Hindawi
Journal of Function Spaces
Volume 2018, Article ID 4983827, 5 pages
https://doi.org/10.1155/2018/4983827

Research Article

Analytic Morrey Spaces and Bloch-Type Spaces

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Received 25 September 2017; Accepted 24 January 2018; Published 11 March 2018

Academic Editor: Alberto Fiorenza

This paper is devoted to characterizing the boundedness of the Riemann-Stieltjes operators from analytic Morrey spaces to Bloch-type spaces. Moreover, the boundedness of the superposition operator and weighted composition operator on analytic Morrey spaces is discussed, respectively.

1. Introduction

The open unit disk, the unit circle, and the area measure in the complex $\mathbb{C}$ are denoted by $\mathbb{D}$, $\mathbb{T}$, and $dA(z)$, respectively. The space of all analytic functions in $\mathbb{D}$ will be written $\mathscr{A}(\mathbb{D})$, and $\mathscr{H}^p$ denotes the Hardy space on the unit disk $\mathbb{D}$.

For $(p-1,\eta) \in [0,\infty)\times[0,\infty)$, the space of analytic Campanato space, denoted by $\mathscr{AL}_{p,\eta}$, is the set of all $f \in \mathscr{H}^p$ satisfying

$$\|f\|_{\mathscr{AL}_{p,\eta,*}} = \sup_{I\subseteq\mathbb{T}}\left(|I|^{-\eta}\int_I \left|f(\xi)-f_I\right|^p \frac{|d\xi|}{2\pi}\right)^{1/p} < \infty, \quad (1)$$

where $\sup_{I\subseteq\mathbb{T}}$ is taken over all subarcs $|I|$ of $\mathbb{T}$ and

$$\begin{aligned}|d\xi| &= \left|de^{i\theta}\right| = d\theta \\ |I| &= (2\pi)^{-1}\int_I |d\xi| \\ f_I &= (2\pi|I|)^{-1}\int_I f(\xi)\,|d\xi|.\end{aligned} \quad (2)$$

Table 1 will help us to understand the structure of $\mathscr{AL}_{p,\eta}$ (see, e.g., [1–3] and [4, p. 209–217] for the real counterparts).

Of course, this value defines a seminorm on $\mathscr{AL}_{p,\eta}$. A complete norm on $\mathscr{AL}_{p,\eta}$ can be equipped by

$$\|f\|_{\mathscr{AL}_{p,\eta}} = \|f\|_p + \|f\|_{\mathscr{AL}_{p,\eta,*}}. \quad (3)$$

For $\alpha > 0$, the α-Bloch space, denoted by $\mathscr{B}_\alpha = \mathscr{B}_\alpha(\mathbb{D})$, is the set of all $f \in \mathscr{A}(\mathbb{D})$ for which

$$\mathscr{B}_\alpha(f) = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \quad (4)$$

The expression $\mathscr{B}_\alpha(f)$ defines a seminorm while the natural norm is given by

$$\|f\|_{\mathscr{B}_\alpha} = |f(0)| + \mathscr{B}_\alpha(f). \quad (5)$$

The Riemann-Stieltjes operator J_g with analytic function symbol g is defined by

$$J_g f(z) = \int_0^z f(w)\,g'(w)\,dw, \quad f \in \mathscr{A}(\mathbb{D}). \quad (6)$$

The corresponding integral operator is defined by

$$I_g f(z) = \int_0^z f'(w)\,g(w)\,dw, \quad f \in \mathscr{A}(\mathbb{D}). \quad (7)$$

Obviously, the multiplication operator M_g is given as follows:

$$\begin{aligned}M_g f(z) &= f(z)\,g(z) \\ &= f(0)\,g(0) + I_g f(z) + J_g f(z).\end{aligned} \quad (8)$$

Continuing from [5–14], in this paper, we consider three operators associated with the analytic Morrey spaces. More precisely, in Section 2, we give two lemmas which are used

Table 1: The structure of analytic Campanato space.

Index (p,η)	Analytic Campanato space $\mathcal{AL}_{p,\eta}$
$\eta = 0$	Analytic Hardy space $\mathcal{H}^p$
$\eta \in (0,1)$	Analytic Morrey space $\mathcal{AL}^{p,1-\eta}$
$\eta = 1$	Analytic John-Nirenberg space $\mathcal{BMOA}$
$\eta \in (1, 1+p]$	Analytic Lipschitz space $\mathcal{A}_{(\eta-1)/p}$
$\eta \in (1+p, \infty)$	Space of constants $\mathbb{C}$

to prove our main results in the paper. In Section 3, we obtain the boundedness of the Riemann-Stieltjes operators J_g and I_g from analytic Morrey spaces to Bloch-type spaces. In Section 4, the boundedness of the defined superposition operator on analytic Morrey spaces is proved. In Section 5, we consider the boundedness of the weighted composition operator from analytic Morrey spaces to Bloch-type spaces.

Notation. Let $\mathsf{A} \lesssim \mathsf{B}$ and $\mathsf{A} \gtrsim \mathsf{B}$ denote that there exists an absolute constant $c > 0$ such that $\mathsf{A} \leq c\mathsf{B}$ and $\mathsf{A} \geq c\mathsf{B}$, respectively. $\mathsf{A} \approx \mathsf{B}$ means that $\mathsf{A} \lesssim \mathsf{B}$ and $\mathsf{A} \gtrsim \mathsf{B}$ hold.

2. Some Lemmas

We now recall some auxiliary results which will be used throughout this paper. Lemmas 1 and 2 are proved, the first by Xiao and Yuan in [15] and the second by Wang in [16].

Lemma 1. *Let $(p-1,\eta) \in [0,\infty) \times (0,1)$. If $f \in \mathcal{AL}^{p,\eta}$, then*

$$\left|f(z)\right| \lesssim \frac{\|f\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|z|^2\right)^{(1-\eta)/p}}, \qquad \left|f'(z)\right| \lesssim \frac{\|f\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|z|^2\right)^{1+(1-\eta)/p}} \tag{9}$$

hold for all $z \in \mathbb{D}$.

Lemma 2. *For any fixed $w \in \mathbb{D}$. Let $(p-1,\eta,\lambda) \in [0,\infty) \times (0,1) \times (1/p,\infty)$. Then functions*

$$f_w(z) = \frac{\left(1-|w|^2\right)^{\lambda-(1-\eta)/p}}{(1-\overline{w}z)^{\lambda}} \tag{10}$$

belong to $\mathcal{AL}^{p,\eta}$. Moreover, f_w is uniformly bounded in $\mathcal{AL}^{p,\eta}$; that is,

$$\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{AL}^{p,\eta}} < \infty. \tag{11}$$

3. Boundedness of the Operators $J_g, I_g : \mathcal{AL}^{p,\eta} \to \mathcal{B}_\alpha$

In this section, we give the boundedness of J_g and I_g from $\mathcal{AL}^{p,\eta} \to \mathcal{B}_\alpha$, respectively.

Theorem 3. *Let $(p-1,\eta) \in [0,\infty) \times [0,1)$. Suppose that $g \in \mathcal{A}(\mathbb{D})$. Then, $J_g : \mathcal{AL}^{p,\eta} \to \mathcal{B}_\alpha$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{a+(\eta-1)/p} \left|g'(z)\right| < \infty. \tag{12}$$

Moreover,

$$\left\|J_g\right\|_{\mathcal{AL}^{p,\eta}\to\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{a+(\eta-1)/p} \left|g'(z)\right|. \tag{13}$$

Proof. Suppose $J_g : \mathcal{AL}^{p,\eta} \to \mathcal{B}_\alpha$ is bounded. Taking $f_w(z) = (1-|w|^2)^{2+(\eta-1)/p}/(1-\overline{w}z)^2$, then

$$f_w'(w) = \frac{2\overline{w}\left(1-|w|^2\right)^{2+(\eta-1)/p}}{(1-\overline{w}z)^3}. \tag{14}$$

Note that

$$\begin{aligned}\left\|J_g f_w\right\|_{\mathcal{B}_\alpha} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|\left(J_g f_w\right)'(z)\right| \\ &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|f_w(z)\right| \left|g'(z)\right| \\ &\gtrsim \left(1-|w|^2\right)^{\alpha} \left|f_w(w)\right| \left|g'(w)\right| \\ &= \left(1-|w|^2\right)^{\alpha} \left(1-|w|^2\right)^{(\eta-1)/p} \left|g'(w)\right| \\ &= \left(1-|w|^2\right)^{\alpha+(\eta-1)/p} \left|g'(w)\right|.\end{aligned} \tag{15}$$

It implies that

$$\begin{aligned}&\left\|J_g f_w\right\|_{\mathcal{B}_\alpha} \leqslant \left\|f_w\right\|_{\mathcal{AL}^{p,\eta}} \left\|J_g\right\|_{\mathcal{AL}^{p,\eta}\to\mathcal{B}_\alpha} \\ &\quad\lesssim \left\|J_g\right\|_{\mathcal{AL}^{p,\eta}\to\mathcal{B}_\alpha}, \\ &\sup_{z\in\mathbb{D}} \left(1-|w|^2\right)^{\alpha+(\eta-1)/p} \left|g'(w)\right| \lesssim \left\|J_g\right\|_{\mathcal{AL}^{p,\eta}\to\mathcal{B}_\alpha}.\end{aligned} \tag{16}$$

Hence,

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{a+(\eta-1)/p} \left|g'(z)\right| < \infty. \tag{17}$$

Conversely, suppose

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{a+(\eta-1)/p} \left|g'(z)\right| < \infty. \tag{18}$$

By Lemma 1, we have that

$$\left|f(z)\right| \lesssim \frac{\|f\|_{\mathcal{AL}_{p,\eta}}}{\left(1-|z|^2\right)^{(1-\eta)/p}} \quad \forall z \in \mathbb{D}. \tag{19}$$

Hence,

$$\begin{aligned}\left\|J_g f\right\|_{\mathcal{B}_\alpha} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|\left(J_g f\right)'(z)\right| \\ &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} \left|f(z)\right| \left|g'(z)\right| \\ &\lesssim \|f\|_{\mathcal{AL}_{p,\eta}} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha+(\eta-1)/p} \left|g'(z)\right| \\ &\lesssim \|f\|_{\mathcal{AL}_{p,\eta}}\end{aligned} \tag{20}$$

implies $J_g : \mathscr{A}\mathscr{L}^{p,\eta} \to \mathscr{B}_\alpha$ is bounded and

$$\left\|J_g\right\|_{\mathscr{A}\mathscr{L}^{p,\eta}\to\mathscr{B}_\alpha} = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{a+(\eta-1)/p}\left|g'(z)\right|. \quad (21)$$

□

Theorem 4. *Let $(p-1,\eta) \in [0,\infty)\times[0,1)$ and $g \in \mathscr{A}(\mathbb{D})$. Then $I_g : \mathscr{A}\mathscr{L}^{p,\eta} \to \mathscr{B}_\alpha$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)| < \infty. \quad (22)$$

Moreover,

$$\left\|I_g\right\|_{\mathscr{A}\mathscr{L}^{p,\eta}\to\mathscr{B}_\alpha} = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)|. \quad (23)$$

Proof. Suppose $I_g : \mathscr{A}\mathscr{L}^{p,\eta} \to \mathscr{B}_\alpha$ is bounded. Let

$$f_w(z) = \frac{\left(1-|w|^2\right)^{2-(1-\eta)/p}}{(1-\overline{w}z)^2}. \quad (24)$$

Then

$$f_w'(w) = \frac{2\overline{w}\left(1-|w|^2\right)^{2+(\eta-1)/p}}{(1-\overline{w}z)^3}. \quad (25)$$

Lemma 2 shows that

$$\begin{aligned}\left\|I_g f_w\right\|_{\mathscr{B}_\alpha} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}\left|\left(I_g f_w\right)'(z)\right| \\ &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}\left|f_w'(z)\right||g(z)| \\ &\geq \left(1-|w|^2\right)^{\alpha}\left|f_w'(w)\right||g(w)| \\ &\gtrsim \left(1-|w|^2\right)^{\alpha}\left(1-|w|^2\right)^{(\eta-1)/p-1}|w||g(w)|.\end{aligned} \quad (26)$$

Taking supremum in the last inequality over the set $1/2 \leq |w| < 1$ and applying to the maximum modulus principle we have

$$\begin{aligned}&\sup_{|w|\leq 1/2}\left(1-|w|^2\right)^{\alpha}\left(1-|w|^2\right)^{(\eta-1)/p-1}|g(w)| \\ &= \sup_{|w|=1/2}\left(1-|w|^2\right)^{\alpha}\left(1-|w|^2\right)^{(\eta-1)/p-1}|g(w)| \\ &\lesssim \sup_{1/2\leq|w|<1}\left(1-|w|^2\right)^{\alpha}\left(1-|w|^2\right)^{(\eta-1)/p-1}|w||g(w)|.\end{aligned} \quad (27)$$

Hence

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)| \leq \left\|I_g\right\|_{\mathscr{A}\mathscr{L}^{p,\eta}\to\mathscr{B}_\alpha}. \quad (28)$$

That is,

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)| < \infty. \quad (29)$$

Conversely, suppose

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)| < \infty. \quad (30)$$

By Lemma 1, we have that

$$\left|f'(z)\right| \lesssim \frac{\|f\|_{\mathscr{A}\mathscr{L}_{p,\eta}}}{\left(1-|z|^2\right)^{1+(1-\eta)/p}} \quad \forall z \in \mathbb{D}. \quad (31)$$

Then

$$\begin{aligned}\left\|I_g f\right\|_{\mathscr{B}_\alpha} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}\left|\left(I_g f\right)'(z)\right| \\ &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}\left|f'(z)\right||g(z)| \\ &\lesssim \|f\|_{\mathscr{A}\mathscr{L}_{p,\eta}}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)| \\ &\lesssim \|f\|_{\mathscr{A}\mathscr{L}_{p,\eta}},\end{aligned} \quad (32)$$

which implies $I_g : \mathscr{A}\mathscr{L}^{p,\eta} \to \mathscr{B}_\alpha$ is bounded and

$$\left\|I_g\right\|_{\mathscr{A}\mathscr{L}^{p,\eta}\to\mathscr{B}_\alpha} = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+(\eta-1)/p-1}|g(z)|. \quad (33)$$

□

Remark 5. When $\alpha < 1 + (1-\eta)/p$ in Theorem 4, the conclusion is equivalent to $g \equiv 0$.

As an application of Theorems 3 and 4, we can obtain the following corollary.

Corollary 6. *Let $(p-1,\eta) \in [0,\infty)\times[0,1)$ and $g \in \mathscr{A}(\mathbb{D})$. Then $M_g : \mathscr{A}\mathscr{L}^{p,\eta} \to \mathscr{B}_\alpha$ is bounded if and only if $g \in \mathscr{H}^\infty$.*

4. Superposition on $\mathscr{A}\mathscr{L}^{p,\eta}$

Let $\mathscr{X}$ and $\mathscr{Y}$ represent two subspaces of $\mathscr{A}(\mathbb{D})$. If ϕ is a complex-valued function $\mathbb{C}$ such that $\phi\circ f \in \mathscr{Y}$ whenever $f \in \mathscr{X}$, then we call that ϕ acts by superposition from $\mathscr{X}$ into $\mathscr{Y}$. If $\mathscr{X}$ and $\mathscr{Y}$ contain the linear functions, then ϕ must be an entire function. The superposition $\mathsf{S}^\phi : \mathscr{X} \to \mathscr{Y}$ with symbol ϕ is then defined by $\mathsf{S}^\phi(f) = \phi\circ f$. A basic question is when S^ϕ map $\mathscr{X}$ into $\mathscr{Y}$ continuously. This question has been studied for many distinct pairs $(\mathscr{X},\mathscr{Y})$—see, for example, [17–20]. In this section, we are interested in the analytic Morrey space and have the following result which extends the case of $p = 2$ in [3].

Theorem 7. *Let $(p-1,\eta) \in [0,\infty)\times(0,1)$. Then S^ϕ is bounded on $\mathscr{A}\mathscr{L}^{p,\eta}$ if and only if $\phi(z) = az + b$ for some $a, b \in \mathbb{C}$.*

Proof. Lemma 1 shows that

$$f_w(z) = \frac{\left(1-|w|^2\right)^{2+(\eta-1)/p}}{(1-\overline{w}z)^2} \Longrightarrow$$
$$\sup_{b\in\mathbb{D}} \|f_w\|_{\mathcal{AL}^{p,\eta}} < \infty;$$
$$f \in \mathcal{AL}^{p,\eta} \Longrightarrow \tag{34}$$
$$\left|f'(z)\right| \lesssim \frac{\|f\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|z|^2\right)^{1+(1-\eta)/p}} \quad \forall z \in \mathbb{D}.$$

If S^ϕ is bounded on $\mathcal{AL}^{p,\eta}$, then for $f \in \mathcal{AL}^{p,\eta}$ we obtain

$$\begin{aligned}\left|\left(\mathsf{S}^\phi(f)\right)'(z)\right| &= \left|\phi'(f(z))\right|\left|f'(z)\right| \\ &\lesssim \frac{\|f\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|z|^2\right)^{1+(1-\eta)/p}} \quad \forall z \in \mathbb{D}.\end{aligned} \tag{35}$$

Taking the following $\mathcal{AL}^{p,\eta}$-function

$$f_{\theta,w}(z) = \frac{e^{i\theta}\left(1-|w|^2\right)^{2+(\eta-1)/p}}{(1-\overline{w}z)^2} \tag{36}$$

in the last inequality, we have

$$\begin{aligned}&\left|\phi'(f_{\theta,w}(z))\right|\left|f'_{\theta,w}(z)\right| \\ &= \left|\phi'(f_{\theta,w}(z))\right|\left(\frac{|w|\left(1-|w|^2\right)^{3+(\eta-1)/p}}{|1-\overline{w}z|^3}\right) \\ &\lesssim \frac{\|f_{\theta,w}\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|z|^2\right)^{(1-\eta)/p}}.\end{aligned} \tag{37}$$

Since

$$\sup_{\theta,w} \|f_{\theta,w}\|_{\mathcal{AL}^{p,\eta}} < \infty, \tag{38}$$

so there is a positive M independent of (θ, w) such that

$$\sup_{\theta,w} \|f_{\theta,w}\|_{\mathcal{AL}^{p,\eta}} \le M. \tag{39}$$

In particular, setting $w = z$ yields

$$\begin{aligned}&\left|\phi'(f_{\theta,w}(w))\right|\left|f'_{\theta,w}(w)\right| = \sup_{w\in\mathbb{D}}\left|\phi'(f_w(w))\right||w| \le M \Longrightarrow \\ &\left|\phi'\left(e^{i\theta}\left(1-|w|^2\right)^{(1-\eta)/p}\right)\right| \le M \\ &\qquad \forall w \in \mathbb{D}.\end{aligned} \tag{40}$$

Setting $|w| \to 1$ in the last estimate and noticing $\eta \in (0,1)$, we obtain that the entire function ϕ is bounded on $\mathbb{C}$. By using the maximum principle, we get that ϕ must be a linear function.

Conversely, if $\phi(z) = az+b$ for some $a, b \in \mathbb{C}$, then $\phi'(z) = a$, and hence $\mathsf{S}^\phi(f) = af+b$. Hence S^ϕ is bounded on $\mathcal{AL}^{p,\eta}$. □

5. Weighted Composition Operator from $\mathcal{AL}^{p,\eta}$ to $\mathcal{B}_\alpha$

Suppose that $\psi \in \mathcal{A}(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$. These maps induce a linear weighted composition operator $W_{\psi,\varphi}$ which is defined by

$$\left(W_{\psi,\varphi}f\right)(z) = \left(M_\psi C_\varphi f\right)(z) = \psi(z) f(\varphi(z)), \quad z \in \mathbb{D}, \tag{41}$$

where M_ψ is the operator of pointwise multiplication by ψ and C_φ is the composition operator $f \mapsto f \circ \varphi$. Our next result will be as follows.

Theorem 8. *Let $0 < \eta < 1$, $0 < \alpha, p - 1 < \infty$, and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\left\|W_{\psi,\varphi}f\right\|_{\mathcal{B}_\alpha} \lesssim \|f\|_{\mathcal{AL}^{p,\eta}}, \quad \forall f \in \mathcal{AL}^{p,\eta}, \tag{42}$$

if and only if

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha\left(\left(1-|\varphi(z)|^2\right)\left|\psi'(z)\right| + |\psi(z)|\left|\varphi'(z)\right|\right)}{\left(1-|\varphi(z)|^2\right)^{(p-\eta+1)/p}} < \infty. \tag{43}$$

Proof. The growth of functions in $\mathcal{AL}^{p,\eta}$ shows that if $f \in \mathcal{AL}^{p,\eta}$ then

$$\begin{aligned}&\left(1-|z|^2\right)^\alpha\left|\left(W_{\psi,\varphi}(f)\right)'(z)\right| \\ &\le \left(1-|z|^2\right)^\alpha\left|\psi'(z)\right||f(\varphi(z))| \\ &\quad + \left(1-|z|^2\right)^\alpha|\psi(z)|\left|f'(\varphi(z))\right|\left|\varphi'(z)\right| \\ &= B_1 + B_2.\end{aligned} \tag{44}$$

It is easy to see

$$\begin{aligned}B_1 &= \left(1-|z|^2\right)^\alpha\left|\psi'(z)\right||f(\varphi(z))| \\ &\le \left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|\frac{\|f\|_{\mathcal{AL}^{p,\eta}}}{\left(1-|\varphi(z)|^2\right)^{(1-\eta)/p}} \\ &= \frac{\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{(1-\eta)/p}}\|f\|_{\mathcal{AL}^{p,\eta}}, \\ B_2 &= \left(1-|z|^2\right)^\alpha|\psi(z)|\left|f'(\varphi(z))\right|\left|\varphi'(z)\right| \\ &\le \frac{\left(1-|z|^2\right)^\alpha|\psi(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{(p-\eta+1)/p}}\|f\|_{\mathcal{AL}^{p,\eta}}.\end{aligned} \tag{45}$$

If

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha\left(\left(1-|\varphi(z)|^2\right)\left|\psi'(z)\right| + |\psi(z)|\left|\varphi'(z)\right|\right)}{\left(1-|\varphi(z)|^2\right)^{(p-\eta+1)/p}} < \infty, \tag{46}$$

then $B_1 < \infty$ and $B_2 < \infty$. It yields that $W_{\psi,\varphi}(f) \in \mathscr{B}_\alpha$ which proves that the sufficiency holds. On the other hand, the necessary follows by taking the test function in $\mathscr{A}\mathscr{L}^{p,\eta}$

$$f_w(z) = \frac{\left(1 - |\varphi(w)|^2\right)^{2+(1-\eta)/p}}{\left(1 - \overline{\varphi(w)}z\right)^2}, \quad \forall z \in \mathbb{D}. \tag{47}$$

□

Remark 9. When $\psi(z) \equiv 1$, the result reduces to Xiao and Yuan in [15]. When $\varphi(z) \equiv z$, $W_{\psi,\varphi}f$ denote the multiplication operator.

$$\|\psi \cdot f\|_{\mathscr{B}_\alpha} \lesssim \|f\|_{\mathscr{A}\mathscr{L}^{p,\eta}}, \quad \forall f \in \mathscr{A}\mathscr{L}^{p,\eta}, \tag{48}$$

if and only if

$$\sup_{z\in\mathbb{D}} \frac{\left(1 - |z|^2\right)\left|\psi'(z)\right| + |\psi(z)|}{\left(1 - |z|^2\right)^{(p-\eta+1)/p-\alpha}} < \infty. \tag{49}$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

The authors were partially supported by the National Natural Science Foundation of China (nos. 11471111, 11571105, and 11671362) and the Natural Science Foundation of Zhejiang Province (nos. LY16A010004 and LY13A010021).

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