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Research Article

Strauss's Radial Compactness and Nonlinear Elliptic Equation Involving a Variable Critical Exponent

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We study existence of a nontrivial solution of $-\Delta_p u(x) + u(x)^{p-1} = u(x)^{q(x)-1}$, $u(x) \geq 0$, $x \in \mathbb{R}^N$, $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, under some conditions on $q(x)$, especially, $\liminf_{|x|\longrightarrow\infty} q(x) = p$. Concerning this problem, we firstly consider compactness and noncompactness for the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$. We point out that the decaying speed of $q(x)$ at infinity plays an essential role on the compactness. Secondly, by applying the compactness result, we show the existence of a nontrivial solution of the elliptic equation.

1. Introduction and Main Results

In this article, we consider the following nonlinear elliptic equation:

$$-\Delta_p u + u^{p-1} = u^{q(\cdot)-1}, \quad u \geq 0 \text{ in } \mathbb{R}^N,$$
$$u \in W^{1,p}_{\mathrm{rad}}\left(\mathbb{R}^N\right), \tag{1}$$

for $1 < p < N$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is p−Laplacian and variable exponent $q(x)$ is a measurable function satisfying $q(x) > p$, $\liminf_{|x|\longrightarrow\infty} q(x) = p$. $p(x)$-Laplacian type elliptic equation is one of the problems with variable exponent and this type equation on is studied by many researchers in both bounded domain case and unbounded domain case. In this paper, we refer the paper studying on $\mathbb{R}^N$, in several subjects: multiplicity of solutions (see, e.g., [1, 2]), existence of solutions of equations involving several nonlinearities (see, e.g., [3, 4]), equations under periodic assumptions (see e.g. [5, 6]), and so on. Moreover, existence of solutions of (1) involving variable exponent touching the critical exponent, that is, $\operatorname{ess\,sup}_{\mathbb{R}^N} q(x) = p^* := Np/(N-p)$, is studied by [7, 8].

Concerning the classical Sobolev embedding in $\mathbb{R}^N$, it is well known that the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{p^*}(\mathbb{R}^N)$ is not compact. And the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ is not compact, either. In the viewpoint of the lack of compactness, consider the case where $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$ is natural as another critical case.

However, even for p-Laplace equation there are no results in this case. Thus we study problem (1) at the opening of this article. In this case, unlike the subcritical case, we need to overcome some difficulties to show the existence of a nontrivial solution of (1). We will explain them more precisely after Remark 5. Thus in advance of study of (1), we consider the related embedding to the equation. Namely, we study the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$.

We define the generalized Sobolev spaces $W^{k,p(x)}(\Omega)$ with variable exponents $p(x)$ according to [9]. For a domain $\Omega \subset \mathbb{R}^N$ and a function $p \in L^\infty(\Omega)$ with $p(x) \geq 1$ we set

$$L^{p(x)}(\Omega) = \Big\{u \text{ is a real measurable function on } \Omega \mid \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$
$$W^{k,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \mid D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq k\Big\}. \tag{2}$$

These $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are Banach spaces with the following norms:

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}, \\ \|u\|_{W^{k,p(x)}} = \|u\|_{p(x)} + \sum_{|\alpha|\le k} \left\|D^\alpha u\right\|_{p(x)}. \tag{3}$$

The Sobolev embedding theorem of $W^{k,p(x)}$ and related subjects have been well studied so far; see, e.g., [10–19], and we refer to the book [20]. For example, in bounded domain case, the Sobolev best constant with a variable critical exponent and the existence of extremals were studied in [21]; see also [22] in the Sobolev trace embedding case. One of the results in [18] is the existence of the compact embedding. They consider the situation when $p(x)$ is uniformly continuous on $\overline{\Omega}$ and $1 < \operatorname{ess\,inf}_{\overline{\Omega}} p(x) \le \operatorname{ess\,sup}_{\overline{\Omega}} p(x) < N/k$. Under this situation there exists a compact embedding from $W^{k,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ for $q(x)$ satisfying $p(x) \le q(x)$ a.e. in Ω and $\operatorname{ess\,inf}_{\overline{\Omega}}(p^*(x) - q(x)) > 0$, where $p^*(x) = Np(x)/(N - kp(x))$. On the other hand, for $W^{1,p}(\Omega)$ Kurata and Shioji [17] consider the critical case, that is, $\operatorname{ess\,sup}_{\overline{\Omega}} q(x) = p^*$. They showed that if there exist $x_0 \in \Omega$, $C_0 > 0$, $\eta > 0$ and $0 < \ell < 1$ such that $\operatorname{ess\,sup}_{\Omega\setminus B_\eta(x_0)} q(x) < p^*$ and

$$q(x) \le p^* - \frac{C_0}{\left|\log\left|x - x_0\right|\right|^\ell} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0), \tag{4}$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact. Conversely, if

$$q(x) \ge p^* - \frac{C_0}{\left|\log\left|x - x_0\right|\right|} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0), \tag{5}$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is not compact.

When $\Omega = \mathbb{R}^N$, Strauss [23] and Lions [24] showed that the radial Sobolev space $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ can be embedded to $L^q(\mathbb{R}^N)$ compactly for $q \in (p, p^*)$. In addition, related results are in [25, 26] and so on. In $p(x)$ case, under the same conditions as those of bounded domain case the compact embedding from $W^{1,p(x)}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is obtained for $q(x)$ satisfying $\operatorname{ess\,inf}_{\mathbb{R}^N}(q(x) - p(x)) > 0$ and $\operatorname{ess\,inf}_{\mathbb{R}^N}(p^*(x) - q(x)) > 0$ by [27]. On the other hand, the critical case, that is, $\operatorname{ess\,inf}_{\mathbb{R}^N}(q(x) - p(x)) = 0$ or $\operatorname{ess\,inf}_{\mathbb{R}^N}(p^*(x) - q(x)) = 0$, has not been treated so far even if $p(x) \equiv p$. In this paper, we fix $p(x) \equiv p$. Our first study is to obtain a sufficiently condition of compactness and noncompactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ for variable exponent $q(x)$ which also includes the excluded case in [27]. Based on these results, as the second study we obtain a nontrivial solution of (1) under the compactness conditions with $\liminf_{|x|\longrightarrow\infty} q(x) = p$.

Before introducing main results, we fix several notations. B_R denote an open ball centered 0 with radius R. ω_{N-1} is an area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Throughout this paper we assume that $q(x) \in L^\infty(\mathbb{R}^N)$ and $q(x) \ge 1$ for a.e. $x \in \mathbb{R}^N$. A letter C denotes various positive constant. If u is a radial function in $\mathbb{R}^N$, then we can write as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$ in $\mathbb{R}_+$. For simplicity we write $u(x) = u(|x|)$ with admitting some ambiguity.

Theorem 1 (noncompactness). *If there exist positive constants R and C_0 and an open set Γ in $\mathbb{S}^{N-1}$ such that*

$$q(x) \le p + \frac{C_0}{\left|\log|x|\right|} \quad \text{for } x \in (R, +\infty) \times \Gamma; \tag{6}$$

then the embedding from $W^{1,p}_{rad}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact.

Theorem 2 (compactness). *If there exist positive constants r, R, C_0, C_1, and $k, l \in (0,1)$ such that*

$$q(x) \le p^* - \frac{C_0}{\left|\log|x|\right|^k} \quad \text{for } x \in B_r, \tag{7}$$

$$q(x) \ge p + \frac{C_1}{\left|\log|x|\right|^\ell} \quad \text{for } x \in \mathbb{R}^N \setminus B_R, \tag{8}$$

then the embedding from $W^{1,p}_{rad}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact.

Remark 3. In Theorem 2, we do not need the constraint $p \le q(x) \le p^*$. $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \subset L^{q(x)}(\mathbb{R}^N)$ holds whenever $q(x)$ satisfies $q(x) \le p^*$ in B_r and $q(x) \ge p$ in $\mathbb{R}^N \setminus B_R$.

Theorem 4. *Assume that $q(x)$ satisfies the hypotheses (7) and (8) in Theorem 2 and* $\operatorname{ess\,inf}_{x\in B_R} q(x) > p$. *Then there exists a nontrivial weak solution $u \in W^{1,p}_{rad}(\mathbb{R}^N)$ of (1) in the sense of*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla \phi + u^{p-1}\phi - u^{q(x)-1}\phi\right) dx = 0 \tag{9}$$

for any $\phi \in W^{1,p}_{rad}(\mathbb{R}^N)$.

Remark 5. If $q(x)$ is a radially symmetric function satisfying the hypotheses of Theorem 4, then we can show that the weak solution u obtained in Theorem 4 satisfies $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. Indeed, since u and $q(x)$ are radially symmetric, it follows that, for all $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\int_0^\infty \left(\left|u'(r)\right|^{p-2} u'(r)\phi'(r) + u^{p-1}\phi - u^{q(r)-1}\phi\right) \cdot r^{N-1} dr = 0, \tag{10}$$

where $r = |x|$. If for any $\psi \in C^\infty_c(\mathbb{R}^N)$ we consider the radial function $\Psi(r) = \int_{\omega\in\mathbb{S}^{N-1}} \psi(r\omega) dS_\omega$, then we have

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla \psi + u^{p-1}\psi - u^{q(x)-1}\psi\right) dx = \int_0^\infty \left(\left|u'(r)\right|^{p-2} u'(r)\Psi'(r) + u^{p-1}\Psi - u^{q(r)-1}\Psi\right) \cdot r^{N-1} dr = 0. \tag{11}$$

Therefore we see that u satisfies (9) even for nonradial functions ϕ. Finally, by Corollary of Theorem 2 in [28] we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. And also, by Theorem 2.5.1 in [29] we have $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

We note the difficulties to obtain Theorem 4 caused by the condition $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$. *Ambrosetti-Rabinowitz condition (AR)* is well known in order to obtain a nontrivial weak solution to the following problem by mountain pass method:

$$-\Delta_p u + |u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{12}$$

$$\begin{aligned}&\text{There are } \mu > p \text{ and } M > 0\\ &\text{such that for } |u| \ge M,\\ &0 < \mu F(x,u) \le u f(x,u),\end{aligned} \tag{AR}$$

where $F(x,u) := \int_0^u f(x,s)ds$. Especially, condition (AR) has been used to establish not only the mountain pass structure of the energy functional but also the Palais-Smale condition. A weaker condition has also been considered, for instance, Liu-Wang [30] studied (SQ) which is called *superquadratic condition*.

$$\lim_{|u|\longrightarrow\infty} \frac{F(x,u)}{|u|^p} = \infty \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{SQ}$$

However, assuming that the nonlinear term $u(x)^{q(x)-1}$ in (1) is a special case of the general nonlinear term $f(x,u)$, this does not satisfy even condition (SQ) when $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$. From these facts, it seems to be difficult to confirm whether the energy functional J (see Section 4) corresponding to (1) satisfies the Palais-Smale condition or not. In more detail, while the fact that bounded Palais-Smale sequence has a convergent subsequence is straightforward from Theorem 2, boundedness of all Palais-Smale sequence is nontrivial. Besides that, satisfying the mountain pass structure for J is not trivial since we cannot apply the fibering map method directly.

To overcome these difficulties, in Section 3, we construct a solution of (1) as a limit of mountain pass solutions of some elliptic equations approaching (1) in the sense of energy functional. In Section 4, we show another proof by using the variant of the mountain pass theorem. More precisely, by introducing the condition (C) (see Section 4) defined in [31] or [32] instead of the Palais-Smale condition, we obtain a solution of (1) in a different way from Section 3.

2. Compactness and Noncompactness of the Embedding

We prove Theorems 1 and 2. Before beginning the proof we recall the point-wise estimate and the compactness theorem introduced in [23, 24] ($p = 2$). For the reader's convenience, the proofs are in Appendix.

Proposition 6. *For any $u \in W^{1,p}_{rad}(\mathbb{R}^N)$ we have*

$$|u(x)| \le \left(\frac{p}{\omega_{N-1}}\right)^{1/p} |x|^{-(N-1)/p} \|u\|^{(p-1)/p}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{1/p}_{L^p(\mathbb{R}^N)}. \tag{13}$$

Proposition 7. *The embedding from $W^{1,p}_{rad}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$.*

Proof of Theorem 1. We shall show Theorem 1 in the same way as [17]. Set $r(x) = q(x) - p$ for $x \in \mathbb{R}^N$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a radial function satisfying $\phi \equiv 1$ on $B_{1/2}$ and $\operatorname{supp}\phi \subset B_1$. For $m \in \mathbb{N}$, we define $\phi_m(x) = m^{-N/p}\phi(x/m)$. Then for any $m \in \mathbb{N}$ we obtain

$$\begin{aligned}\|\phi_m\|_{L^p(\mathbb{R}^N)} &= \|\phi\|_{L^p(B_1)},\\ \|\nabla\phi_m\|_{L^p(\mathbb{R}^N)} &= m^{-1}\|\nabla\phi\|_{L^p(B_1)}.\end{aligned} \tag{14}$$

Since $\{\phi_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is reflexive (see, e.g., Proposition 3.20. in [33]), there exist a weakly convergent subsequence $\{\phi_{m_j}\}_{j=1}^\infty$ and $\phi_\infty \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\phi_{m_j} \rightharpoonup \phi_\infty$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \longrightarrow \infty$. By compactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ for $p < r < p^*$, we have $\phi_{m_j} \longrightarrow \phi_\infty$ in $L^r(\mathbb{R}^N)$ and $\phi_{m_j} \longrightarrow \phi_\infty$ a.e. in $\mathbb{R}^N$ which yields that $\phi_\infty \equiv 0$. On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)} dx &= \int_{B_m} m^{-(N/p)(p+r(x))} \left|\phi\left(\frac{x}{m}\right)\right|^{q(x)} dx\\ &= \int_{B_1} m^{-(N/p)r(my)} |\phi(y)|^{q(my)} dy\\ &\ge \int_{B_{1/2}\setminus B_{1/4}} m^{-(N/p)r(my)} dy.\end{aligned} \tag{15}$$

Since Γ is open in $\mathbb{S}^{N-1}$, there exists a smooth subset $D \subset \mathbb{S}^{N-1}$ such that $D \subset \Gamma$. By using the polar coordinates as $y = s\omega$ ($s > 0, \omega \in \mathbb{S}^{N-1}$) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)} dx \ge \int_{1/4}^{1/2} \int_{\omega\in D} m^{-(N/p)r(ms\omega)} s^{N-1} ds\, dS_\omega.\end{aligned} \tag{16}$$

By the assumption (6), we obtain $r(ms\omega) \le C_0 \, |\log ms|^{-1}$ for large m, $s \in (1/4, 1/2)$, and $\omega \in D \subset \Gamma$. Moreover for $s \in (1/4, 1/2)$ and large m, it holds that $\log ms = \log m + \log s \ge$

$(1/2)\log m$ which yields

$$r(ms\omega) \le \frac{2C_0}{\log m}. \tag{17}$$

Therefore we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\,dx \\
&\quad \ge \int_{1/4}^{1/2}\int_{\omega\in D} e^{-(N/p)\log m(2C_0/\log m)} s^{N-1} ds\, dS_\omega \\
&\quad = \mathscr{H}^{N-1}(D)\, e^{-2C_0N/p}\frac{2^{-N}-4^{-N}}{N} > 0
\end{aligned} \tag{18}$$

for large m, where $\mathscr{H}^d$ is the d−dimensional Hausdorff measure. Thus, if we assume that the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact, then we have $\int_{\mathbb{R}^N}|\phi_\infty|^{q(x)}dx > 0$ which contradicts $\phi_\infty \equiv 0$. Hence the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact. □

Proof of Theorem 2 . We assume that $r < R$ without loss of generality. Let $\{u_m\}_{m=1}^\infty$ be a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. We shall show the existence of a strongly convergence subsequence of $\{u_m\}_{m=1}^\infty$ in $L^{q(x)}(\mathbb{R}^N)$. By the reflexivity of $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, there exist a subsequence $\{u_{m_j}\}_{j=1}^\infty$ and $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \longrightarrow \infty$. Especially it also holds that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N)$ as $j \longrightarrow \infty$. And also, by Proposition 7 we have $u_{m_j} \longrightarrow u_0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ and

$$u_{m_j} \longrightarrow u_0 \quad \text{a.e. in } \mathbb{R}^N \text{ as } j \longrightarrow \infty. \tag{19}$$

Furthermore, $\{u_{m_j}|_{B_r}\}_{j=1}^\infty \subset W^{1,p}(B_r)$ is a bounded sequence and the embedding from $W^{1,p}(B_r)$ to $L^{q(x)}(B_r)$ is compact by the assumption (7) (see Remark 2 in [17]). Thus there exist a subsequence of $\{u_{m_j}|_{B_r}\}_{j=1}^\infty$ (we use $\{u_{m_j}|_{B_r}\}_{j=1}^\infty$ again for simplicity) and $v_0 \in L^{q(x)}(B_r)$ such that the following hold true:

$$\begin{aligned}
u_{m_j}\big|_{B_r} &\rightharpoonup v_0 \quad \text{in } W^{1,p}(B_r), \\
u_{m_j}\big|_{B_r} &\longrightarrow v_0 \quad \text{in } L^{q(x)}(B_r), \\
u_{m_j}\big|_{B_r} &\longrightarrow v_0 \quad \text{in } L^{p}(B_r), \\
u_{m_j}\big|_{B_r} &\longrightarrow v_0 \quad \text{a.e. in } B_r
\end{aligned} \tag{20}$$

$$\text{as } j \longrightarrow \infty. \tag{21}$$

By (19) and (20), we can check that $u_0|_{B_r} = v_0$ a.e. in B_r which yields

$$u_{m_j}\big|_{B_r} \longrightarrow u_0\big|_{B_r} \quad \text{in } L^{q(x)}(B_r) \text{ as } j \longrightarrow \infty. \tag{22}$$

In the similar way as above, we also obtain the following:

$$\begin{aligned}
u_{m_j}\big|_{B_K\setminus B_r} &\rightharpoonup u_0\big|_{B_K\setminus B_r} \quad \text{in } W^{1,p}_{\mathrm{rad}}(B_K\setminus B_r), \\
u_{m_j}\big|_{B_K\setminus B_r} &\longrightarrow u_0\big|_{B_K\setminus B_r} \quad \text{in } L^{s}(B_K\setminus B_r), \\
u_{m_j}\big|_{B_K\setminus B_r} &\longrightarrow u_0\big|_{B_K\setminus B_r} \quad \text{a.e. in } B_K\setminus B_r
\end{aligned} \tag{23}$$

for any $K > 0$ and any $s \ge 1$ as $j \longrightarrow \infty$ since the embedding from $W^{1,p}_{\mathrm{rad}}(B_K\setminus B_r)$ to $L^s(B_K\setminus B_r)$ is compact for any K, s.

Set $v_{m_j} := u_{m_j} - u_0$. In order to make good use of (22) and (23) we divide $\int_{\mathbb{R}^N}|v_{m_j}(x)|^{q(x)}dx$ into three terms as follows:

$$\begin{aligned}
\int_{\mathbb{R}^N}\left|v_{m_j}(x)\right|^{q(x)}dx &= \int_{B_r}\left|v_{m_j}(x)\right|^{q(x)}dx \\
&\quad + \int_{B_K\setminus B_r}\left|v_{m_j}(x)\right|^{q(x)}dx \\
&\quad + \int_{\mathbb{R}^N\setminus B_K}\left|v_{m_j}(x)\right|^{q(x)}dx \\
&=: I_1(j) + I_2(j,K) + I_3(j,K),
\end{aligned} \tag{24}$$

where K is sufficiently large.

Firstly, by (22) we have

$$I_1(j) = o(1) \quad \text{as } j \longrightarrow \infty. \tag{25}$$

Next, for $I_2(j,K)$ we have

$$\begin{aligned}
I_2(j,K) &= \int_{B_K\setminus B_r}\left|v_{m_j}(x)\right|^{q(x)}dx \\
&\le \int_{B_K\setminus B_r}\left|v_{m_j}(x)\right|dx \\
&\quad + \int_{B_K\setminus B_r}\left|v_{m_j}(x)\right|^{\|q\|_{L^\infty(\mathbb{R}^N)}}dx.
\end{aligned} \tag{26}$$

Thus, by (23) we obtain

$$I_2(j,K) = o(1) \quad \text{as } j \longrightarrow \infty \text{ for fixed } K > 0. \tag{27}$$

Finally we shall estimate $I_3(j,K)$. Since

$$\begin{aligned}
\left|v_{m_j}(x)\right| &\le \left(\frac{p}{\omega_{N-1}}\right)^{1/p}\left\|v_{m_j}\right\|_{W^{1,p}(\mathbb{R}^N)}|x|^{-(N-1)/p} \\
&\le C|x|^{-(N-1)/p}
\end{aligned} \tag{28}$$

by Proposition 6 and the boundedness of $\{v_{m_j}\}_{j=1}^{\infty}$, we can assume $|v_{m_j}(x)| \leq 1$ for $x \in \mathbb{R}^N \setminus B_K$ with large K. Therefore by assumption (8) we obtain

$$\begin{aligned}
I_3(j,K) &= \int_{\mathbb{R}^N\setminus B_K} \left|v_{m_j}\right|^{q(x)} dx \\
&\leq \int_{\mathbb{R}^N\setminus B_K} \left|v_{m_j}\right|^{p+C_1(\log|x|)^{-\ell}} dx \\
&\leq \sum_{n=2}^{\infty} \int_{B_{K^n}\setminus B_{K^{n-1}}} \left|v_{m_j}\right|^{p+C_1(n\log K)^{-\ell}} dx \\
&\leq \sum_{n=2}^{\infty} \int_{B_{K^n}\setminus B_{K^{n-1}}} \left|v_{m_j}\right|^{p} \left(C|x|^{-(N-1)/p}\right)^{C_1(n\log K)^{-\ell}} dx \\
&\leq C^{C_1(2\log K)^{-\ell}} \left\|v_{m_j}\right\|^{p}_{W^{1,p}(\mathbb{R}^N)} \\
&\cdot \sum_{n=2}^{\infty} K^{-((N-1)/p)(n-1)C_1(n\log K)^{-\ell}} \leq C\sum_{n=2}^{\infty}\delta_1^{(n-1)^{1-\ell}} \\
&= C\sum_{n=1}^{\infty}\delta_1^{n^{1-\ell}},
\end{aligned} \tag{29}$$

where $\delta_1 = \delta_1(K) := K^{-((N-1)/p)C_1(\log K)^{-\ell}} \longrightarrow 0$ as $K \longrightarrow \infty$. Since $\sum_{n=1}^{\infty}\delta_1^{n^{1-\ell}} = \delta_1 + \int_1^{\infty}\delta_1^{x^{1-\ell}}dx < \infty$ for each $\delta_1 \in (0,1)$, we have

$$\sum_{n=1}^{\infty}\delta_1^{n^{1-\ell}} \longrightarrow 0 \quad \text{as } K \longrightarrow \infty. \tag{30}$$

Hence we have

$$I_3(j,K) = o(1) \quad \text{uniformly in } j \text{ as } K \longrightarrow \infty. \tag{31}$$

We go back to (24) and by (25), (27), and (31) we have

$$\lim_{j\longrightarrow\infty}\int_{\mathbb{R}^N}\left|v_{m_j}(x)\right|^{q(x)} dx = 0. \tag{32}$$

As a consequence we obtain $u_{m_j} \longrightarrow u_0$ in $L^{q(x)}(\mathbb{R}^N)$. □

3. Approximation Method: Proof of Theorem 4

In this section, we show Theorem 4 by using Theorem 2. First, we prepare the mountain pass theorem (Theorem 8) introduced in [34, 35] and so on which are based on [36]. Let V be a Banach space and $E \in C^1(V,\mathbb{R})$. We define a Palais-Smale sequence for E as $\{u_m\} \subset V$ satisfying $|E(u_m)| \leq c$ uniformly in m, and $E'(u_m) \longrightarrow 0$ in V^*, where $E'(\cdot)$ is Fréchet derivative and V^* is the dual space of V. We say that E satisfies (P.-S.) condition if any Palais-Smale sequence has a strongly convergent subsequence.

Theorem 8 (see [34, 35]). *Suppose $E \in C^1(V,\mathbb{R})$ satisfies (P.-S.) condition. Assume that*

(i) *E(0)=0*

(ii) *There exist $\rho > 0$, $\alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.*

(iii) *There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.*

Define

$$P = \{p \in C([0,1],V) \mid p(0) = 0,\ p(1) = u_1\}. \tag{33}$$

Then

$$\beta = \inf_{p\in P}\sup_{0\leq t\leq 1} E(p(t)) \tag{34}$$

is a critical value.

Proof of Theorem 4 .

Step 1. We may assume that R in the hypotheses of Theorem 2 is sufficiently large such that $\operatorname{ess\,inf}_{x\in B_R} q(x) \geq p + C_1(\log R)^{-\ell}$. Because, if not, we take a sufficiently large $\widetilde{R}$ which satisfies it instead of R. For $m \in \mathbb{N}$ let $\{R_m\}$ be a sequence such that $R_1 = R$ and $R_m \longrightarrow \infty$ as $m \longrightarrow \infty$. Then we set functions as

$$q_m(x) = \begin{cases} q(x) & \text{if } q(x) \geq p + C_1(\log R_m)^{-\ell}, \\ p + C_1(\log R_m)^{-\ell} & \text{if } q(x) < p + C_1(\log R_m)^{-\ell}. \end{cases} \tag{35}$$

Define a functional J_m from $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ by

$$J_m(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p + |u|^p\right)dx - \int_{\mathbb{R}^N}\frac{1}{q_m(x)}u_+^{q_m(x)}dx. \tag{36}$$

We can check that $J_m \in C^1(W^{1,p}_{\text{rad}}(\mathbb{R}^N),\mathbb{R})$. Indeed, for fixed m and any $u,\phi \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$

$$\begin{aligned}
D_G J_m(u)[\phi] &:= \left.\frac{d}{dt}\right|_{t=0} J_m(u+t\phi) \\
&= \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla\phi + |u|^{p-2}u\phi\,dx \\
&\quad - \int_{\mathbb{R}^N}u_+^{q_m(x)-1}\phi\,dx.
\end{aligned} \tag{37}$$

Then we see that J_m is Gâteaux differentiable in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. By the Vitali convergence theorem, we see that $D_G J_m$ is continuous from $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ to its dual space $(W^{1,p}_{\text{rad}}(\mathbb{R}^N))^*$. Hence $J_m \in C^1(W^{1,p}_{\text{rad}}(\mathbb{R}^N),\mathbb{R})$. Moreover, for each m, J_m satisfies as follows:

(i) J_m satisfies (P.-S.) condition.

(ii) $J_m(0) = 0$,

(iii) There exist positive constants α, ρ such that $J_m(u) \geq \alpha$ for any $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \rho$,

(iv) There exists $v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$, $J_m(v) < \alpha$.

By Theorem 8 there exists a critical point $u_m \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ of J_m such that

$$J_m(u_m) = \beta_m, \tag{38}$$

where β_m is defined in the same way as β in Theorem 8. Thus u_m is a nontrivial weak solution of

$$-\Delta_p w + |w|^{p-2} w = w_+^{q_m(x)-1} \quad \text{in } \mathbb{R}^N. \tag{39}$$

We can also see that $u_m \geq 0$ by multiplying both sides of (39) by $(u_m)_-$.

Proposition 9. *$\{u_m\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N)$.*

We will prove this proposition at the end of this section.

Step 2. Since $\{u_m\}$ is a bounded sequence, there exists $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Put

$$\begin{aligned} G_m &= \left\langle |\nabla u_m|^{p-2} \nabla u_m - |\nabla u_0|^{p-2} \nabla u_0, \nabla u_m - \nabla u_0 \right\rangle_{\mathbb{R}^N} \\ &\quad + \left(u_m^{p-1} - u_0^{p-1}\right)(u_m - u_0). \end{aligned} \tag{40}$$

Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} G_m\, dx &= \int_{\mathbb{R}^N} \left(|\nabla u_m|^p + u_m^p\right) dx \\ &\quad - \int_{\mathbb{R}^N} \left(|\nabla u_m|^{p-2} \nabla u_m \nabla u_0 + u_m^{p-1} u_0\right) dx + h_m, \end{aligned} \tag{41}$$

where $h_m = \int_{\mathbb{R}^N} [|\nabla u_0|^{p-2} \nabla u_0 (\nabla u_0 - \nabla u_m) + u_0^{p-1}(u_0 - u_m)] dx = o(1)$ as $m \longrightarrow \infty$. Moreover, from (56) and (57) in the proof of Proposition 9 it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_m|^p + u_m^p\right) dx \\ &\quad - \int_{\mathbb{R}^N} \left(|\nabla u_m|^{p-2} \nabla u_m \nabla u_0 + u_m^{p-1} u_0\right) dx \\ &= \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1} \left((u_m)_+ - u_0\right) dx \\ &\leq 2 \left\| u_m^{q_m(x)-1} \right\|_{q(x)/(q(x)-1)} \|u_m - u_0\|_{q(x)} \\ &= 2 \|u_m\|_{q(x)} \|u_m - u_0\|_{q(x)} \end{aligned} \tag{42}$$

by the generalized Hölder inequality (see, e.g., [9] Theorem 2.1). By the boundedness of $\{u_m\}$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and Theorem 2 we have $\|u_m\|_{q(x)} \|u_m - u_0\|_{q(x)} = o(1)$ as $m \longrightarrow \infty$. Hence

$$\int_{\mathbb{R}^N} G_m\, dx = o(1) \tag{43}$$

as $m \longrightarrow \infty$. Recall that, for $p \geq 1$, $a, b \in \mathbb{R}^d$, we have

$$\begin{aligned} &\left\langle |b|^{p-2} b - |a|^{p-2} a,\ b - a \right\rangle \\ &\geq \begin{cases} 2^{2-p} |b-a|^p & \text{if } p \geq 2, \\ (p-1)|b-a|^2 \left(1 + |a|^2 + |b|^2\right)^{(p-2)/2} & \text{if } 1 \leq p \leq 2. \end{cases} \end{aligned} \tag{44}$$

From this inequality and (43) it follows that

$$\int_{\mathbb{R}^N} \left(|\nabla u_m - \nabla u_0|^p + |u_m - u_0|^p\right) dx = o(1) \tag{45}$$

which is equivalent to $u_m \longrightarrow u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$. Thus u_0 satisfies

$$-\Delta_p u_0 + u_0^{p-1} = u_0^{q(x)-1}, \quad u_0 \geq 0 \text{ in } \mathbb{R}^N. \tag{46}$$

Step 3. Finally, we have to show $u_0 \not\equiv 0$. From the boundedness of $\{u_m\}$ and Proposition 6, we see that $u_m \leq 1$ in $\mathbb{R}^N \setminus B_L$ for large L. Therefore we have

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\nabla u_m|^p + u_m^p\right) dx &= \int_{\mathbb{R}^N} u_m^{q_m(x)} dx \\ &\leq \int_{\mathbb{R}^N} u_m^p dx + \int_{B_r} u_m^{p^*} dx + \int_{B_L \setminus B_r} u_m^{\|q\|_{L^\infty(\mathbb{R}^N)}} dx. \end{aligned} \tag{47}$$

By the Sobolev inequality it follows that

$$\begin{aligned} \int_{B_r} u_m^{p^*} dx &\leq \int_{\mathbb{R}^N} u_m^{p^*} dx \\ &\leq S^{-p^*/p} \left(\int_{\mathbb{R}^N} |\nabla u_m|^p dx\right)^{p^*/p}. \end{aligned} \tag{48}$$

Moreover, we have

$$\begin{aligned} &\int_{B_L \setminus B_r} u_m^{\|q\|_{L^\infty(\mathbb{R}^N)}} dx \\ &\leq C \left[\int_{B_L \setminus B_r} \left(|\nabla u_m|^p + |u_m|^p\right) dx\right]^{\|q\|_{L^\infty(\mathbb{R}^N)}/p} \\ &\leq C \left[\int_{B_L \setminus B_r} |\nabla u_m|^p dx + \left(\int_{B_L \setminus B_r} |u_m|^{p^*} dx\right)^{p/p^*} \right. \\ &\quad \left. \cdot |B_L \setminus B_r|^{1-p/p^*}\right]^{\|q\|_{L^\infty(\mathbb{R}^N)}/p} \\ &\leq C \left(\int_{\mathbb{R}^N} |\nabla u_m|^p dx\right)^{\|q\|_{L^\infty(\mathbb{R}^N)}/p}. \end{aligned} \tag{49}$$

Put $q_* := \min\{p^*, \|q\|_{L^\infty(\mathbb{R}^N)}\}$. From (47), (48), and (49), we obtain

$$C \leq \left(\int_{\mathbb{R}^N} |\nabla u_m|^p dx\right)^{(q_*-p)/p}, \tag{50}$$

where we used that $u_m \not\equiv 0$. By Theorem 2 we have

$$\begin{aligned} C &\leq \lim_{m\to\infty} \int_{\mathbb{R}^N} |\nabla u_m|^p \, dx \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^N} \left(-u_m^p + u_m^{q_m(x)}\right) dx \leq \int_{\mathbb{R}^N} u_0^{q(x)} dx. \end{aligned} \tag{51}$$

Consequently we have $u_0 \not\equiv 0$. □

Proof of Proposition 9. We take a smooth radial function $\widehat{u} > 0$ on $\mathbb{R}^N$. Since

$$\begin{aligned} J_m(K\widehat{u}) &\leq \frac{K^p}{p} \int_{\mathbb{R}^N} \left(|\nabla \widehat{u}|^p + |\widehat{u}|^p\right) dx \\ &\quad - \int_{B_R} \frac{K^{q(x)}}{q(x)} \widehat{u}_+^{q(x)} dx \\ &\leq \frac{K^p}{p} \int_{\mathbb{R}^N} \left(|\nabla \widehat{u}|^p + |\widehat{u}|^p\right) dx \\ &\quad - \frac{K^{p+C_1(\log R)^\ell}}{\operatorname{ess\,sup}_{B_R} q(x)} \int_{B_R} \widehat{u}_+^{q(x)} dx \longrightarrow -\infty \end{aligned} \tag{52}$$

as $K \longrightarrow +\infty$, there exists $\widehat{K} > 0$ independent of m such that $J_m(\widehat{K}\widehat{u}) < 0$. If we set $\widehat{p}(t) = t\widehat{K}\widehat{u}$ for $t \in [0,1]$, then we see that

$$\begin{aligned} \widehat{p} \in \widehat{P} = \{ p \in C\left([0,1], W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\right) \mid p(0) \\ = 0,\ p(1) = \widehat{K}\widehat{u} \}. \end{aligned} \tag{53}$$

Moreover, we have

$$\begin{aligned} \beta_m &= \inf_{p\in\widehat{P}} \max_{0\leq t\leq 1} J_m(p(t)) \leq \max_{0\leq t\leq 1} J_m(\widehat{p}(t)) \\ &= \max_{0\leq t\leq \widehat{K}} \left[\frac{t^p}{p} \int_{\mathbb{R}^N} \left(|\nabla \widehat{u}|^p + |\widehat{u}|^p\right) dx \right. \\ &\quad \left. - \int_{B_R} \frac{t^{q(x)}}{q(x)} \widehat{u}_+^{q(x)} dx \right] \leq C. \end{aligned} \tag{54}$$

On the other hand, since u_m is a critical point of J_m at β_m we have

$$\begin{aligned} \beta_m &= \frac{1}{p} \int_{\mathbb{R}^N} \left(|\nabla u_m|^p + |u_m|^p\right) dx \\ &\quad - \int_{\mathbb{R}^N} \frac{1}{q_m(x)} (u_m)_+^{q_m(x)} dx \end{aligned} \tag{55}$$

and, for any $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_m|^{p-2} \nabla u_m \nabla\phi + |u_m|^{p-2} u_m \phi\right) dx \\ &\quad - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1} \phi \, dx = 0. \end{aligned} \tag{56}$$

In particular,

$$\int_{\mathbb{R}^N} \left(|\nabla u_m|^p + |u_m|^p\right) dx - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)} dx = 0. \tag{57}$$

From (54), (55), and (57), it follows that

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)}\right) (u_m)_+^{q_m(x)} dx \leq C. \tag{58}$$

Furthermore, by $q(x) \leq q_m(x)$ we have

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)}\right) (u_m)_+^{q_m(x)} dx \leq C. \tag{59}$$

Thus for any $L > 0$ there exists a positive constant $C(L)$ such that

$$\int_{B_L} (u_m)_+^{q_m(x)} dx \leq C(L). \tag{60}$$

Here, we take a constant $R_0 > R$ sufficiently large (this R_0 will be chosen again later) and we have

$$\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} \leq C(R_0) + \int_{\mathbb{R}^N\setminus B_{R_0}} (u_m)_+^{q_m(x)} dx \tag{61}$$

by (57) and (60). Set $\delta = C_1(\log R_0)^{-\ell}$ and $A_{n,m} := \{x \in B_{R_0^n} \setminus B_{R_0^{n-1}} \mid q_m(x) \leq p + \delta\}$. Then we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_{R_0}} (u_m)_+^{q_m(x)} dx \\ &\quad \leq \int_{\{x\in\mathbb{R}^N \mid q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx \\ &\qquad + \int_{\{x\in\mathbb{R}^N\setminus B_{R_0} \mid q_m(x) \leq p+\delta\}} (u_m)_+^{q_m(x)} dx \\ &\quad \leq \int_{\{x\in\mathbb{R}^N \mid q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx \\ &\qquad + \sum_{n=2}^{\infty} \int_{A_{n,m}} (u_m)_+^{p+C_1(n\log R_0)^{-\ell}} dx \\ &\qquad + \sum_{n=2}^{\infty} \int_{A_{n,m}} (u_m)_+^{p+\delta} dx =: L_1 + L_2 + L_3, \end{aligned} \tag{62}$$

where the third inequality comes from the assumption (8). We shall estimate L_1, L_2, and L_3. For L_1, by (59) we have

$$\begin{aligned} &L_1 \\ &\leq \left(\frac{1}{p} - \frac{1}{p+\delta}\right)^{-1} \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)}\right) (u_m)_+^{q_m(x)} dx \\ &= C. \end{aligned} \tag{63}$$

In order to estimate L_2 and L_3, we prepare an estimate of $\|u_m\|_{L^p(A_{n,m})}$. For each $n, m \in \mathbb{N}$ we have

$$\int_{A_{n,m}} u_m^p \, dx \leq 2 \|u_m\|^p_{L^{q_m(x)}(A_{n,m})} \|1\|_{L^{r_m(x)}(A_{n,m})} \tag{64}$$

by the generalized Hölder inequality, where $r_m(x) := q_m(x)/(q_m(x)-p)$. Now we assume $\|u_m\|_{L^{q_m(x)}(A_{n,m})} > 1$ and $\|1\|_{L^{r_m(x)}(A_{n,m})} > 1$ (if not, the proof is much simpler). By Proposition 2.2. in [27] we have

$$\begin{aligned} \|u_m\|_{L^{q_m(x)}(A_{n,m})} &\le \left(\int_{A_{n,m}} u_m^{q_m(x)} dx\right)^{(\text{ess.inf}_{x\in A_{n,m}} q_m(x))^{-1}} \\ &\le \left(\int_{A_{n,m}} u_m^{q_m(x)} dx\right)^{1/(p+(n\log R_0)^{-\ell})}. \end{aligned} \tag{65}$$

Since

$$\begin{aligned} \int_{A_{n,m}} u_m^{q_m(x)} dx &\le \left(\frac{1}{p} - \frac{1}{p+(n\log R_0)^{-\ell}}\right)^{-1} \\ &\quad \cdot \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)}\right) u_m^{q_m(x)} dx \le C(n\log R_0)^{\ell}, \end{aligned} \tag{66}$$

we obtain

$$\|u_m\|_{L^{q_m(x)}(A_{n,m})} \le C(n\log R_0)^{\ell/(p+(n\log R_0)^{-\ell})}. \tag{67}$$

In the same way as above, we have

$$\begin{aligned} \|1\|_{L^{r_m(x)}(A_{n,m})} &\le \left(\int_{A_{n,m}} dx\right)^{(\text{ess.inf}_{x\in A_{n,m}} r_m(x))^{-1}} \\ &\le |A_{n,m}|^{1/(1+pC_1^{-1}(\log R_0)^{\ell})} \\ &\le CR_0^{n/(1+pC_1^{-1}(\log R_0)^{\ell})}, \end{aligned} \tag{68}$$

where the second inequality comes from

$$\begin{aligned} \operatorname*{essinf}_{x\in A_{n,m}} r_m(x) &= 1 + \frac{p}{\text{esssup}_{x\in A_{n,m}} q_m(x) - p} \ge 1 + \frac{p}{\delta} \\ &= 1 + pC_1^{-1}(\log R_0)^{\ell}. \end{aligned} \tag{69}$$

From (67) and (68) we obtain

$$\begin{aligned} &\int_{A_{n,m}} u_m^p dx \\ &\quad \le CR_0^{n/(1+pC_1^{-1}(\log R_0)^{\ell})} (n\log R_0)^{p\ell/(p+(n\log R_0)^{-\ell})}. \end{aligned} \tag{70}$$

For L_2, by using (70) and Proposition 6, we have

$$\begin{aligned} L_2 &\le C\sum_{n=2}^{\infty} \|u_m\|_{W^{1,p}(\mathbb{R}^N)}^{C_1(n\log R_0)^{-\ell}} R_0^{(-(N-1)/p)(n-1)C_1(n\log R_0)^{-\ell}} \int_{A_{n,m}} u_m^p dx \\ &\le C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} R_0^{-((N-1)C_1/2^{\ell}p)(n-1)^{1-\ell}(\log R_0)^{-\ell}} \left(\int_{A_{n,m}} u_m^p dx\right)^{C_1(n\log R_0)^{-\ell}/p} \\ &\le C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} R_0^{-((N-1)C_1/2^{\ell}p)(n-1)^{1-\ell}(\log R_0)^{-\ell}+C_1 n^{1-\ell}(\log R_0)^{-\ell}/(p+p^2C_1^{-1}(\log R_0)^{-\ell})} (n\log R_0)^{(\ell C_1/(p+(n\log R_0)^{-\ell}))(n\log R_0)^{-\ell}} \\ &= C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} \delta_1(n,R_0)^{(n-1)^{1-\ell}} \delta_2(n,R_0). \end{aligned} \tag{71}$$

Since

$$\begin{aligned} \delta_2(n,R_0) = (n\log R_0)^{(\ell C_1/(p+(n\log R_0)^{-\ell}))(n\log R_0)^{-\ell}} &\longrightarrow 1 \\ \text{as } n \longrightarrow \infty \text{ or } R_0 &\longrightarrow \infty, \end{aligned} \tag{72}$$

there exists a positive constant $\widetilde{C}$ which is independent of n and R_0 such that

$$\delta_2(n,R_0) \le \widetilde{C}. \tag{73}$$

On the other hand, for large R_0 we obtain

$$\begin{aligned} &\delta_1(n,R_0) \\ &\quad = R_0^{-(C_1/p)(\log R_0)^{-\ell}[(N-1)/2^{\ell} - (1/(1+pC_1^{-1}(\log R_0)^{\ell}))(n/(n-1))^{1-\ell}]} \\ &\quad \le R_0^{-(C_1(N-1)/2^{\ell+1}p)(\log R_0)^{-\ell}} \end{aligned} \tag{74}$$

which yields

$$\delta_1 = \delta_1(n,R_0) \longrightarrow 0 \quad \text{uniformly in } n \text{ as } R_0 \longrightarrow \infty. \tag{75}$$

From (73) and (75) we have

$$\begin{aligned} L_2 \le C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} \delta_1^{(n-1)^{1-\ell}} &= o(1)\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \\ \text{as } R_0 &\longrightarrow \infty \end{aligned} \tag{76}$$

in the same way as the proof of Theorem 2. Thus for sufficiently large R_0 we have

$$L_2 \le \frac{1}{3}\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p. \tag{77}$$

In the same way as L_2, we obtain the estimate of L_3 for large R_0 as follows:

$$
\begin{aligned}
L_3 &\leq C \|u_m\|^{\delta}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{(-(N-1)/p)(n-1)\delta} \int_{A_{n,m}} u_m^p dx \\
&\leq C \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \\
&\quad \cdot \sum_{n=2}^{\infty} R_0^{-((N-1)/p)C_1(n-1)(\log R_0)^{-\ell}} \left(\int_{A_{n,m}} u_m^p dx \right)^{(C_1/p)(\log R_0)^{-\ell}} \\
&\leq C \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \\
&\quad \cdot \sum_{n=2}^{\infty} R_0^{-((N-1)/p)C_1(n-1)(\log R_0)^{-\ell} + (n/(1+pC_1^{-1}(\log R_0)^{-\ell}))(C_1/p)(\log R_0)^{-\ell}} (n \\
&\quad \cdot \log R_0)^{\ell C_1/(p(\log R_0)^{-\ell}+n^{-\ell})} \leq C \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \\
&\quad \cdot \sum_{n=2}^{\infty} R_0^{-((N-1)/2p)C_1(n-1)(\log R_0)^{-\ell}} (n \log R_0)^{\ell C_1/(p(\log R_0)^{-\ell}+n^{-\ell})} \\
&\leq C \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{-((N-1)/4p)C_1(n-1)(\log R_0)^{-\ell}},
\end{aligned}
\tag{78}
$$

where the last inequality comes from

$$
\begin{aligned}
&(n \log R_0)^{\ell C_1/(p(\log R_0)^{\ell}+n^{-\ell})} \\
&\quad = o\left(R_0^{((N-1)/4p)C_1(n-1)(\log R_0)^{-\ell}} \right)
\end{aligned}
\tag{79}
$$

$$\text{as } n \longrightarrow \infty \text{ or } R_0 \longrightarrow \infty.$$

Therefore for sufficiently large R_0 we have

$$L_3 \leq \frac{1}{3} \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}. \tag{80}$$

From (61), (63), (77), and (80) we have

$$\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \leq C + \frac{2}{3} \|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}. \tag{81}$$

As a consequence $\{u_m\}$ is bounded. □

4. Mountain Pass Theorem under the Condition (C): Proof of Theorem 4

In this section, we show Theorem 4 by a different method from Section 3.

Cerami [31] and Bartolo-Benci-Fortunato [32] have proposed a variant of (P.-S.) condition. In this paper, we use the condition (C) introduced by [31, 32] and the mountain pass theorem under the condition (C) (Theorem 11). Let V be a real Banach space and $E \in C^1(V, \mathbb{R})$. First, we define the condition (C) based on [31, 32].

Definition 10 ([31, 32] Definition 1.1.). We say that E satisfies the condition (C) in (c_1, c_2), $(-\infty \leq c_1 < c_2 \leq +\infty)$, if

(i) every bounded sequence $\{u_k\} \subset E^{-1}((c_1, c_2))$, for which $\{E(u_k)\}$ is bounded and $E'(u_k) \longrightarrow 0$, possesses a convergent subsequence, and

(ii) for any $c \in (c_1, c_2)$ there exist $\sigma, \rho, \alpha > 0$ such that $[c-\sigma, c+\sigma] \subset (c_1, c_2)$ and for any $u \in E^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \geq \rho$, $\|E'(u)\|_* \|u\| \geq \alpha$.

Theorem 11 (mountain pass theorem under the condition (C)). *Let E satisfy the condition (C) in $(0, +\infty)$. Assume that*

(i) *E(0)=0*

(ii) *There exist $\rho > 0$, $\alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.*

(iii) *There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.*

Define

$$P = \{p \in C([0,1], V) \mid p(0) = 0, \ p(1) = u_1\}. \tag{82}$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t)) \geq \alpha \tag{83}$$

is a critical value.

For $c \in \mathbb{R}$, we set

$$
\begin{aligned}
E_c &= \{u \in V \mid E(u) < c\}, \\
K_c &= \{u \in V \mid E'(u) = 0, \ E(u) = c\}.
\end{aligned}
\tag{84}
$$

Note that Theorem 11 can be shown in the same way as the proof of Theorem 6.1 in p.109 in [35] by substituting the following deformation theorem under the condition (C) for Theorem 3.4 in p.83 in [35].

Theorem 12 ([32] Theorem 1.3.). *Let E satisfy the condition (C) in (c_1, c_2). If $\beta \in (c_1, c_2)$ and N is any neighborhood of K_β, there exist a bounded homeomorphism η of V onto V and constants $\bar{\varepsilon} > \varepsilon > 0$ such that $[\beta - \bar{\varepsilon}, \beta + \bar{\varepsilon}] \subset (c_1, c_2)$, satisfying the following properties:*

(I) $\eta\ (E_{\beta+\varepsilon} \setminus N) \subset E_{\beta-\varepsilon}$

(II) $\eta\ (E_{\beta+\varepsilon}) \subset E_{\beta-\varepsilon}$ *if* $K_\beta = \emptyset$

(III) $\eta\ (u) = u$ *if* $|E(u) - \beta| \geq \bar{\varepsilon}$.

We set an energy functional from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ as

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p \right) dx - \int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx. \tag{85}$$

We can check that $J \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$ in the same way as the proof of Theorem 4.

Proposition 13. *Assume that $q(x)$ satisfies the hypotheses (7) and (8) in Theorem 2 and* $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. *Then J satisfies the condition (C) on $\mathbb{R}$.*

Proof. We take $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ arbitrary. First, we shall show that J satisfies (i) in Definition 10. Let $\{u_m\} \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ be a bounded sequence satisfying $J(u_m) \in (c_1, c_2)$

and $\|J'(u_m)\|_* \longrightarrow 0$ as $m \longrightarrow +\infty$. Then the following holds true for any $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} \left(|\nabla u_m|^{p-2} \nabla u_m \nabla \phi + |u_m|^{p-2} u_m \phi \right) dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)-1} \phi \, dx = o(1). \quad (86)$$

In particular, since $\{u_m\}$ is bounded it follows that

$$\int_{\mathbb{R}^N} \left(|\nabla u_m|^p + |u_m|^p \right) dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)} dx = o(1). \quad (87)$$

Likewise since $\{u_m\}$ is bounded, there exists a subsequence written as $\{u_m\}$ for simplicity and $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}(\mathbb{R}^N)$. Put

$$G_m = \left\langle |\nabla u_m|^{p-2} \nabla u_m - |\nabla u_0|^{p-2} \nabla u_0, \nabla u_m - \nabla u_0 \right\rangle_{\mathbb{R}^N} + \left(u_m^{p-1} - u_0^{p-1} \right) (u_m - u_0) \quad (88)$$

as in Section 3. In the same way as Step 2 in the proof of Theorem 4 in Section 3 by substituting (86) and (87) for (56) and (57), respectively, we have

$$\int_{\mathbb{R}^N} G_m dx = o(1) \quad (89)$$

as $m \longrightarrow \infty$ by Theorem 2. Recalling that

$$\left\langle |b|^{p-2} b - |a|^{p-2} a, b - a \right\rangle \geq \begin{cases} 2^{2-p} |b-a|^p & \text{if } p \geq 2, \\ (p-1) |b-a|^2 \left(1 + |a|^2 + |b|^2 \right)^{(p-2)/2} & \text{if } 1 \leq p \leq 2, \end{cases} \quad (90)$$

consequently we have

$$\lim_{m \longrightarrow \infty} \int_{\mathbb{R}^N} \left(|\nabla (u_m - u_0)|^p + |u_m - u_0|^p \right) dx \leq C \lim_{m \longrightarrow \infty} \int_{\mathbb{R}^N} G_m dx = 0. \quad (91)$$

This implies that $u_m \longrightarrow u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Next, we shall show (ii). For any $c \in (c_1, c_2)$, we take some σ with $[c - \sigma, c + \sigma] \subset (c_1, c_2)$. We will choose suitable $\rho > 0$ again later. By deriving a contradiction, we show that there exists $\alpha > 0$ such that for any $u \in J^{-1}([c - \sigma, c + \sigma])$ with $\|u\| \geq \rho$, $\|J'(u)\|_* \|u\| \geq \alpha$. We assume that there exists $\{u_m\} \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_m \in J^{-1}([c - \sigma, c + \sigma])$ with $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$ and $\|J'(u_m)\|_* \|u_m\|_{W^{1,p}(\mathbb{R}^N)} =: \alpha_m \longrightarrow 0$ as $m \longrightarrow +\infty$. Since $J'(u_m) u_m \longrightarrow 0$ as $m \longrightarrow +\infty$, we have

$$\left| \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (u_m)_+^{q(x)} dx \right| \leq \alpha_m \quad (92)$$

which yields

$$c + \sigma \geq J(u_m) \geq \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)} \right) (u_m)_+^{q(x)} dx - \alpha_m. \quad (93)$$

Moreover, in the same way as the proof of Proposition 9, for large m we have

$$\int_{A_n} u_m^p dx \leq C R_0^{n/(1+pC_1^{-1}(\log R_0)^{\ell})} (n \log R_0)^{p\ell/(p+(n\log R_0)^{-\ell})}, \quad (94)$$

where $A_n := \{x \in B_{R_0^n} \setminus B_{R_0^{n-1}} \mid q(x) \leq p + \delta\}$ for $n \geq 2$ and R_0 is the same as the proof of Proposition 9. By substituting (93) and (94) for (59) and (70), we obtain the following estimates:

$$\begin{aligned} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \alpha_m &\leq \int_{B_{R_0}} (u_m)_+^{q(x)} dx + \int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q(x)} dx \\ &\leq C(R_0)(c + \sigma + \alpha_m) + \frac{2}{3} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)}, \end{aligned} \quad (95)$$

where $C(R_0)$ is a positive constant independent of ρ. Therefore we have

$$\begin{aligned} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} &\leq 3 \{\alpha_m + C(R_0)(c + \sigma + \alpha_m)\} \\ &\leq 3 \{1 + C(R_0)(c_2 + 1)\} \end{aligned} \quad (96)$$

for large m. If we choose sufficiently large ρ satisfying $\rho > 3^{1/p}\{1 + C(R_0)(c_2 + 1)\}^{1/p}$, then we see that (96) contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$.

The proof of Proposition 13 is now complete. □

Proposition 14. *Assume that $q(x)$ satisfies the hypotheses (7) and (8) in Theorem 2 and* $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. *Then J has the mountain pass geometry, that is, J satisfies (i), (ii), and (iii) in Theorem 11.*

Proof. (i) is obvious. We prove (ii). Let S be the best constant of the Sobolev inequality: $S\|v\|^p_{L^{p^*}(\mathbb{R}^N)} \leq \|\nabla v\|^p_{L^p(\mathbb{R}^N)}$ for $v \in C_c^\infty(\mathbb{R}^N)$. Set $q^* = \max\{p^*, p^2, \|q\|_{L^\infty(\mathbb{R}^N)}\}$. Note that $q^* \geq p^* > pN/(N-1)$. For $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx &\leq \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx + \frac{1}{p} \left[\int_{B_r} |u|^{p^*} dx + \int_{\mathbb{R}^N \setminus B_r} |u|^{q^*} dx \right] \leq \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \\ &+ \frac{1}{p} \left[\left(S^{-1} \int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{p^*/p} + \|u\|^{q^*((p-1)/p)}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{q^*/p}_{L^p(\mathbb{R}^N)} K(r) \right] \leq \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \\ &+ \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx \left[S^{-p^*/p} \gamma^{p^*-p} + K(r) \gamma^{q^*-p} \right], \end{aligned} \quad (97)$$

where $K(r) = (p/\omega_{N-1})^{q^*/p} \int_{\mathbb{R}^N \setminus B_r} |x|^{-q^*(N-1)/p} dx < \infty$ and the second inequality comes from Proposition 6. From this if γ is sufficiently small, we have

$$J(u) \geq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx \left[1 - S^{-p^*/p} \gamma^{p^*-p} - K(r) \gamma^{q^*-p} \right] > 0. \tag{98}$$

For $\{u_m\} \subset W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ and γ satisfying $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$ and (98), we assume that $J(u_m) \longrightarrow 0$ and derive a contradiction. From (98) it follows that $\int_{\mathbb{R}^N} |\nabla u_m|^p dx \longrightarrow 0$. In addition, for sufficiently large R we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{q(x)} (u_m)_+^{q(x)} \, dx &\leq \frac{1}{p} \left(\int_{B_r} |u_m|^{q(x)} \, dx + \int_{B_R \setminus B_r} |u_m|^{q(x)} \, dx + \int_{\mathbb{R}^N \setminus B_R} |u_m|^{q(x)} \, dx \right) \\ &\leq \frac{1}{p} \left[\int_{B_r} |u_m|^{p^*} \, dx + \int_{B_R} |u_m|^p \, dx + \int_{\mathbb{R}^N \setminus B_r} |u_m|^{q^*} \, dx + \int_{\mathbb{R}^N \setminus B_R} |u_m|^{p + C_1 (\log|x|)^{-\ell}} \, dx \right] \\ &= \frac{1}{p} (H_1 + H_2 + H_3 + H_4). \end{aligned} \tag{99}$$

By using the estimates in the calculation of $\int_{\mathbb{R}^N} (u)_+^{q(x)}/q(x) dx$ to show (98) we have $H_1 = o(1)$ and $H_3 = o(1)$ as $m \longrightarrow \infty$. For H_2 we have

$$H_2 \leq |B_R|^{1-p/p^*} S^{-1} \int_{\mathbb{R}^N} |\nabla u_m|^p \, dx = o(1). \tag{100}$$

We can show that H_4 is bounded uniformly for m and $H_4 \longrightarrow 0$ as $R \longrightarrow \infty$ in the same way as the estimate of $I_3(j, K)$ in the proof of Theorem 2. Therefore

$$\int_{\mathbb{R}^N} \frac{1}{q(x)} |u_m|^{q(x)} \, dx \longrightarrow 0 \tag{101}$$

as $m \longrightarrow \infty$, which implies that $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \longrightarrow 0$ since $J(u_m) \longrightarrow 0$ as $m \longrightarrow \infty$. This contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$.

Finally, we prove (iii). We take a smooth radial function v such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, $v > 0$ in B_R, where R is in the hypothesis (8). Recall that $\underline{q} := \operatorname{ess\,inf}_{x \in B_R} q(x) > p$. By taking sufficiently large t we have

$$\begin{aligned} J(tv) &= \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) \, dx - \int_{\mathbb{R}^N} \frac{t^{q(x)}}{q(x)} v_+^{q(x)} dx \\ &\leq \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) \, dx - t^{\underline{q}} \int_{B_R} \frac{1}{q(x)} v_+^{q(x)} dx < 0. \end{aligned} \tag{102}$$

Since $\|tv\|_{W^{1,p}(\mathbb{R}^N)} > \gamma$ we prove (iii). □

Proof of Theorem 4 . From Propositions 13 and 14 and Theorem 11, we can show the existence of a nontrivial critical point $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ which is a weak solution to $-\Delta_p u + |u|^{p-2} u = u_+^{q(x)-1}$ in $\mathbb{R}^N$. Then we also see that $u \geq 0$ in $\mathbb{R}^N$. □

Appendix

In this section we show Propositions 6 and 7.

Proof of Proposition 6 . It is sufficient to show that (13) holds for $f \in C_c^\infty(\mathbb{R}^N)$ with radially symmetric. We have

$$r^{N-1} |f(r)|^p = -\int_r^\infty \frac{d}{ds} \left(s^{N-1} |f(s)|^p \right) ds. \tag{A.1}$$

By direct calculation we have

$$\left(s^{N-1} |f(s)|^p \right)' = (N-1) s^{N-2} |f(s)|^p + p s^{N-1} |f(s)|^{p-2} f(s) f(s)'. \tag{A.2}$$

Thus it follows that

$$\begin{aligned} r^{N-1} |f(r)|^p &= -(N-1) \int_r^\infty s^{N-2} |f(s)|^p \, ds - p \int_r^\infty s^{N-1} |f(s)|^{p-2} f(s) f(s)' \, ds \\ &\leq p \int_r^\infty s^{N-1} |f(s)|^{p-1} |f(s)'| \, ds \\ &\leq \frac{p}{\omega_{N-1}} \|f\|_{L^p(\mathbb{R}^N)}^{p-1} \|\nabla f\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{A.3}$$

Consequently (13) follows immediately. □

Proof of Proposition 7 . By (13) we have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |u|^q \, dx &\leq C_u \int_{\mathbb{R}^N \setminus B_R} |x|^{-((N-1)/p)q} \, dx \\ &= C_u \int_R^\infty r^{-(N-1)(q/p-1)} dr, \end{aligned} \tag{A.4}$$

where $C_u = (p/\omega_{N-1})^{q/p} \|u\|_{L^p(\mathbb{R}^N)}^{q(p-1)/p} \|\nabla u\|_{L^p(\mathbb{R}^N)}^{q/p}$. When $(N-1)(q/p-1) > 1$, that is, $q > pN/(N-1)$, we have

$$\int_{\mathbb{R}^N \setminus B_R} |u|^q \, dx \leq C_u R^{-(N-1)(q/p-1)+1}. \tag{A.5}$$

Let $\{u_m\}$ be a sequence such that $u_m \rightharpoonup 0$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Firstly we show that the case of $q \in (pN/(N-1), p^*)$. In this case we have

$$\int_{\mathbb{R}^N} |u_m|^q \, dx \leq \int_{B_R} |u_m|^q \, dx + C_{u_m} R^{-(N-1)(q/p-1)+1}. \tag{A.6}$$

Since C_{u_m} is bounded from above uniformly, letting $m \longrightarrow \infty$ and $R \longrightarrow \infty$ we have $u_m \longrightarrow 0$ strongly in $L^q(\mathbb{R}^N)$.

Next, for $q \in (p, pN/(N-1)]$ using interpolation of L^q space, we have

$$\|u_m\|_{L^q(\mathbb{R}^N)} \le \|u_m\|^{\lambda}_{L^p(\mathbb{R}^N)} \|u_m\|^{1-\lambda}_{L^r(\mathbb{R}^N)}, \tag{A.7}$$

where $r \in (pN/(N-1), p^*)$. Since $\|u_m\|_{L^r(\mathbb{R}^N)} \longrightarrow 0$ and $\|u_m\|_{L^p(\mathbb{R}^N)}$ is bounded we have $\|u_m\|_{L^q(\mathbb{R}^N)} \longrightarrow 0$. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

An earlier abstract of this manuscript was presented in Osaka University Differential Equation Seminar in 2017.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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