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Research Article

Class of Analytic Function Related with Uniformly Convex and Janowski's Functions

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In this paper, we introduce a new subclass of analytic functions in open unit disc. We obtain coefficient estimates, extreme points, and distortion theorem. We also derived the radii of close-to-convexity and starlikeness for this class.

1. Introduction

Let $\mathscr{A}$ denote the class of normalized analytic functions in open unit disc $\Delta = \{z : |z| < 1\}$ and having Taylor series of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Silverman [1] introduced and studied a subclass $\mathscr{A}^-$ of $\mathscr{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0, \ z \in \Delta. \tag{2}$$

A complex valued function $f(z)$ is said to be univalent in Δ if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$, for all $z_1, z_2 \in \Delta$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ composed of univalent functions in Δ. By $\mathscr{ST}$, $\mathscr{CV}$, and $\mathscr{K}$, we mean the well-known subclasses of $\mathscr{S}$ that are starlike, convex, and close-to-convex functions, respectively; for detail see [2, 3].

In 1991, Goodman [2, 3] introduced classes $\mathscr{UCV}$ and $\mathscr{UST}$ of uniformly convex and uniformly starlike functions, respectively. A function $f \in \mathscr{S}$ is uniformly convex if $f(z)$ maps every circular arc γ contained in Δ with center $\zeta \in \Delta$ onto a convex arc. The function $f \in \mathscr{S}$ is uniformly starlike if $f(z)$ maps every circular arc ζ contained in Δ with center $\zeta \in \Delta$ onto a starlike arc with respect to $f(\zeta)$. A more useful representation of $\mathscr{UCV}$ and $\mathscr{UST}$ was given in [2, 3]; see also [4–7] as

$$\begin{gathered} f \in \mathscr{UCV} \Longleftrightarrow \\ f \in \mathscr{A} \\ \text{and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta, \end{gathered} \tag{3}$$

and

$$\begin{gathered} f \in \mathscr{UST} \Longleftrightarrow \\ f \in \mathscr{A} \\ \text{and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \Delta, \end{gathered} \tag{4}$$

In 2011, Noor et. al [8] introduced and studied a class $k - \mathscr{UST}(A, B)$, $k \geq 0$, $-1 \leq B < A \leq 1$ as follows.

A function $f(z) \in \mathscr{A}$ is said to be in the class $k - \mathscr{UST}(A, B)$ if and only if

$$\begin{aligned} &\operatorname{Re}\left[\frac{(B-1)\left(zf'(z)/f(z)\right) - (A-1)}{(B+1)\left(zf'(z)/f(z)\right) - (A+1)}\right] \\ &\quad > k\left|\frac{(B-1)\left(zf'(z)/f(z)\right) - (A-1)}{(B+1)\left(zf'(z)/f(z)\right) - (A+1)} - 1\right|. \end{aligned} \tag{5}$$

The abovementioned few classes were widely investigated by many authors in the last decades; see [4, 8–15] and the references cited therein. By taking motivation from cited work, we define a unified class of analytic functions as follows.

Definition 1. A function $f(z) \in \mathcal{A}$ is said to be in the class $k-\mathcal{UST}(A,B,\delta,\lambda)$ if and only if

$$\begin{aligned}
&\operatorname{Re}\left[\frac{(B-1)\left(\left(zf'(z)+\delta\left((1+2\lambda)z^2f''(z)+\lambda z^3f'''(z)\right)\right)/\left((1-\delta)f(z)+\delta\left(zf'(z)+\lambda z^2f''(z)\right)\right)\right)-(A-1)}{(B+1)\left(\left(zf'(z)+\delta\left((1+2\lambda)z^2f''(z)+\lambda z^3f'''(z)\right)\right)/\left((1-\delta)f(z)+\delta\left(zf'(z)+\lambda z^2f''(z)\right)\right)\right)-(A+1)}\right] \\
&\quad > k\left|\frac{(B-1)\left(\left(zf'(z)+\delta\left((1+2\lambda)z^2f''(z)+\lambda z^3f'''(z)\right)\right)/\left((1-\delta)f(z)+\delta\left(zf'(z)+\lambda z^2f''(z)\right)\right)\right)-(A-1)}{(B+1)\left(\left(zf'(z)+\delta\left((1+2\lambda)z^2f''(z)+\lambda z^3f'''(z)\right)\right)/\left((1-\delta)f(z)+\delta\left(zf'(z)+\lambda z^2f''(z)\right)\right)\right)-(A+1)}\right. \\
&\quad \left.-1\right|,
\end{aligned} \tag{6}$$

where $0 \le \delta \le 1$, $0 \le \lambda < 1$, $k \ge 0$, $-1 \le B < A \le 1$.

We further let $k-\mathcal{VUST}(A,B,\delta,\lambda) = k-\mathcal{UST}(A,B,\delta,\lambda) \cap \mathcal{A}^-$.

Special Cases

(i) $k-\mathcal{UST}(A,B,0,\lambda) = k-\mathcal{UST}(A,B)$ [8].

(ii) $k-\mathcal{UST}(A,B,1,0) = k-\mathcal{UCV}(A,B)$ [8].

(iii) $k-\mathcal{UST}(1-2\alpha,-1,0,\lambda) = k-\mathcal{UST}(\alpha)$ [16].

(iv) $k-\mathcal{UST}(1-2\alpha,-1,1,0) = k-\mathcal{UCV}(\alpha)$[16].

(v) $k-\mathcal{UST}(1,-1,0,\lambda) = k-\mathcal{UST}$ [14].

Throughout this paper $0 \le \delta \le 1$, $0 \le \lambda < 1$, $k \ge 0-$, and $-1 \le B < A \le 1$, unless otherwise stated.

In the next section we shall present our major findings as theorems and corollaries.

2. Main Results

Theorem 2. *Let $f \in \mathcal{A}$ be given by (1); then f is in $k-\mathcal{UST}(A,B,\delta,\lambda)$ if*

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)\Phi_n+\left|(B+1)\Psi_n-(A+1)\Phi_n\right|\right\} \\
&\quad\cdot\left|a_n\right| \le |B-A|,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\Phi_n &= (1-\delta)+\delta n(1+\lambda(n-1)) \\
\text{and } \Psi_n &= (n+\delta n(n-1)(1+\lambda n)).
\end{aligned} \tag{8}$$

Proof. Assume that inequality (7) holds true. Then we have

$$\begin{aligned}
&k\left|\frac{(B-1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A-1)}{(B+1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A+1)}-1\right| \\
&\quad-\operatorname{Re}\left[\frac{(B-1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A-1)}{(B+1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A+1)}-1\right] \\
&\quad\le (k+1)\left|\frac{(B-1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A-1)}{(B+1)\left(\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)/\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)\right)-(A+1)}-1\right| \\
&\quad= 2(k+1)\left|\frac{(1-\delta)\left(f-zf'\right)-\delta\left((1+\lambda)z^2f''+\lambda z^3f'''\right)}{(B+1)\left((1-\delta)zf'+\delta\left(zf'+(1+2\lambda)z^2f''+\lambda z^3f'''\right)\right)-(A+1)\left((1-\delta)f+\delta\left(zf'+\lambda z^2f''\right)\right)}\right| \\
&\quad= 2(k+1)\left|\frac{\sum_{n=2}^{\infty}(n-1)\Phi_na_nz^n}{(B-A)z+\sum_{n=2}^{\infty}\left\{(B+1)\Psi_n-(A+1)\Phi_n\right\}a_nz^n}\right| \le 2(k+1) \\
&\quad\cdot\frac{\sum_{n=2}^{\infty}\Phi_n\left|a_n\right|}{|B-A|-\sum_{n=2}^{\infty}\left|(B+1)\Psi_n-(A+1)\Phi_n\right|\left|a_n\right|}.
\end{aligned} \tag{9}$$

This expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)\Phi(n,\delta,\lambda) + |(B+1)\Psi(n,\delta,\lambda)-(A+1)\Phi(n,\delta,\lambda)|\}|a_n| \le |B-A|. \tag{10}$$

□

When $\delta = 0$, we obtain the main result proved by Noor et at. [8] stated as follows.

Corollary 3. *A function $f \in \mathcal{A}$ given by (1) is in the class $k-\mathcal{USST}[A,B]$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|(B+1)n-(A+1)|\}|a_n| \le |B-A|. \tag{11}$$

When $\delta = 1$, $\lambda = 0$, then we have the following known result proved by Noor et at. [8] stated as follows.

Corollary 4. *A function $f \in \mathcal{A}$ given by (1) is in the class $k-\mathcal{USST}[A,B]$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} n\{2(k+1)(n-1)+|(B+1)n-(A+1)|\}|a_n| \le |B-A|. \tag{12}$$

Theorem 5. *A necessary and sufficient condition for any function $f(z)$ given by (2) to be in the class $k-\mathcal{VUSST}(A,B,\delta,\lambda)$ is that*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)\Phi_n + |(B+1)\Psi_n-(A+1)\Phi_n|\} \cdot a_n \le |B-A|. \tag{13}$$

Proof. In view of Theorem 2, we need only to prove the necessity. If $f \in k-\mathcal{VUSST}(A,B,\delta,\lambda)$ and z is real, then, by relation (6), we have

$$\frac{(B-A)-\sum_{n=2}^{\infty}[(B-1)\Psi_n-(A-1)\Phi_n]a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}[(B+1)\Psi_n-(A+1)\Phi_n]a_n z^{n-1}} \ge 2k\left|\frac{(n-1)\Phi_n a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}[(B+1)\Psi_n-(A+1)\Phi_n]a_n z^{n-1}}\right|. \tag{14}$$

□

Letting $z \longrightarrow 1^-$ along the real axis, we obtain the required inequality (13).

Corollary 6. *Let $f(z) \in k-\mathcal{VUSST}(A,B,\delta,\lambda)$ and be of the form (2). Then*

$$a_n \le \frac{|B-A|}{2(k+1)(n-1)\Phi_n + |(B+1)\Psi_n-(A+1)\Phi_n|}, \quad n \ge 2. \tag{15}$$

Next, we give the growth and distortion theorem for the class $k-\mathcal{VUSST}(A,B,\delta,\lambda)$.

Theorem 7. *Let the function $f(z)$ given by (2) be in the class $k-\mathcal{VUSST}(A,B,\delta,\lambda)$. Then, for $|z|<1$, we have*

$$|f(z)| \ge r - \frac{|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r^2, \tag{16}$$

and

$$|f(z)| \le r + \frac{|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r^2. \tag{17}$$

Proof. From Theorem 5, for $n \ge 2$, we have

$$2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|\sum_{n=2}^{\infty}a_n \le 2(k+1)(n-1)\Phi_n + |(B+1)\Psi_n-(A+1)\Phi_n|\sum_{n=2}^{\infty}a_n \le |B-A|. \tag{18}$$

□

For $f(z)$ given by (2) and using the triangle inequality we have

$$|f(z)| \ge z - \sum_{n=2}^{\infty}a_n|z|^n \ge r - r^2\sum_{n=2}^{\infty}a_n \ge r - \frac{|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r^2, \tag{19}$$

and

$$|f(z)| \le z + \sum_{n=2}^{\infty}a_n|z|^n \le r + \sum_{n=2}^{\infty}a_n r^n|f(z)| \le r + \frac{|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r^2. \tag{20}$$

Theorem 8. *Let $f(z) \in k-\mathcal{VUSST}(A,B,\delta,\lambda)$ be of the form (1). Then, for $|z|<1$, we have*

$$1 - \frac{2|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r \le |f'(z)| \le 1 + \frac{2|B-A|}{2(k+1)\Phi_2 + |(B+1)\Psi_2-(A+1)\Phi_2|}r. \tag{21}$$

Proof. By differentiating (2) and after some simplification we have

$$|f'(z)| \ge 1 - \sum_{n=2}^{\infty}na_n z^{n-1} \ge 1 - r\sum_{n=2}^{\infty}na_n, \tag{22}$$

and

$$\left|f'(z)\right| \leq 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n. \tag{23}$$

As $f \in k - \mathcal{VUST}(A, B, \delta, \lambda)$ so, from Theorem 5,

$$\begin{aligned} &\frac{2(k+1)\Phi_2 + \left|(B+1)\Psi_2 - (A+1)\Phi_2\right|}{2} \sum_{n=2}^{\infty} n a_n \\ &\quad \leq \left(2(k+1)(n-1)\Phi_n \right. \\ &\quad \left. + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right) \sum_{n=2}^{\infty} n a_n \leq |B - A|, \end{aligned} \tag{24}$$

or equivalently

$$\begin{aligned} &\sum_{n=2}^{\infty} n \left|a_n\right| \\ &\quad \leq \frac{2|B-A|}{\left[2(k+1)\Phi_2 + \left|(B+1)\Psi_2 - (A+1)\Phi_2\right|\right]}. \end{aligned} \tag{25}$$

Using (25) in (22) and (23) yields required inequality (23). □

Theorem 9. *Let $\mu_\nu \geq 0$, for $\nu = 1, 2, \ldots, l$ and $\sum_{\nu=1}^{l} \mu_\nu \leq 1$. If the functions $F_\nu(z) = z - \sum_{n=2}^{\infty} a_{n,\nu} z^n$, $a_{n,\nu} \geq 0$ are in the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$ then*

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} A_n z^n \\ A_n &= \sum_{\nu=1}^{l} \mu_\nu a_{n,\nu}, \end{aligned} \tag{26}$$

is in class $k - \mathcal{VUST}(A, B, \delta, \lambda)$.

Proof. The proof follows immediately by using Theorem 5. □

Theorem 10. *Let $\mu_n \geq 0$, $\sum_{n=1}^{\infty} \mu_n = 1$, $f_1(z) = z$, and*

$$\begin{aligned} f_n(z) &= z \\ &- \sum_{n=2}^{\infty} \frac{|B-A|}{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)} z^n, \\ &\qquad n \geq 2. \end{aligned} \tag{27}$$

Then $f(z)$ is in the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{28}$$

Proof. Consider

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) = z \\ &- \sum_{n=2}^{\infty} \frac{|B-A|}{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)} \mu_n z^n \\ &\qquad n \geq 2. \end{aligned} \tag{29}$$

Then, by Theorem 5, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{|B-A|} \\ &\quad \times \frac{|B-A|}{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)} \\ &\quad \cdot \mu_n = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 < 1, \end{aligned} \tag{30}$$

and hence $f(z) \in k - \mathcal{VUST}(A, B, \delta, \lambda)$.

Conversely assume that $f(z)$ given by (2) is in the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$. Then

$$\begin{aligned} &a_n \\ &\quad \leq \frac{|B-A|}{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)} \\ &\qquad n \geq 2. \end{aligned} \tag{31}$$

By setting

$$\begin{aligned} &\mu_n \\ &\quad = \frac{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{|B-A|} \\ &\qquad n \geq 2, \end{aligned} \tag{32}$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$, we can see that $f(z)$ can be expressed in the form (28), which completes the proof. □

Theorem 11. *Let the function $f(z)$ of the form (2) be in the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$. Then $f(z)$ is close to convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$,*

$$r_1 = \inf_{n \geq 2} \left\{ \frac{(1-\rho)\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{n|B-A|} \right\}^{1/(n-1)}. \tag{33}$$

Proof. It is well-known that $f \in \mathcal{K}$ for $|z| < r_1$ if

$$\left|f'(z) - 1\right| \leq 1 - \rho \quad \text{for } |z| < r_1 \tag{34}$$

(see [12]), where r_1 is given by (33).
From (2), we have

$$\left|f'(z) - 1\right| \leq \sum_{n=2}^{\infty} n a_n z^{n-1}. \tag{35}$$

Clearly $|f'(z) - 1| \leq 1 - \rho$, if

$$\sum_{n=2}^{\infty} \frac{n}{1-\rho} |z|^{n-1} \leq 1. \tag{36}$$

As $f \in k - \mathcal{VUST}(A, B, \delta, \lambda)$, from Theorem 5, inequality (36) will be true if

$$\frac{n}{1-\rho} |z|^{n-1} \leq \frac{2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|}{|B-A|}, \tag{37}$$

and this implies

$$|z| \leq \left\{ \frac{(1-\rho)\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{n|B-A|} \right\}^{1/(n-1)}. \tag{38}$$

□

Theorem 12. *Let $f(z)$ of form (2) be in the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$,*

$$r_2 = \left\{ \frac{(1-\rho)\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{(n-\rho)|B-A|} \right\}^{1/(n-1)}. \tag{39}$$

Proof. To prove $f \in \mathcal{ST}(\rho)$, for $|z| < r_2$ it is sufficient to show

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho \quad \text{for } |z| < r_2 \tag{40}$$

(see [12]). From (2) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}. \tag{41}$$

Thus $|zf'(z)/f(z) - 1| \leq 1 - \rho$, if

$$\sum_{n=2}^{\infty} \frac{(n-\rho)}{1-\rho} a_n |z|^{n-1} \leq 1. \tag{42}$$

As $f \in k - \mathcal{VUST}(A, B, \delta, \lambda)$, (42) must hold if

$$\frac{(n-\rho)}{1-\rho} |z|^{n-1} \leq \frac{\left(2(k+1)(n-1)\Phi_n + \left|(B+1)\Psi_n - (A+1)\Phi_n\right|\right)}{|B-A|}, \tag{43}$$

which gives us $|z| < r_2$, where r_2 is given by (39). □

Next we prove that the class $k - \mathcal{VUST}(A, B, \delta, \lambda)$ is closed under generalized Bernardi-Livingston operator defined as

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, \quad c > -1 \tag{44}$$

(see [17]).

Theorem 13. *If $f(z) \in k - \mathcal{VUST}(A, B, \delta, \lambda)$ then $L_c(f) \in k - \mathcal{VUST}(A, B, \delta, \lambda)$.*

Proof. Consider

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, \tag{45}$$

using (2) and, after integration, we obtain

$$L_c(f) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n. \tag{46}$$

Since $f(z) \in k - \mathcal{VUST}(A, B, \delta, \lambda)$, so an easy calculation leads to

$$\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)\Phi_n+\left|(B+1)\Psi_n-(A+1)\Phi_n\right|\right\}\cdot\left(\frac{c+1}{c+n}\right)a_n\leq\sum_{n=2}^{\infty}\left(2(k+1)(n-1)\Phi_n+\left|(B+1)\Psi_n-(A+1)\Phi_n\right|\right)a_n\leq|B-A|, \tag{47}$$

and, therefore, $L_c(f)\in k-\mathscr{VUST}(A,B,\delta,\lambda)$. □

Data Availability

There is no data for this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors jointly work on results, and they read and approved the final manuscript.

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