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Research Article

On the Effective Reducibility of a Class of Quasi-Periodic Linear Hamiltonian Systems Close to Constant Coefficients

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In this paper, we consider the effective reducibility of the quasi-periodic linear Hamiltonian system $\dot{x} = (A + \varepsilon Q(t, \varepsilon))x$, $\varepsilon \in (0, \varepsilon_0)$, where A is a constant matrix with possible multiple eigenvalues and $Q(t, \varepsilon)$ is analytic quasi-periodic with respect to t. Under nonresonant conditions, it is proved that this system can be reduced to $\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t, \varepsilon))y$, $\varepsilon \in (0, \varepsilon^*)$, where R^* is exponentially small in ε, and the change of variables that perform such a reduction is also quasi-periodic with the same basic frequencies as Q.

1. Introduction

The question about the reducibility of quasi-periodic systems plays an important role in the theory of ordinary differential equations. In general, in order to understand the qualitative behavior of a system, we need to obtain the information about the existence and stability of solutions. During the last two decades, the study of the existence of solutions for differential equations has attracted the attention of many researchers; see [1–10] and the references therein. Some classical tools have been used to study the existence of solutions for differential equations in the literature, including the method of upper and lower solutions, degree theory, some fixed point theorems in cones for completely continuous operators, Schauder's fixed point theorem, and a nonlinear Leray-Schauder alternative principle.

Compared with the existence of solutions, the study on the dynamical stability behaviors of such equations is more difficult, and the results are fewer in the literature. Here we refer the reader to [11–16].

Before stating our problem, we give some definitions and notations. A function f is said to be a quasi-periodic function with a vector of basic frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$ if $f(t) = F(\theta_1, \theta_2, \ldots, \theta_r)$, where F is 2π periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, 2, \ldots, r$. Moreover, if $F(\theta)$ $(\theta = (\theta_1, \theta_2, \ldots, \theta_r))$ is analytic on $D_\rho = \{\theta \in \mathbb{C}^r : |\mathrm{Im}\theta_j| \leq \rho, j = 1, 2, \ldots, r\}$, we say that $f(t)$ is analytic quasi-periodic on D_ρ.

It is well known that an analytic quasi-periodic function $f(t)$ can be expanded as Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}^r} f_k e^{\langle k, \omega \rangle \sqrt{-1} t} \tag{1}$$

with Fourier coefficients defined by

$$f_k = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} F(\theta_1, \theta_2, \ldots, \theta_r) e^{-\langle k, \theta \rangle \sqrt{-1}} d\theta. \tag{2}$$

We denote by $\|f\|_\rho$ the norm

$$\|f\|_\rho = \sum_{k \in \mathbb{Z}^r} |f_k| e^{|k|\rho}. \tag{3}$$

An $n \times n$ matrix $Q(t) = (q_{ij})_{1 \leq i, j \leq n}$ is said to be analytic quasi-periodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$, if all $q_{ij}(i, j = 1, 2, \ldots, n)$ are analytic quasi-periodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$. Define the norm of Q by

$$\|Q\|_\rho = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left\|q_{ij}\right\|_\rho. \tag{4}$$

It is easy to see that $\|Q_1Q_2\|_\rho \le \|Q_1\|_\rho\|Q_2\|_\rho$. If Q is a constant matrix, write $\|Q\| = \|Q\|_\rho$ for simplicity. Denote the average of $Q(t)$ by $\overline{Q} = (\overline{q}_{ij})_{1\le i,j\le n}$, where

$$\overline{q}_{ij} = \lim_{T\longrightarrow\infty}\frac{1}{2T}\int_{-T}^{T} q_{ij}(t)\,dt, \tag{5}$$

for the existence of the limit, see [17].

Let $A(t)$ be an $n\times n$ quasi-periodic matrix; the differential equations $\dot{x} = A(t)x$, $x \in \mathbb{R}^n$, are called reducible if there exists a nonsingular quasi-periodic change of variables $x = \phi(t)y$, such that $\phi(t)$ and $\phi^{-1}(t)$ are quasi-periodic and bounded, which changes $\dot{x} = A(t)x$ to $\dot{y} = By$, where B is a constant matrix. The well-known Floquet theorem states that any periodic differential equations $\dot{x} = A(t)x$ can be reduced to constant coefficient differential equations $\dot{y} = By$ by means of a periodic change of variables with the same period as $A(t)$. But this is not true for the quasi-periodic coefficient system; see [18]. Johnson and Sell [19] proved that $\dot{x} = A(t)x$ is reducible if the quasi-periodic coefficient matrix $A(t)$ satisfies "full spectrum" condition.

Recently, many authors [20–23] considered the reducibility of the following system which is close to constant coefficients matrix:

$$\dot{x} = (A + \varepsilon Q(t))\,x. \tag{6}$$

This problem was first considered by Jorba and Simó in [20]. Suppose that A is a constant matrix with different eigenvalues; they proved that if the eigenvalues of A and the frequencies of Q satisfy some nonresonant conditions, then for sufficiently small $\varepsilon_0 > 0$, there exists a nonempty Cantor set $E \subset (0,\varepsilon_0)$, such that, for any $\varepsilon \in E$, system (6) is reducible. Moreover, the relative measure of the set $(0,\varepsilon_0)\backslash E$ in $(0,\varepsilon_0)$ is exponentially small in ε_0. In [23], Xu obtained the similar result for the multiple eigenvalues case.

In [21], Jorba and Simó extended the conclusion of the linear system to the nonlinear system

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))\,x + \varepsilon g(t) + h(x,t), \quad x \in \mathbb{R}^n. \tag{7}$$

Suppose that A has n different nonzero eigenvalues; they proved that, under some nonresonant conditions and nondegeneracy conditions, there exists a nonempty Cantor set $E \subset (0,\varepsilon_0)$, such that, for all $\varepsilon \in E$, system (7) is reducible. Later, in [24], Wang and Xu considered the nonlinear quasi-periodic system

$$\dot{x} = Ax + f(x,t), \quad x \in \mathbb{R}^2, \tag{8}$$

and they proved without any nondegeneracy condition that one of two results holds: (1) system (8) is reducible to $\dot{y} = By + O(y)$ for all $\varepsilon \in (0,\varepsilon_0)$; (2) there exists a nonempty Cantor set $E \subset (0,\varepsilon_0)$, such that system (8) is reducible to $\dot{y} = By + O(y^2)$ for all $\varepsilon \in E$.

These papers above all deal with a total reduction to constant coefficients. In [25], instead of a total reduction to constant coefficients, Jorba, Ramirez-ros, and Villanueva considered the effective reducibility of the following quasi-periodic system:

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))\,x, \quad |\varepsilon| \le \varepsilon_0, \tag{9}$$

where A is a constant matrix with different eigenvalues. They proved that, under nonresonant conditions, by a quasi-periodic transformation, system (9) is reducible to a quasi-periodic system

$$\dot{y} = \left(A^*(\varepsilon) + \varepsilon R^*(t,\varepsilon)\right) y, \quad |\varepsilon| \le \varepsilon_* \le \varepsilon_0, \tag{10}$$

where R^* is exponentially small in ε. In [26], Li and Xu obtained the similar result for Hamiltonian systems.

In this paper, we consider the case that A has multiple eigenvalues. Under some nonresonant conditions, we can obtain the effective reducibility for system (9) similar to [25, 26].

Now we are in a position to state the main result.

Theorem 1. *Consider the following linear Hamiltonian system:*

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))\,x, \quad x \in \mathbb{R}^n, \tag{11}$$

where A is a constant matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$, $Q(t,\varepsilon)$ is an analytic quasi-periodic function on D_ρ with the frequencies $\omega = (\omega_1,\ldots,\omega_r)$, and $\varepsilon \in (0,\varepsilon_0)$ is a small parameter.

If $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ and $\omega = (\omega_1,\omega_2,\ldots,\omega_r)$ satisfy the nonresonant conditions,

$$\left|(k,\omega)\sqrt{-1} - \lambda_i + \lambda_j\right| \ge \frac{\alpha}{|k|^\tau} \tag{12}$$

for all $k \in \mathbb{Z}^r\backslash\{0\}$, $0 \le i, j \le n$, where $\alpha > 0$ is a small constant and $\tau > r-1$. In addition, we assume that $A+\varepsilon\overline{Q}$ has n different eigenvalues $\mu_1,\ldots,\mu_n$, and $\delta := \min\{\varepsilon^{-1}|\mu_i - \mu_j| : 0 \le i, j \le n, i\ne j\}$ is a positive constant independent of ε.

Then there exists some $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0,\varepsilon^*)$, there is an analytic quasi-periodic symplectic transformation $x = \psi(t,\varepsilon)y$ on D_ρ, where $\psi(t,\varepsilon)$ has same frequencies as $Q(t,\varepsilon)$, which changes system (11) into the following linear system:

$$\dot{y} = \left(A^*(\varepsilon) + \varepsilon R^*(t,\varepsilon)\right) y, \tag{13}$$

where A^* is a constant matrix with

$$\left\|A^* - A\right\| \le \frac{e(\beta+1)q}{e-1}\varepsilon, \tag{14}$$

$R^*(t,\varepsilon)$ is an analytic quasi-periodic function on D_ρ with the frequencies ω, and

$$\left\|R^*(t,\varepsilon)\right\|_{\rho-s} \le \frac{e^2\beta^2 q}{e-1}\exp\left(-\left(\frac{d}{\varepsilon^{1/2}}\right)^{1/\tau} s\right), \quad s \in (0,\rho]. \tag{15}$$

Furthermore, a general explicit computation of ε^* and d is possible:

$$\varepsilon^* = \min\left\{\varepsilon_0, \left(\frac{\delta(e-1)}{(3n-1)e\beta q}\right)^2, \frac{1}{\beta^2}\right\}, \quad d = \frac{\alpha}{12e\beta q}, \tag{16}$$

where β is the condition number of a matrix S such that $S^{-1}(A+\varepsilon\overline{Q})S$ is diagonal, that is, $\beta = C(S) = \|S^{-1}\|\|S\|$, and the constant q is the bound of $Q(t,\varepsilon)$ on D_ρ, that is, $\|Q(t,\varepsilon)\|_\rho \le q$.

Remark 2. In general, Q depends on ε, so does the average $\overline{Q}$. Below for simplicity, we do not indicate this dependence explicitly.

Remark 3. In Hamiltonian system (11), n is an even number. In fact, a Hamiltonian system is $2m$-dimensional; moreover, the eigenvalues $\lambda_1, \ldots, \lambda_{2m}$ of a $2m \times 2m$ Hamiltonian matrix may be ordered so that $\lambda_{k+m} = -\lambda_k \quad (k = 1, \ldots, m)$.

Now we give some remarks on this result. Firstly, here we deal with the Hamiltonian system and have to find the *symplectic* transformation, which is different from that in [20, 23, 25]. Secondly, compared with [26], we can allow the matrix A to have multiple eigenvalues. Of course, if the eigenvalues of A are different, the nondegeneracy condition holds naturally, then our result is just the same as in [26].

2. Some Lemmas

We need some lemmas which are provided in this section for the proof of Theorem 1.

Lemma 4. *Let $Q(t) = \sum_{k\in\mathbb{Z}^r} Q_k e^{\langle k,\omega\rangle\sqrt{-1}t}$ be analytic quasi-periodic on D_ρ with frequencies ω. Let $\widetilde{Q}(t) = Q(t) - \overline{Q}$,*

$$Q^{\geq M}(t) = \sum_{k\in\mathbb{Z}^r, |k|\geq M} Q_k e^{\langle k,\omega\rangle\sqrt{-1}t}, \tag{17}$$

and $\widetilde{Q}^M = \widetilde{Q} - Q^{\geq M}$, where $M > 0$. Then we have the following results:

(1) $\|\overline{R}\|$, $\|\widetilde{Q}\|_\rho$, $\|\widetilde{Q}^M\|_\rho \leq \|Q\|_\rho$.

(2) $\|Q^{\geq M}\|_{\rho-s} \leq \|Q\|_\rho e^{-Ms}$, $\forall s \in (0, \rho]$.

This lemma can be seen in [25].

The next lemma will be used to show the convergence.

Lemma 5. *Let $(q_m)_m$, $(a_m)_m$, and $(r_m)_m$ be sequences defined by*

$$\begin{aligned} q_{m+1} &= q_m^2, \\ a_{m+1} &= a_m + q_{m+1}, \\ r_{m+1} &= \frac{2+q_m}{2-q_m} r_m + q_{m+1}, \end{aligned} \tag{18}$$

with initial values $q_1 = a_1 = r_1 = e^{-1}$. Then $(q_m)_m$ is decreasing to zero and $(a_m)_m$, $(r_m)_m$ are increasing and convergent to some values a_∞ and r_∞, respectively, with $a_\infty < 1/(e-1)$, $r_\infty < e/(e-1)$.

The proof of this lemma can be found in [25].

Lemma 6. *Let D be an $n \times n$ diagonal matrix with different eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\delta = \min_{i\neq j}(|\lambda_i - \lambda_j|)$. Then if A verifies $\|A - D\| \leq b \leq \delta/(3n-1)$, the following conditions hold:*

(1) A has n different eigenvalues $\mu_1, \ldots, \mu_n$ and $|\lambda_j - \mu_j| \leq b$, $j = 1, \ldots, n$.

(2) There exists a regular matrix S such that $S^{-1}AS = D^* = diag(\mu_1, \ldots, \mu_n)$ satisfying $C(S) \leq 2$.

This lemma can be seen in [20].

3. Proof of Theorem 1

By the assumptions of Theorem 1, $A + \varepsilon\overline{Q}$ has n different eigenvalues $\mu_1, \cdots, \mu_n$, then there exists a symplectic matrix S such that

$$S^{-1}\left(A + \varepsilon\overline{Q}\right)S = D = \operatorname{diag}(\mu_1, \ldots, \mu_n). \tag{19}$$

Under the change of variables $x = Sx_1$, system (11) is changed into

$$\dot{x}_1 = \left(D + \varepsilon S^{-1}\left(Q(t) - \overline{Q}\right)S\right)x_1 = \left(D + \varepsilon\widetilde{Q}(t)\right)x_1, \tag{20}$$

where $\widetilde{Q}(t) = S^{-1}(Q(t) - \overline{Q})S$; it is easy to see that $\overline{\widetilde{Q}} = 0$.

Now we can consider the iteration step.

In the m-th step, we consider the system

$$\dot{x}_m = \left(A_m(\varepsilon) + \varepsilon Q_m(t) + \varepsilon R_m(t)\right)x_m, \quad m \geq 1, \tag{21}$$

where $A_1 = D$, $Q_1 = \widetilde{Q}^M$, $R_1 = \widetilde{Q}^{\geq M}$ are Hamiltonian. Suppose A_m, Q_m, and R_m are Hamiltonian. Assume

$$\begin{aligned} \|A_m - D\| &\leq q^* a_m \varepsilon^{3/2}, \\ \|Q_m\|_\rho &\leq q^* q_m, \\ \|R_m\|_{\rho-s} &\leq q^* r_m e^{-M(\varepsilon)s}, \end{aligned} \tag{22}$$

where $q^* = \beta eq$, $s \in (0, \rho]$, a_m, q_m, and r_m are defined in Lemma 5.

Let the change of variables be $x_m = e^{\varepsilon P_m}x_{m+1}$; under this symplectic transformation, system (21) is changed to

$$\begin{aligned} \dot{x}_{m+1} &= \Bigg(e^{-\varepsilon P_m}\left(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m\right)e^{\varepsilon P_m} \\ &\quad + e^{-\varepsilon P_m}\left(\varepsilon\dot{P}_m e^{\varepsilon P_m} - \frac{d}{dt}\left(e^{\varepsilon P_m}\right)\right) + \varepsilon e^{-\varepsilon P_m}R_m e^{\varepsilon P_m}\Bigg) \\ &\quad \cdot x_{m+1} = \Bigg(\left(I - \varepsilon P_m + \widetilde{B}_m\right)\left(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m\right) \\ &\quad \cdot \left(I + \varepsilon P_m + B_m\right) + e^{-\varepsilon P_m}\left(\varepsilon\dot{P}_m e^{\varepsilon P_m} - \frac{d}{dt}\left(e^{\varepsilon P_m}\right)\right) \\ &\quad + \varepsilon e^{-\varepsilon P_m}R_m e^{\varepsilon P_m}\Bigg)x_{m+1} = \Big(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m \\ &\quad + \varepsilon A_m P_m - \varepsilon P_m A_m + \varepsilon Q_m^* + \varepsilon e^{-\varepsilon P_m}R_m e^{\varepsilon P_m}\Big)x_{m+1}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \varepsilon Q_m^* &= -\varepsilon^2 P_m\left(Q_m - \dot{P}_m\right) + \varepsilon^2\left(Q_m - \dot{P}_m\right)P_m \\ &\quad - \varepsilon^2 P_m\left(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m\right)P_m \\ &\quad - \varepsilon P_m\left(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m\right)B_m \\ &\quad + \left(A_m + \varepsilon Q_m - \varepsilon\dot{P}_m\right)B_m \end{aligned}$$

$$\begin{aligned}
&+\widetilde{B}_m\left(A_m+\varepsilon Q_m-\varepsilon \dot{P}_m\right)e^{\varepsilon P_m}\\
&+e^{-\varepsilon P_m}\left(\varepsilon \dot{P}_m e^{\varepsilon P_m}-\frac{d}{dt}e^{\varepsilon P_m}\right),\\
e^{\varepsilon P_m}&=I+\varepsilon P_m+B_m,\\
e^{-\varepsilon P_m}&=I-\varepsilon P_m+\widetilde{B}_m,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
B_m&=\frac{(\varepsilon P_m)^2}{2!}+\frac{(\varepsilon P_m)^3}{3!}+\cdots,\\
\widetilde{B}_m&=\frac{(\varepsilon P_m)^2}{2!}-\frac{(\varepsilon P_m)^3}{3!}+\cdots.
\end{aligned} \tag{25}$$

We would like to have

$$Q_m+A_mP_m-\dot{P}_m-P_mA_m=0, \tag{26}$$

and this is equivalent to

$$\dot{P}_m=A_mP_m-P_mA_m+Q_m. \tag{27}$$

Now we want to solve (27) to obtain an analytic quasi-periodic Hamiltonian solution $P_m(t)$ on D_ρ with the frequencies ω.

From (22), it follows that

$$\|A_m-D\|\le q^* a_m \varepsilon^{3/2}\le \frac{\delta\varepsilon}{3n-1},\quad \varepsilon\in(0,\varepsilon^*). \tag{28}$$

Thus by Lemma 6, A_m has n different eigenvalues $\lambda_1^m,\dots,\lambda_n^m$ and

$$\left|\lambda_i^m-\mu_i\right|\le q^* a_m\varepsilon^{3/2}. \tag{29}$$

Since A_m is Hamiltonian, from the discussion in Section 15 of [17], it follows that there exists a symplectic matrix S_m such that

$$S_m^{-1}A_mS_m=D_m=\operatorname{diag}\left(\lambda_1^m,\dots,\lambda_n^m\right); \tag{30}$$

moreover, $C(S_m)\le 2$, where we let $D_1=D$, $\lambda_i^1=\mu_i$ $(i=1,2,\dots,n)$.

If

$$\left|\langle k,\omega\rangle\sqrt{-1}-\lambda_i^m+\lambda_j^m\right|\ge L\varepsilon^{1/2} \tag{31}$$

for $0<|k|<M$, $1\le i,j\le n$, where $L>0$, $M>0$ are constants.

Making the change of variable $P_m=S_mX_mS_m^{-1}$ and defining $Y_m=S_m^{-1}Q_mS_m$, (27) becomes

$$\dot{X}_m=D_mX_m-X_mD_m+Y_m,\quad \overline{Y}_m=0. \tag{32}$$

Expand X_m and Y_m into Fourier series

$$\begin{aligned}
X_m(t)&=\sum_{k\in\mathbb{Z}^r,0<|k|<M} x_m^k e^{\langle k,\omega\rangle\sqrt{-1}t},\\
Y_m(t)&=\sum_{k\in\mathbb{Z}^r,0<|k|<M} y_m^k e^{\langle k,\omega\rangle\sqrt{-1}t},
\end{aligned} \tag{33}$$

where $x_m^k=(x_{ij,m}^k)_{1\le i,j\le n}$ and $y_m^k=(y_{ij,m}^k)_{1\le i,j\le n}$.

Thus the coefficients must be

$$x_{ij,m}^k=\frac{y_{ij,m}^k}{\langle k,\omega\rangle\sqrt{-1}-\lambda_i^m+\lambda_j^m}. \tag{34}$$

By (31), we have

$$\begin{aligned}
\|X_m\|_\rho&\le\left(L\varepsilon^{1/2}\right)^{-1}\|Y_m\|_\rho\le\left(L\varepsilon^{1/2}\right)^{-1}C(S_m)\|Q_m\|_\rho\\
&\le 2\left(L\varepsilon^{1/2}\right)^{-1}\|Q_m\|_\rho,
\end{aligned} \tag{35}$$

which implies

$$\|P_m\|_\rho\le C(S_m)\|X_m\|_\rho\le 4\left(L\varepsilon^{1/2}\right)^{-1}\|Q_m\|_\rho. \tag{36}$$

Now we prove that P_m is Hamiltonian. To this end, we only need to prove that X_m is Hamiltonian. Since D_m and Y_m are Hamiltonian, then $D_m=JD_{mJ}$ and $Y_m=JY_{mJ}$, where D_{mJ} and Y_{mJ} are symmetric. Let $X_{mJ}=J^{-1}X_m$, if X_{mJ} is symmetric, then X_m is Hamiltonian. Below we prove that X_{mJ} is symmetric. Substituting $X_m=JX_{mJ}$ into (32) yields that

$$\dot{X}_{mJ}=D_{mJ}JX_{mJ}-X_{mJ}JD_{mJ}+Y_{mJ}, \tag{37}$$

and transposing (37), we get

$$\dot{X}_{mJ}^T=D_{mJ}JX_{mJ}^T-X_{mJ}^TJD_{mJ}+Y_{mJ}. \tag{38}$$

It is easy to see that JX_{mJ} and JX_{mJ}^T are solutions of (32); moreover, $\overline{JX_{mJ}}=\overline{JX_{mJ}^T}=0$. Since the solution of (32) with $\overline{X}_m=0$ is unique, we have that $JX_{mJ}=JX_{mJ}^T$, which implies that X_m is Hamiltonian. Since S_m is symplectic, it is easy to see that $P_m=S_mX_mS_m^{-1}$ is Hamiltonian.

Thus, under the symplectic transformation $x_m=e^{\varepsilon P_m}x_{m+1}$, system (21) is changed into the system

$$\dot{x}_{m+1}=\left(A_m+\varepsilon Q_m^*+\varepsilon e^{-\varepsilon P_m}R_me^{\varepsilon P_m}\right)x_{m+1}, \tag{39}$$

where

$$\begin{aligned}
\varepsilon Q_m^*=&-\varepsilon^2P_m\left(-A_mP_m+P_mA_m\right)\\
&+\varepsilon^2\left(-A_mP_m+P_mA_m\right)P_m\\
&-\varepsilon^2P_m\left(A_m+\varepsilon P_mA_m-\varepsilon A_mP_m\right)P_m\\
&-\varepsilon P_m\left(A_m+\varepsilon P_mA_m-\varepsilon A_mP_m\right)B_m\\
&+\left(A_m+\varepsilon P_mA_m-\varepsilon A_mP_m\right)B_m\\
&+\widetilde{B}_m\left(A_m+\varepsilon P_mA_m-\varepsilon A_mP_m\right)e^{\varepsilon P_m}\\
&+e^{-\varepsilon P_m}\left(\varepsilon\dot{P}_me^{\varepsilon P_m}-\frac{d}{dt}e^{\varepsilon P_m}\right).
\end{aligned} \tag{40}$$

System (39) can be written in the following system:

$$\dot{x}_{m+1}=\left(A_{m+1}+\varepsilon Q_{m+1}+\varepsilon e^{-\varepsilon P_m}R_me^{\varepsilon P_m}\right)x_{m+1}, \tag{41}$$

where $A_{m+1} = A_m + \varepsilon\overline{Q}_m^*$, $Q_{m+1} = (Q_m^* - \overline{Q}_m^*)^M$, and $R_{m+1} = (Q_m^* - \overline{Q}_m^*)^{\geq M} + e^{-\varepsilon P_m} R_m e^{\varepsilon P_m}$ are Hamiltonian and analytic quasi-periodic on D_ρ with the frequencies ω.

Now we prove the convergence of the iteration as $m \longrightarrow \infty$.

We first prove (22) holds by mathematical induction. By Lemma 4, it is easy to verify that

$$\begin{aligned} \|A_1 - D\| &= 0 \leq q^* a_1 \varepsilon^{3/2}, \\ \|Q_1\|_\rho &\leq q^* q_1, \\ \|R_1\|_\rho &\leq q^* r_1 e^{-M(\varepsilon)s}, \end{aligned} \tag{42}$$

where $a_1 = q_1 = r_1 = e^{-1}$, $\beta = C(S)$, $q^* = \beta e q$, $s \in (0, \rho]$, $\varepsilon \in (0, \varepsilon^*)$.

Assume that (22) holds at the m-th step. By (22) and (36), we have

$$\|P_m\|_\rho \leq 4\left(L\varepsilon^{1/2}\right)^{-1} \|Q_m\|_\rho \leq 4\left(L\varepsilon^{1/2}\right)^{-1} q^* q_m. \tag{43}$$

Hence

$$\left\|\varepsilon^{1/2} P_m\right\|_\rho \leq \frac{q_m}{2} < \frac{1}{2}, \quad \varepsilon \in (0, \varepsilon^*), \tag{44}$$

where $L = 8q^*$ is a constant. It is easy to see that

$$\|\varepsilon P_m\|_\rho \leq \left\|\varepsilon^{1/2} P_m\right\|_\rho \leq \frac{q_m}{2} < \frac{1}{2}, \quad \varepsilon \in (0, \varepsilon^*). \tag{45}$$

Thus

$$\left\|e^{\pm\varepsilon P_m}\right\|_\rho < 2. \tag{46}$$

From (22), (44), (46), Lemmas 4 and 5, it follows that

$$\|Q_m^*\|_\rho \leq c\varepsilon^{1/2} q^* q_m^2 \varepsilon^{1/2} \leq q^* q_m^2 \varepsilon^{1/2} = q^* q_{m+1} \varepsilon^{1/2}, \tag{47}$$

where c is a positive constant,

$$\begin{aligned} \|A_{m+1} - D\| &\leq \|A_m - D\| + \|\varepsilon Q_m^*\|_\rho \\ &\leq q^* (a_m + q_{m+1}) \varepsilon^{3/2} = q^* a_{m+1} \varepsilon^{3/2}, \\ \|Q_{m+1}\|_\rho &\leq \|Q_m^*\|_\rho \leq q^* q_{m+1} \varepsilon^{1/2} \leq q^* q_{m+1}, \\ \|R_{m+1}\|_{\rho-s} &\leq \left\|(Q_m^*)^{\geq M}\right\|_{\rho-s} + \frac{1 + \|\varepsilon P_m\|}{1 - \|\varepsilon P_m\|} \|R_m\|_{\rho-s} \\ &\leq \left(q_{m+1} + \frac{2 + q_m}{2 - q_m} r_m\right) q^* e^{-M(\varepsilon)s} \\ &= q^* r_{m+1} e^{-M(\varepsilon)s}, \quad s \in (0, \rho]. \end{aligned} \tag{48}$$

By the mathematical induction, then (22) holds.

Below we prove (31) holds. If $k \in \mathbb{Z}^r$ and $0 < |k| < M(\varepsilon) = (d/\varepsilon^{1/2})^{1/\tau}$, from the nonresonant conditions of Theorem 1 and (29), it follows that

$$\begin{aligned} &\left|\langle k, \omega\rangle \sqrt{-1} - \lambda_i^m + \lambda_j^m\right| \\ &\quad \geq \left|\langle k, \omega\rangle \sqrt{-1} - \lambda_i + \lambda_j\right| - \left|\lambda_i^m - \mu_i\right| - \left|\lambda_j^m - \mu_j\right| \\ &\qquad - \left|\mu_i - \lambda_i\right| - \left|\mu_j - \lambda_j\right| \\ &\quad \geq \frac{\alpha}{|k|^\tau} - 2q^* a_m \varepsilon^{3/2} - 2q\beta\varepsilon \\ &\quad \geq \frac{\alpha}{|k|^\tau} - 2q^* a_\infty \varepsilon^{3/2} - 2q\beta\varepsilon \\ &\quad \geq \frac{\alpha}{|k|^\tau} - 2q^* a_\infty \varepsilon^{1/2} - 2q\varepsilon^{1/2} > \left(\frac{\alpha}{d} - 4q^*\right)\varepsilon^{1/2} \\ &\quad = L\varepsilon^{1/2}, \end{aligned} \tag{49}$$

where $d = \alpha/12q^*$ and $L = 8q^*$. So for any $m \geq 1$, (31) holds.

Consequently, the iterative process can be carried out. The composition of all of the changes $e^{\varepsilon P_m}$ is convergent because $\|e^{\varepsilon P_m}\|_\rho \leq 1 + q_m$. That is, there exists an analytic quasi-periodic function $\varphi(t, \varepsilon)$ on D_ρ with the frequencies ω, such that the composition of all of the changes $e^{\varepsilon P_m}$ converges to $\varphi(t, \varepsilon)$ as $m \longrightarrow \infty$.

From (22) and Lemma 5, it follows that

$$\lim_{m \longrightarrow \infty} Q_m = 0. \tag{50}$$

By (22) and (41), we have

$$\|A_{m+1} - A_m\| \leq \varepsilon \left\|\overline{Q}_m^*\right\| \leq \varepsilon \|Q_m^*\|_\rho \leq q^* q_{m+1} \varepsilon^{3/2}. \tag{51}$$

Hence, according to Lemma 5, A_m and R_m are convergent as $m \longrightarrow \infty$. Let

$$\begin{aligned} \lim_{m \longrightarrow \infty} A_m &= A_\infty, \\ \lim_{m \longrightarrow \infty} R_m &= R_\infty. \end{aligned} \tag{52}$$

Then the final equation is

$$\dot{x}_\infty = \left(A_\infty(\varepsilon) + \varepsilon R_\infty(t, \varepsilon)\right) x_\infty, \quad \varepsilon \in (0, \varepsilon^*). \tag{53}$$

By (22) and Lemma 5, we have

$$\|A_\infty(\varepsilon) - D\| \leq q^* a_\infty \varepsilon^{3/2} \leq \frac{e\beta q}{e - 1} \varepsilon^{3/2} \tag{54}$$

and

$$\begin{aligned} \|R_\infty(t, \varepsilon)\|_{\rho-s} &\leq q^* r_\infty e^{-M(\varepsilon)s} \\ &\leq \frac{e^2 \beta q}{e - 1} \exp\left(-\left(\frac{d}{\varepsilon^{1/2}}\right)^{1/\tau} s\right), \\ &\qquad s \in (0, \rho], \end{aligned} \tag{55}$$

where $d = \alpha/12\beta e q$.

Under the change of variables $x_\infty = S^{-1}y$, system (53) is changed into (13) with

$$A^* = SA_\infty S^{-1}, \quad R^* = SR_\infty S^{-1}. \tag{56}$$

Moreover,

$$\begin{aligned}\|A^* - A\| &= \left\|A^* - \left(A + \varepsilon\overline{Q}\right) + \varepsilon\overline{Q}\right\| \\ &\le \left\|SA_\infty S^{-1} - SDS^{-1}\right\| + \|\varepsilon Q\|_\rho \\ &\le \beta\frac{e\beta q}{e-1}\varepsilon^{3/2} + \varepsilon q \le \frac{e(\beta+1)q}{e-1}\varepsilon, \\ &\qquad \varepsilon \in (0, \varepsilon^*). \\ \|R^*\|_{\rho-s} &\le \frac{e^2\beta^2 q}{e-1}\exp\left(-\left(\frac{d}{\varepsilon^{1/2}}\right)^{1/\tau} s\right), \\ &\qquad s \in (0, \rho], \ \varepsilon \in (0, \varepsilon^*),\end{aligned} \tag{57}$$

where $d = \alpha/12e\beta q$.

Thus, under the symplectic transformation $x = \varphi(t, \varepsilon)S^{-1}y = \psi(t,\varepsilon)y$, Hamiltonian system (11) is changed into Hamiltonian system (13). Therefore, Theorem 1 is proved completely.

Data Availability

There is no additional data in the manuscript, because the main result is theoretical proof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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