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*Research Article*

# On Some Properties of Cowen-Douglas Class of Operators

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We will consider multiplication operators on a Hilbert space of analytic functions on a domain $\Omega \subset \mathbb{C}$. For a bounded analytic function $\varphi$ on $\Omega$, we will give necessary and sufficient conditions under which the complement of the essential spectrum of $M_\varphi$ in $\varphi(\Omega)$ becomes nonempty and this gives conditions for the adjoint of the multiplication operator $M_\varphi$ belongs to the Cowen-Douglas class of operators. Also, we characterize the structure of the essential spectrum of a multiplication operator and we determine the commutants of certain multiplication operators. Finally, we investigate the reflexivity of a Cowen-Douglas class operator.

## 1. Introduction

In this section we include some preparatory material which will be needed later.

For a positive integer $n$ and a domain $U \subset \mathbb{C}$, the Cowen-Douglas class $B_n(U)$ consists of bounded linear operators $T$ on any fixed separable infinite dimensional Hilbert space $X$ with the following properties:

(a) $U$ is a subset of the spectrum of $T$.

(b) $\mathrm{Ran}(\lambda - T) = X$ for every $\lambda \in U$.

(c) $\dim[\ker(\lambda - T)] = n$ for every $\lambda \in U$.

(d) $\mathrm{Span}\{\ker(\lambda - T) : \lambda \in U\} = X$.

Here Span denotes the closed linear span of a collection of sets in $X$. The classes $B_n(U)$ were introduced by Cowen-Douglas (see [1]), and each element of $B_n(U)$ is called a Cowen-Douglas class operator. By $B_n$, we mean $B_n(U)$ for some complex domain $U$. For the study of Cowen-Douglas classes $B_n$, we mention [1–7].

Recall that a bounded linear operator $A$ on a Hilbert space is a Fredholm operator if and only if $\mathrm{ran}\, A$ is closed and both $\ker A$ and $\ker A^*$ are finite dimensional. We use $\sigma(A)$ and $\sigma_e(A)$ to denote, respectively, the spectrum of $A$ and the essential spectrum of $A$.

Now let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall that if $A \in \mathscr{B}(\mathscr{H})$, then $\mathrm{Lat}(A)$ is by definition the lattice of all invariant subspaces of $A$, and $\mathrm{AlgLat}(A)$ is the algebra of all operators $B$ in $\mathscr{B}(\mathscr{H})$ such that $\mathrm{Lat}(A) \subset \mathrm{Lat}(B)$. An operator $A$ in $\mathscr{B}(\mathscr{H})$ is said to be reflexive if $\mathrm{AlgLat}(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathscr{B}(\mathscr{H})$ that contains $A$ and the identity $I$ and is closed in the weak operator topology.

Also, if $\mathscr{H}$ is a Hilbert space of functions analytic on a plane domain $\Omega$, a complex-valued function $\varphi$ on $\Omega$ for which $\varphi f \in \mathscr{H}$ for every $f \in \mathscr{H}$ is called a multiplier of $\mathscr{H}$ and the multiplier $\varphi$ on $\mathscr{H}$ determines a multiplication operator $M_\varphi$ on $\mathscr{H}$ by $M_\varphi f = \varphi f$, $f \in \mathscr{H}$. The set of all multipliers of $\mathscr{H}$ is denoted by $M(\mathscr{H})$. Clearly $M(\mathscr{H}) \subset H^\infty(\Omega)$, where $H^\infty(\Omega)$ is the space of all bounded analytic function on $\Omega$. In fact $\|\varphi\|_\infty \le \|M_\varphi\|$ (see [8]).

Let $\mathscr{H}$ be a Hilbert space of functions analytic on a domain $\Omega \subset \mathbb{C}$ satisfying the following axioms:

*Axiom 1.* For every point $\omega \in \Omega$, the functional of point evaluation at $\omega$, is a nonzero bounded linear functional on $\mathscr{H}$.

*Axiom 2.* Every function $\varphi \in H^\infty(\Omega)$ is a multiplier of $\mathscr{H}$.

*Axiom 3.* If $f \in \mathscr{H}$ and $f(\lambda) = 0$, then there is a function $g \in \mathscr{H}$ such that $(z - \lambda)g = f$.

A space $\mathscr{H}$ satisfying the above conditions is called *Hilbert space of analytic functions on* $\Omega$ (see [3, 9]). The Hardy and

Bergman spaces are examples for Hilbert spaces of analytic functions on the open unit disk.

Note that, by Axiom 1, there exists a reproducing kernel $k_w \in \mathscr{H}$ such that $f(\omega) = \langle f, k_\omega \rangle$ for all $f \in \mathscr{H}$. Also, by using Axiom 2 and the closed-graph theorem, the operator of multiplication by $\varphi$, $M_\varphi$, is a bounded linear operator on $\mathscr{H}$. So Axiom 2 says that $M(\mathscr{H}) = H^\infty(\Omega)$. If $M_z$ is polynomially bounded on $\mathscr{H}$ and $\Omega$ is the open unit disk, then $M(\mathscr{H}) = H^\infty(\Omega)$ (see [9, Theorem 1]). In the rest of the paper we assume that $\mathscr{H}$ is a Hilbert space of analytic function on a bounded plane domain $\Omega$.

In this paper, we want to study some properties of operators in $B_n$. We see that complement of the essential spectrum of a multiplication operator $M_\varphi$ is nonempty if and only if the adjoint of $M_\varphi$ belongs to some $B_n$. Also, we investigate the intertwining multiplication operators and reflexivity of the multiplication operator on $B_n$. For some other source on these topics one can see [10–16].

## 2. Multiplication Operators with Adjoint in $B_n$ and Its Spectra

Recall that if $T$ is a Cowen-Douglas class operator, then it should be $\sigma(T) \setminus \sigma_e(T) \neq \emptyset$. For $\varphi \in H^\infty(\Omega)$, we would like to give some necessary and sufficient conditions so that $\sigma(M_\varphi) \setminus \sigma_e(M_\varphi)$ becomes a nonempty open set. This implies a sufficient condition for the adjoint of the multiplication operator $M_\varphi$ to be a Cowen-Douglas class operator.

**Theorem 1.** *Let $\varphi$ be a nonconstant function in $H^\infty(\Omega)$, $\sigma(M_\varphi) \setminus \sigma_e(M_\varphi) \neq \emptyset$, and $k_z/\|k_z\| \to 0$ weakly as $dist(z, \partial\Omega) \to 0$. Then there exist a domain $V \subset \varphi(\Omega)$ and a positive integer $n$ such that $\Omega \cap \varphi^{-1}(\lambda)$ consists of $n$ points (counting multiplicity) for every $\lambda \in V$.*

*Proof.* First note that if $\lambda \in \varphi(\Omega)$, then $\lambda = \varphi(\omega)$ for some $\omega \in \Omega$. But by Axiom 1, the functional of evaluation at $\omega$ is a bounded point evaluation; thus the reproducing kernel $k_\omega$ is defined and we have

$$M_\varphi^* k_\omega = \overline{\varphi(\omega)} k_\omega. \tag{1}$$

Thus $\lambda \in \sigma(M_\varphi)$ and clearly $\overline{\varphi(\Omega)} \subset \sigma(M_\varphi)$. Now let $\lambda \notin \overline{\varphi(\Omega)}$. Then $\varphi - \lambda$ is an invertible element of $H^\infty(\Omega)$. But by Axiom 2, we have $M(\mathscr{H}) = H^\infty(\Omega)$; thus $M_{\varphi-\lambda}$ is invertible. This implies that $\sigma(M_\varphi) \subset \overline{\varphi(\Omega)}$; thus indeed $\sigma(M_\varphi) = \overline{\varphi(\Omega)}$. Now we prove that

$$\sigma(M_\varphi) \setminus \sigma_e(M_\varphi) = \varphi(\Omega) \setminus \sigma_e(M_\varphi). \tag{2}$$

For this it is sufficient to show that $\partial\varphi(\Omega) \subset \sigma_e(M_\varphi)$. Let $\lambda \notin \sigma_e(M_\varphi)$. If $\lambda \in \partial\varphi(\Omega)$, then there exists a sequence $\{z_n\}_n \subset \Omega$ such that $\varphi(z_n) \to \lambda$. By passing to a subsequence if necessary, we may assume that $\{z_n\}_n$ converges to a point in $\partial\Omega$ and so by our assumptions $k_{z_n}/\|k_{z_n}\| \to 0$ weakly. On the other hand we have

$$(M_\varphi - \lambda)^* \left( \frac{k_{z_n}}{\|k_{z_n}\|} \right) = \frac{\left(\overline{\varphi(z_n)} - \overline{\lambda}\right) k_{z_n}}{\|k_{z_n}\|} \tag{3}$$

for all $n \in \mathbb{N}$. So we get

$$\left\| (M_\varphi - \lambda)^* \left( \frac{k_{z_n}}{\|k_{z_n}\|} \right) \right\| \longrightarrow 0 \tag{4}$$

which contradicts the fact that $(M_\varphi - \lambda)^*$ is Fredholm. Thus we have

$$\sigma(M_\varphi) \setminus \sigma_e(M_\varphi) = \varphi(\Omega) \setminus \sigma_e(M_\varphi). \tag{5}$$

Now, let $V$ be a connected component of the open set $\varphi(\Omega) \setminus \sigma_e(M_\varphi)$. Since $V \cap \sigma_e(M_\varphi) = \emptyset$, thus $M_\varphi - \lambda$ is Fredholm for every $\lambda$ in $V$. Also, note that if $(M_\varphi - \lambda)f = 0$, then $f = 0$ on $\Omega \setminus (\varphi - \lambda)^{-1}\{0\}$ which is open. Hence $f \equiv 0$ and so $M_\varphi - \lambda$ is injective. Thus

$$\operatorname{index}(M_\varphi - \lambda)^* = \dim\left[\ker(M_\varphi - \lambda)^*\right] \tag{6}$$

for all $\lambda$ in $V$. But the index function is continuous from the set of semi-Fredholm operators into $\mathbb{Z} \cup \{\pm\infty\}$ with discrete topology; thus, index$(M_\varphi - \lambda)^*$ is constant for all $\lambda$ in $V$. Put

$$\dim\left[\ker(M_\varphi - \lambda)^*\right] = n. \tag{7}$$

If $z \in V$, then $\lambda = \varphi(\lambda_0)$ for some $\lambda_0 \in \Omega$ and so $M_\varphi^* k_{\lambda_0} = \overline{\lambda} k_{\lambda_0}$. Thus $k_{\lambda_0} \in \ker(M_\varphi - \lambda)^*$. Since a finite subset of points $\omega$ in $\Omega$ yields a linearly set independent set of functions $k_\omega$ in $\mathscr{H}$, thus $\Omega \cap \varphi^{-1}(\lambda)$ consist of at most $n$ points for all $\lambda$ in $V$. So for each fixed $\lambda \in V$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_m$ in $\Omega$ and $n_1, n_2, \ldots, n_m$ in $\mathbb{N}$ such that $m \leq n$ and for all $z \in \Omega$ we have

$$\varphi(z) - \lambda = \psi(z)(z - \lambda_1)^{n_1}(z - \lambda_2)^{n_2} \cdots (z - \lambda_m)^{n_m}, \tag{8}$$

where $\psi$ belongs to $H^\infty(\Omega)$ and is nonvanishing on $\Omega$. Now by a method used in the proof of [3, Proposition 3.1] we show that the function $\psi$ is also bounded below on $\Omega$. For this choose $r > 0$ such that $\overline{B(\lambda, r)}$ is contained in $V$. Put $K = \varphi^{-1}(\overline{B(\lambda, r)})$, and thus $K$ is a compact subset of $\Omega$ and so it has a positive distance $\delta$ to $\partial\Omega$. Now if $\psi$ is not bounded below on $\Omega$, then there exists a sequence $\{z_i\}$ in $\Omega - \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\psi(z_i) \to 0$ as $i \to \infty$. Since $\psi$ is nonvanishing on $\Omega$ implies that $\varphi(z_i) \to \lambda$, so there exists a positive integer $N$ such that $\varphi(z_i) \in \overline{B(\lambda, r)}$ for all $i > N$. Hence $z_i \in K$ for all $i > N$ that is contradiction to $z_i \to \partial\Omega$. Thus the function $\psi$ is indeed bounded below on $\Omega$. Now since $\psi$ is bounded below and bounded above on $\Omega$ it is an invertible element of $H^\infty(\Omega)$ and so the operator $M_\psi$ is invertible on $\mathscr{H}$ because $M(\mathscr{H}) = H^\infty(\Omega)$. Thus index$(M_\psi) = 0$. Note that since

$$\begin{aligned} &M_\varphi - \lambda \\ &\quad = M_\psi (M_z - \lambda_1)^{n_1} (M_z - \lambda_2)^{n_2} \cdots (M_z - \lambda_m)^{n_m}, \end{aligned} \tag{9}$$

we get

$$\operatorname{index}(M_\varphi - \lambda) = \sum_{j=1}^{m} n_j \left(\operatorname{index}(M_z - \lambda_j)\right). \tag{10}$$

But $M_\varphi - \lambda$ is injective for all $\lambda \in V$; thus

$$\operatorname{index}\left(M_\varphi - \lambda\right) = -\dim\left[\ker\left(M_\varphi - \lambda\right)^*\right] = -n. \tag{11}$$

Clearly, $M_z - \lambda_j$ is injective; thus

$$\operatorname{index}\left(M_z - \lambda_j\right) = -\dim\left[\ker\left(M_z - \lambda_j\right)^*\right] \tag{12}$$

for $j = 1, \ldots, m$. Note that, by Axiom 3 on $\mathscr{H}$, $\ker(M_z - \lambda_j)^*$ is one-dimensional (see [17]); thus $\sum_{j=1}^m n_j = n$ and therefore $\Omega \cap \varphi^{-1}(\lambda)$ consists of exactly $n$ points (counting multiplicity) for every $\lambda \in V$ and now the proof is complete. □

From the proof of Theorem 1, we can conclude the following result.

**Corollary 2.** *Let $\varphi$ be a nonconstant function in $H^\infty(\Omega)$ and $k_z/\|k_z\| \to 0$ weakly as $dist(z, \partial\Omega) \to 0$. Then $\partial\varphi(\Omega) \subset \sigma_e(M_\varphi)$.*

Note that, by Axiom 3, for every $\lambda \in \Omega$ the operator $M_{z-\lambda}$ is bounded below on $\mathscr{H}$ and also the space $\mathscr{H} \ominus (z - \lambda)\mathscr{H}$ is one-dimensional (see [3]). So the Hilbert space under consideration, $\mathscr{H}$, satisfies the conditions assumed by Zhu in [7].

The following result was stated by Zhu in [7, Proposition 5.2], but its proof is left to readers. For this reason we sketch a proof of this proposition and although our proof might seem more straightforward than the one stated by Zhu, we emphasise that our main idea is given from [7].

**Proposition 3.** *Suppose $\varphi \in H^\infty(\Omega)$ and $V$ is a domain contained in $\varphi(\Omega)$. If there exists a positive integer $n$ such that $\Omega \cap \varphi^{-1}(\lambda)$ consists of $n$ points (counting multiplicity) for every $\lambda \in V$, then the adjoint of the operator $M_\varphi : \mathscr{H} \to \mathscr{H}$ belongs to the Cowen-Douglas class $B_n(U)$, where $U = \{\bar{z} : z \in V\}$.*

*Proof.* Let $\lambda = \varphi(\omega) \in V$. Then there exist an invertible function $\psi \in H^\infty(\Omega)$ and $z_1, z_2, \ldots, z_m \in \Omega \cap \varphi^{-1}(\lambda)$ such that

$$\begin{aligned}&\left(M_\varphi - \lambda\right)^* \\ &\quad = M_\psi^* \left(M_z - z_1\right)^{*^{k_1}} \left(M_z - z_2\right)^{*^{k_2}} \cdots \left(M_z - z_m\right)^{*^{k_m}},\end{aligned} \tag{13}$$

where $\sum_{i=1}^m k_i = n$. Axiom 3 implies that for all $i = 1, \ldots, m$, $(M_z - z_i)^*$ is onto (see [17]); thus for all $\lambda \in V$, $(M_\varphi - \lambda)^*$ is onto since $M_\psi$ is invertible. Also, by Axiom 3, $\dim[\ker(M_z - z_i)^*] = 1$ for $i = 1, \ldots, m$ and so

$$\begin{aligned}\dim\left[\ker\left(M_\varphi - \lambda\right)^*\right] &= \sum_{i=1}^m k_i \dim\left[\ker\left(M_z - z_i\right)^*\right] \\ &= \sum_{i=1}^m k_i = n.\end{aligned} \tag{14}$$

Finally, we note that

$$\begin{aligned}&\operatorname{Span}\left\{k_\omega : \omega \in \varphi^{-1}(V)\right\} \\ &\quad \subset \operatorname{Span}\left\{\ker\left(M_\varphi - \lambda\right)^* : \lambda \in V\right\}.\end{aligned} \tag{15}$$

Now, since $\varphi^{-1}(V)$ is open, $\operatorname{Span}\{k_\omega : \omega \in \varphi^{-1}(V)\} = \mathscr{H}$ and so the proof is complete. □

**Corollary 4.** *Under the conditions of Theorem 1, there exist a positive integer $n$ and a domain $U$ in the complex plane such that $M_\varphi^* \in B_n(U)$.*

*Proof.* By Theorem 1 and Proposition 3 it is clear. □

Now we investigate the converse of Theorem 1.

**Corollary 5.** *Let $\varphi$ be a nonconstant function in $H^\infty(\Omega)$. If there exists a domain $V \subset \varphi(\Omega)$ and a positive integer $n$ such that $\Omega \cap \varphi^{-1}(\lambda)$ consists of $n$ points (counting multiplicity) for every $\lambda \in V$; then $V \subset \sigma(M_\varphi) \setminus \sigma_e(M_\varphi)$.*

*Proof.* By Proposition 3, the adjoint of the operator $M_\varphi : \mathscr{H} \to \mathscr{H}$ belongs to the Cowen-Douglas class $B_n(U)$, where $U = \{\bar{z} : z \in V\}$. Hence for all $\lambda \in U$, $M_\varphi^* - \lambda$ is Fredholm and so clearly $V \subset \sigma(M_\varphi) \setminus \sigma_e(M_\varphi)$. □

**Corollary 6.** *Let $M_\varphi^* \in B_n(U)$ for some positive integer $n$ and a complex domain $U$. If $k_z/\|k_z\| \to 0$ weakly as $dist(z, \partial\Omega) \to 0$, then $\Omega \cap \varphi^{-1}(\lambda)$ consists of $n$ point (counting multiplicity) for every $\lambda \in V$ where $V = \{\bar{z} : z \in U\}$.*

*Proof.* First note that $M_\varphi^* - \lambda$ is Fredholm for all $\lambda \in U$; thus

$$V \subset \sigma\left(M_\varphi\right) \setminus \sigma_e\left(M_\varphi\right) = \overline{\varphi(\Omega)} \setminus \sigma_e\left(M_\varphi\right). \tag{16}$$

But by Corollary 2, $\partial\varphi(\Omega) \subset \sigma_e(M_\varphi)$; thus, $V \subset \varphi(\Omega)$. Now if $\lambda \in V$, then $\lambda = \varphi(\omega)$ for some $\omega \in \Omega$ and clearly $k_\omega \in \ker(M_\varphi - \lambda)^*$. Since $\dim[\ker(M_\varphi - \lambda)^*] = n$ and a finite subset of points $\omega$ in $\Omega$ yields a linearly independent set of functions $k_\omega$ in $\mathscr{H}$, thus $\Omega \cap \varphi^{-1}(\lambda)$ consist of at most n points for all $\lambda \in V$. Now by the same method used in the proof of Theorem 1, we can see that $\Omega \cap \varphi^{-1}(\lambda)$ consists of exactly $n$ points (counting multiplicity) for every $\lambda \in V$. □

*Example 7.* Consider the Hilbert Bergman space $L_a^2(\mathbb{D})$ where $\mathbb{D}$ is the open unit disc in the complex domain. Then $L_a^2(\mathbb{D})$ holds in Axioms 1, 2, and 3 (see [17, Theorem 8.5, page 67]). For the Bergman reproducing kernel function, $k_z$, clearly we can see that $\|k_z\| \to \infty$ as $\operatorname{dist}(z, \partial\mathbb{D}) \to 0$. So if $p$ is a polynomial, then

$$\left\langle p, \frac{k_z}{\|k_z\|} \right\rangle = \frac{p(z)}{\|k_z\|} \longrightarrow 0 \tag{17}$$

as $\operatorname{dist}(z, \partial\mathbb{D}) \to 0$. But polynomials are dense in $L_a^2(\mathbb{D})$; thus $k_z/\|k_z\| \to 0$ weakly as $\operatorname{dist}(z, \partial\mathbb{D}) \to 0$. Now by Theorem 1 and the proof of Corollary 5, we can see that $M_\varphi^* \in B_n(U)$ for some positive integer $n$ and a complex domain $U$ if and only if $\sigma(M_\varphi) \setminus \sigma_e(M_\varphi) \neq \emptyset$.

**Proposition 8.** *Let $\varphi$ be a nonconstant function in $H^\infty(\Omega)$ and $k_z/\|k_z\| \to 0$ weakly as $dist(z, \partial\Omega) \to 0$. Then*

$$\sigma_e\left(M_\varphi\right) = \cap_n \varphi\left(\left\{z \in \Omega : dist\left(z, \partial\Omega\right) < \frac{1}{n}\right\}\right). \tag{18}$$

*Proof.* Let $\lambda \notin \sigma_e(M_\varphi)$; then $M_\varphi - \lambda$ is Fredholm. Now we show that $\varphi - \lambda$ is bounded away from zero near $\partial\Omega$. By way of contradiction, let $\{z_n\}_n \subset \Omega$ be a sequence such that $\varphi(z_n) \to \lambda$ and $\{z_n\}_n$ converges to a point in $\partial\Omega$. Note that by our assumptions $k_{z_n}/\|k_{z_n}\| \to 0$ weakly and

$$\left(M_\varphi - \lambda\right)^* \left(\frac{k_{z_n}}{\left\|k_{z_n}\right\|}\right) = \frac{\left(\overline{\varphi\left(z_n\right)} - \overline{\lambda}\right) k_{z_n}}{\left\|k_{z_n}\right\|} \tag{19}$$

for all $n \in \mathbb{N}$. So we get

$$\left\|\left(M_\varphi - \lambda\right)^* \left(\frac{k_{z_n}}{\left\|k_{z_n}\right\|}\right)\right\| \longrightarrow 0. \tag{20}$$

This is a contradiction because $(M_\varphi-\lambda)^*$ is Fredholm. Hence, $\varphi - \lambda$ is bounded away from zero near $\partial\Omega$ and so there exists $m \in \mathbb{N}$ large enough such that

$$\inf\left\{\left|\varphi(z) - \lambda\right| : \operatorname{dist}(z, \partial\Omega) < \frac{1}{m}\right\} > 0. \tag{21}$$

This implies that

$$\lambda \notin \cap_n \varphi\left(\left\{z \in \Omega : \operatorname{dist}(z, \partial\Omega) < \frac{1}{n}\right\}\right). \tag{22}$$

Conversely, if

$$\lambda \notin \cap_n \varphi\left(\left\{z \in \Omega : \operatorname{dist}(z, \partial\Omega) < \frac{1}{n}\right\}\right), \tag{23}$$

then $\varphi-\lambda$ is bounded away from zero near $\partial\Omega$. Since the zeros of an analytic function are isolated, thus the zeros of $\varphi - \lambda$ are finite. Let $\lambda_1, \lambda_2, \ldots, \lambda_j$ be all zeros (counting multiplicity) of $\varphi - \lambda$ in $\Omega$ such that

$$\varphi(z) - \lambda = \psi(z)\left(z - \lambda_1\right)\left(z - \lambda_2\right)\cdots\left(z - \lambda_j\right). \tag{24}$$

Clearly the function $\psi$ is invertible on $\Omega$ and so $M_\psi$ is bounded below. Also, by Axiom 3 on $\mathscr{H}$, $M_z - \lambda_j$ is Fredholm for all $i = 1, \ldots, j$. This implies that $M_\varphi - \lambda$ is Fredholm and so $\lambda \notin \sigma_e(M_\varphi)$. So the proof is complete. □

## 3. Intertwining Multiplication Operators

The following characterization of the commutant $\{T\}'$ of $T$ is given in Theorem 3.7 of [2], which is stated for the convenience of the reader. Note that $K$ is the reproducing kernel for a coanalytic functional Hilbert space $\mathscr{K}$ defined in [2].

**Theorem 9.** *If $S$ is in $B_n(\Omega)$ and the operator $X$ commutes with $S$, then there exists an analytic function $\Phi : \Omega \to B_n(\mathbb{C}^n)$ such that $XK(\lambda, \cdot) = K(\lambda, \cdot)\Phi(\lambda)$ (all $\lambda \in \Omega$) and for every $f \in \mathscr{K}$, $X^* f(\cdot) = (\Phi(\cdot))^* f(\cdot)$.*

In the following let $\Omega$ be such that if $\lambda \in \Omega$ then $-\lambda \in \Omega$. Also we assume that the composition operator $C_{-z} : \mathscr{H} \to \mathscr{H}$ defined by $C_{-z} f = f(-z)$ is bounded.

**Proposition 10.** *Suppose that $\varphi \in H^\infty(\Omega)$ and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(\omega)$ is a singleton for every $\omega \in V$. If $\varphi$ is odd, $SM_{\varphi^2} = M_{\varphi^2}S$ and $SM_{\varphi^{2n-1}} - M_{\varphi^{2n-1}}S$ is compact for some natural number $n$; then $S = M_h$ for some $h \in H^\infty(\Omega)$.*

*Proof.* Note that, by Proposition 3, the adjoint of the operator $M_\varphi : \mathscr{H} \to \mathscr{H}$ belongs to the Cowen-Douglas class $B_1(U)$, where $U = \{\overline{z} : z \in V\}$. If $n = 1$, all conditions of Theorem 4 in [5] hold and so there exists $h \in H^\infty(\Omega)$ such that $S = M_h$. For $n > 1$, put

$$T_1 = SM_{\varphi^{2n-1}} - M_{\varphi^{2n-1}}S. \tag{25}$$

Clearly $T_1 M_\varphi = -M_\varphi T_1$ and so by Proposition 3 in [5], there exists $h \in H^\infty(\Omega)$ such that $T_1 = M_h C_{-z}$. But $M_h = M_h C_{-z} \circ C_{-z}$ is compact; thus by the Fredholm Alternative Theorem, $h = 0$ and so $T_1 = 0$. Hence $SM_{\varphi^{2n-1}} = M_{\varphi^{2n-1}}S$. Now we show that $SM_{\varphi^{2n-3}} = M_{\varphi^{2n-3}}S$. Put

$$T_2 = SM_{\varphi^{2n-3}} - M_{\varphi^{2n-3}}S. \tag{26}$$

And note that $M_{\varphi^2}T_2 = 0$. This implies that $T_2 = 0$, since $\varphi$ is analytic and the zeros of $\varphi$ are at most countable. Therefore $SM_{\varphi^{2n-3}} = M_{\varphi^{2n-3}}S$. Now if $n = 2$, then $SM_\varphi = M_\varphi S$ and so by Proposition 4.1 in [7] the proof is complete. Else, by continuing this manner, we can conclude that $SM_\varphi = M_\varphi S$ which implies that $S = M_\varphi$ for some $h \in H^\infty(\Omega)$. □

**Proposition 11.** *Suppose that $\varphi \in H^\infty(\Omega)$ and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(\omega)$ is a singleton for every $\omega \in V$. If $\varphi$ is odd, $SM_{\varphi^2} = M_{\varphi^2}S$, and $SM_{\varphi^{2n-1}} + M_{\varphi^{2n-1}}S$ is compact for some natural number $n$, then $S = M_h C_{-z}$ for some $h \in H^\infty(\Omega)$.*

*Proof.* If $n = 1$, put

$$T_1 = SM_\varphi + M_\varphi S. \tag{27}$$

Then $T_1 M_\varphi = M_\varphi T_1$. Thus $T_1 = M_{h_1}$ for some $h_1 \in H^\infty(\Omega)$. But $M_{h_1}$ is compact, hence $h_1 = 0$ and so $T_1 = 0$. This implies that $SM_\varphi = -M_\varphi S$. Now by Proposition 3 in [5], $S = M_h C_{-z}$ for some $h \in H^\infty(\Omega)$. If $n > 1$, put $T_2 = SM_{\varphi^{2n-1}} + M_{\varphi^{2n-1}}S$. Then, clearly $M_\varphi T_2 = T_2 M_\varphi$ from which we can conclude that $T_2 = M_{h_1}$ for some $h_1 \in H^\infty(\Omega)$. The compactness of $M_{h_1}$ implies that $h_1 = 0$ and so $T_2 = 0$.

Thus $SM_{\varphi^{2n-1}} = -M_{\varphi^{2n-1}}S$. Put

$$T_3 = SM_{\varphi^{2n-3}} + M_{\varphi^{2n-3}}S. \tag{28}$$

Hence $M_{\varphi^2}T_3 = 0$ which implies that $T_3 = 0$. Therefore, $SM_{\varphi^{2n-3}} = -M_{\varphi^{2n-3}}S$. If $n = 2$, then $SM_\varphi = -M_\varphi S$ and the proof is complete. If $n > 2$, by continuing this manner, finally we can see that $SM_\varphi = -M_\varphi S$ and this completes the proof. □

## 4. Reflexivity in Cowen-Douglas Class of Operators

It is shown in [4] that, under sufficient conditions, an operator $T$ in the Cowen-Douglas class $B_n(\Omega)$ can be reflexive, where

$\Omega$ is a special bounded plane domain. In this section we give some sufficient conditions so that the associated canonical model is reflexive. This answers Question 5.6 in [9, p. 98]. Indeed, we investigate the reflexivity of $B_n(\Omega)$, when $\Omega$ is an arbitrary bounded domain.

It is well known that every operator in the class $B_n(\Omega)$ is unitarily equivalent to the adjoint of the canonical model associated with a generalized Bergman kernel (g.B.k. for brevity) $K$ (see [2, 6]). Actually $K$ is the reproducing kernel for a coanalytic functional Hilbert space $\mathscr{K}_{\mathscr{K}}$ (briefly $\mathscr{K}$) on which we can define the operator $T_{\overline{z}}$ of multiplication by $\overline{z}$. The operator $T = T_{\overline{z}}^*$ acting on $\mathscr{K}$ is called the canonical model associated with $K$. We know that, for every $\lambda$ in $\Omega$, $T - \lambda$ is onto and

$$\ker(T-\lambda) = \operatorname{ran} K(\lambda,\cdot) = \{K(\lambda,\cdot)\xi : \xi \in \mathbb{C}^n\}, \tag{29}$$

and $\dim\ker(T-\lambda) = n$.

Recall that a compact subset $F$ of the plane is a spectral set for a bounded operator $A$ if $F$ contains $\sigma(A)$ and $\|f(A)\| \le \sup_{z\in F}|f(z)|$ for all rational functions $f$ with poles off $F$. Also, an open connected subset $G$ of the plane is called a Carathéodory region if its boundary equals the boundary of the unbounded component of $\mathbb{C} - \overline{G}$.

It is proved in [4] that if $T$ is in $B_1(\Omega)$ and $T^*$ is an injective unilateral weighted shift, then $T$ is reflexive. Also, it has been shown that if $T$ is in $B_n(\Omega)$, where $\Omega$ is a Carathéodory region such that $\sigma(T) = \overline{\Omega}$ is a spectral set for $T$, then $T$ is reflexive (see [4, Theorem 2]). This implies that if $T$ is a contraction in $B_n(\mathbb{D})$ where $\mathbb{D}$ is the open unit disk, then $T$ is reflexive. Here we want to investigate the reflexivity of $T$ on $B_n(\Omega)$, where $\Omega$ is an arbitrary domain in $\mathbb{C}$.

**Theorem 12.** *If $T$ is in $B_n(\Omega)$, where $\Omega \subset \mathbb{C}$ is an arbitrary domain, then there exists a total set $Y$ such that the weak closure of the set $\{p(T)y$: $p$ is a polynomial, $y \in Y\}$ contains AlgLat$(T)$.*

*Proof.* Let $K$ be a g.B.k. on $\Omega$ and let $X \in \mathrm{AlgLat}(T)$. Then by Theorem 9 and [4, Lemma 1], there exists $\psi \in H^\infty(\Omega)$ such that $XK(\lambda,\cdot) = \psi(\lambda)K(\lambda,\cdot)$ for all $\lambda$ in $\Omega$. Now let $F = \{\lambda_n\}_{n=1}^\infty$ be dense in $\Omega$ and choose $\xi_i \in \mathbb{C}^n$ such that $K(\lambda_i,\cdot)\xi_i \ne 0$ for $i = 1, 2, \ldots$. Put $\mathscr{K}_i = \vee\{K(\lambda_i,\cdot)\xi_i\}$ for $i = 1, 2, \ldots$. Define

$$\begin{gathered}\mathscr{K}'_\infty = \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \mathscr{K}_3 \oplus \cdots = \bigoplus_{i=1}^\infty \mathscr{K}_i,\\ T_\infty = T|_{\mathscr{K}_1} \oplus T|_{\mathscr{K}_2} \oplus T|_{\mathscr{K}_3} \oplus \cdots = \bigoplus_{i=1}^\infty T_i,\\ X_\infty = X \oplus X \oplus X \oplus \cdots = \bigoplus_{i=1}^\infty X_i.\end{gathered} \tag{30}$$

Fix $f = \bigoplus_{i=1}^\infty c_i K(\lambda_i,\cdot)\xi_i$ satisfying

$$\sum_{i=1}^\infty |c_i|^2 \|K(\lambda_i,\cdot)\xi_i\|^2 < \infty, \tag{31}$$

where $c_i \ne 0$ for all $i$. Thus $f \in \mathscr{K}'_\infty$. Define

$$\mathscr{M} = \mathrm{cl}\left\{\bigoplus_{i=1}^\infty p(T)\,c_i K(\lambda_i,\cdot)\,\xi_i : p \text{ is a polynomial}\right\}. \tag{32}$$

Since $f \in \mathscr{M}$, $\mathscr{M} \ne \emptyset$. Now clearly $\mathscr{M}$ is closed subspace of $\mathscr{K}'_\infty$ and we have

$$T_\infty f = \bigoplus_{i=1}^\infty c_i TK(\lambda_i,\cdot)\,\xi_i = \bigoplus_{i=1}^\infty c_i\lambda_i K(\lambda_i,\cdot)\,\xi_i. \tag{33}$$

Thus $T_\infty f \in \mathscr{M}$ and so $\mathscr{M} \in \mathrm{Lat}(T_\infty)$. But $\mathrm{Lat}(T) \subseteq \mathrm{Lat}(X)$,;thus $\mathrm{Lat}(T_\infty) \subseteq \mathrm{Lat}(X_\infty)$ and we get $\mathscr{M} \in \mathrm{Lat}(X_\infty)$. Therefore $X_\infty f \in \mathscr{M}$ and so there exists a sequence $\{p_n\}_n$ of polynomials such that

$$\begin{aligned}&\oplus_i c_i p_n(T)\,c_i K(\lambda_i,\cdot)\,\xi_i \longrightarrow\\ &\qquad X_\infty f = \oplus_i c_i\psi(T)\,c_i K(\lambda_i,\cdot)\,\xi_i\end{aligned} \tag{34}$$

in $\mathscr{K}'_\infty$. Thus $\oplus_i c_i(p_n(\lambda_i) - \psi(\lambda_i))K(\lambda_i,\cdot)\xi_i \to 0$ in $\mathscr{K}'_\infty$ and since for all $i$

$$\begin{aligned}&\|\oplus_i c_i(p_n(\lambda_i) - \psi(\lambda_i))\,K(\lambda_i,\cdot)\,\xi_i\|\\ &\quad\ge \|c_i(p_n(\lambda_i) - \psi(\lambda_i))\,K(\lambda_i,\cdot)\,\xi_i\|,\end{aligned} \tag{35}$$

we get $\sup_i\|c_i(p_n(\lambda_i) - \psi(\lambda_i))K(\lambda_i,\cdot)\xi_i\| \to 0$ as $n \to \infty$. But

$$\begin{aligned}&\|c_i(p_n(\lambda_i) - \psi(\lambda_i))\,K(\lambda_i,\cdot)\,\xi_i\|^2\\ &\quad= |c_i(p_n(\lambda_i) - \psi(\lambda_i))|^2\,\langle K(\lambda_i,\lambda_i)\,\xi_i,\xi_i\rangle\\ &\quad= |c_i|^2\,|p_n(\lambda_i) - \psi(\lambda_i)|^2\,\left\|K(\lambda_i,\lambda_i)^{1/2}\xi_i\right\|^2\end{aligned} \tag{36}$$

and $K(\lambda_i,\lambda_i)$ is invertible; thus for all $i$, $|(p_n - \psi)(\lambda_i)| \to 0$ as $n \to \infty$. This implies that $(p_n(T) - X)g \to 0$ for all $g$ in the finite linear combinations of

$$Y = \{K(\lambda_i,\cdot)\,\xi : i \in \mathbb{N},\ \xi \in \mathbb{C}^n\} \tag{37}$$

that is a total subset of $\mathscr{K}$. At this time the proof is complete. □

Let $\psi$, $\{p_n\}_n$, and $F = \{\lambda_i\}_i$ be defined as in the proof of Theorem 12. At the end of the proof of Theorem 12, we saw that, for all $i$, $|(p_n - \psi)(\lambda_i)| \to 0$ as $n \to \infty$. Now we ask the following question.

*Question 13.* In the proof of Theorem 12, is it true that $\sup_i|(p_n - \psi)(\lambda_i)| \to 0$ as $n \to \infty$?

If the answer of Question 13 is positive, then $\|p_n\|_F \le M$ for some $M > 0$ and we may have the following corollary. Note that the special case of this corollary has been proved as Theorem 2 in [4], only whenever $\Omega$ is a Carathéodory region.

**Corollary 14.** *If $T$ is in $B_n(\Omega)$ where $\Omega \subset \mathbb{C}$ is a domain such that $\sigma(T) = \overline{\Omega}$ is a spectral set for $T$, then $T$ is reflexive.*

*Proof.* Let $K$ be a g.B.k. on $\Omega$ and let $X \in \mathrm{AlgLat}(T)$. By Theorem 12, there exists a sequence of polynomials $\{p_n\}_n$ such that $p_n(T)$ converges in the finite linear combinations of

$$\{K(\lambda,\cdot)\xi : \lambda \in F,\ \xi \in \mathbb{C}^n\} \tag{38}$$

that is a total subset of $\mathscr{K}$, where $F = \{\lambda_n\}_{n=1}^{\infty}$ was a dense set in $\Omega$. Also, $\sup_i|(p_n - \psi)(\lambda_i)| \to 0$ as $n \to \infty$. This implies that $\|p_n\|_\Omega = \|p_n\|_F \le M$ for some $M > 0$. Now since $\sigma(T) = \overline{\Omega}$ is a spectral set for $T$, we conclude that $\|p_n(T)\| \le M$. Since the unit ball of $B(\mathscr{K})$ is compact in the weak operator topology, by passing to a subsequence if necessary, we may assume that $p_n(T) \to A$ in the weak operator topology. Therefore, $p_n(T)K(\lambda,\cdot)\xi \to AK(\lambda,\cdot)\xi$ weakly. But

$$p_n(T)K(\lambda,\cdot)\xi = p_n(\lambda)K(\lambda,\cdot)\xi \longrightarrow \psi(\lambda)K(\lambda,\cdot)\xi. \tag{39}$$

Hence, $AK(\lambda,\cdot)\xi = \psi(\lambda)K(\lambda,\cdot)\xi$, where, by the proof of Theorem 12, $\psi$ is a function in $H^\infty(\Omega)$ and satisfies $XK(\lambda,\cdot) = \psi(\lambda)K(\lambda,\cdot)$ for all $\lambda$ in $\Omega$. From this we conclude that $A = X$, so $X \in W(T)$. Therefore, $\mathrm{AlgLat}(T) \subset W(T)$ and $T$ is reflexive. This completes the proof. □

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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