominated by a constant multiple of

$$\sup_{|\varphi(z)|>r}\left|\mathfrak{D}^\varphi_\mu(z)\right|\rho(\varphi(z),\psi(z)). \tag{44}$$

If $|\varphi(z)| > r$, then we see that the right hand side of (41) is dominated by a constant multiple of

$$\begin{aligned}&\sup_{|\varphi(z)|>r}\left|\mathfrak{D}^\varphi_\mu(z)\right|\rho(\varphi(z),\psi(z))\\&\quad+\sup_{|\varphi(z)|\wedge|\psi(z)|>r}\left|\mathfrak{D}^\varphi_\mu(z)-\mathfrak{D}^\psi_\mu(z)\right|.\end{aligned} \tag{45}$$

Thus

$$\begin{aligned}&\limsup_{r\to 1}\limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{A}^p_\alpha}}\sup_{|\varphi(z)|>r}\mu(z)\\&\quad\cdot\left|\left(\left(C_\varphi-C_\psi\right)(I-L_k)f\right)'(z)\right|\lesssim\limsup_{|\varphi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)\right|\\&\quad\cdot\rho(\varphi(z),\psi(z))\\&\quad+\limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)-\mathfrak{D}^\psi_\mu(z)\right|.\end{aligned} \tag{46}$$

Similarly, we can show that

$$\begin{aligned}&\limsup_{r\to 1}\limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{A}^p_\alpha}}\sup_{|\psi(z)|>r}\mu(z)\\&\quad\cdot\left|\left(\left(C_\varphi-C_\psi\right)(I-L_k)f\right)'(z)\right|\lesssim\limsup_{|\psi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)\right|\\&\quad\cdot\rho(\varphi(z),\psi(z))\\&\quad+\limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)-\mathfrak{D}^\psi_\mu(z)\right|.\end{aligned} \tag{47}$$

Combining (31), (32), (40), (46), and (47), we have the fact that

$$\begin{aligned}&\left\|C_\varphi-C_\psi\right\|_{e,\mathscr{A}^p_\alpha\to\mathscr{B}_\mu}\\&\quad\lesssim\limsup_{|\varphi(z)|\wedge|\psi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)-\mathfrak{D}^\psi_\mu(z)\right|\\&\quad+\limsup_{|\varphi(z)|\to 1}\left|\mathfrak{D}^\varphi_\mu(z)\right|\rho(\varphi(z),\psi(z))\\&\quad+\limsup_{|\psi(z)|\to 1}\left|\mathfrak{D}^\psi_\mu(z)\right|\rho(\varphi(z),\psi(z)).\end{aligned} \tag{48}$$

Combining (29) and (48), we get the desired result. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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